Mixingales on Riesz spaces[1]

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Introduction

- Mixingales are stochastic processes which combine the concepts of martingales and mixing sequences.
- McLeish introduced the term *mixingale* at the 4^{th} Conference of Stochastic Processes and Applications, at York University, Toronto in 1974.
- We generalize the concept of a mixingale to the measure-free Riesz space setting. This generalizes all of the $L^p, 1 \leq p \leq \infty$ variants.
- We also generalize the concept of uniform integrability to the Riesz space setting and prove that a weak law of large numbers holds for Riesz space mixingales.

Background - McLeish

- McLeish defines mixingales using the L^2-norm.
- In McLeish[2] proves invariance principles under strong mixing conditions.
- McLeish[3] also proves a strong law for large numbers for dependent sequences under various conditions.

[2] D.L. McLeish, Invariance Principles for Dependent Variables, *Z. Wahrscheinlichkeitstheorie vew. Gebiete.*, **32** (1975), 165 - 178.

[3] D.L. McLeish, A maximal inequality and dependent strong laws, *The Annals of Probability*, **3** (1975), 829-839.

Background - Andrews and de Jong

- In 1988, Donald W. K. Andrews[4] defined an L^1 analogue of McLeish's mixingales and presented a weak laws of large number for L^1-mixingales.
- The L^1-mixingale condition of Andrews is weaker than McLeish's mixingale condition.
- Andrews makes no restriction on the decay rate of the mixingale numbers, as was assumed by McLeish.
- Mixingales have also been considered in a general $L^p, 1 \leq p < \infty$, by de Jong[5] [6].

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[4] D.W. ANDREWS, Laws of large numbers for dependent non-identically distributed random variables, *Econometric Theory,* **4** (1988), 458-467.

[5] R.M. DE JONG, Weak laws of large numbers for dependent random variables, *Annals of Economics and Statistics,* **51** (1998), 209-225.

[6] R.M. DE JONG, Laws of large numbers for dependent heterogeneous processes, *Econometic Theory,* **11** (1995), 347-358.

Outline

- We define mixingales in a Riesz space and present a weak law of large numbers for mixingales in this setting.
- This order approach highlights the underlying mechanisms of the theory.
- This develops on the work of Kuo, Labuschagne, Vardy and Watson [7] [8] [9] on formulating the theory of stochastic processes in Riesz spaces.
- Other closely related generalizations were given by Stoica [10] and Troitsky [11].

[7] W.-C. Kuo, C.C.A. Labuschagne, B.A. Watson, Discrete time stochastic processes on Riesz spaces, *Indag. Math. N.S.,* **15** (2004), 435-451.

[8] W.-C. Kuo, C.C.A. Labuschagne, B.A. Watson, Conditional expectations on Riesz spaces, *J. Math. Anal. Appl.*, **303** (2005), 509-521.

[9] J.J. Vardy, B.A. Watson, Markov process in Riesz spaces, *Positivity*, **16** (2012), 373-391.

[10] G. Stoica, Martingales in vector lattices, *Bull. Math. Soc. Sci. Math. Roumanie. (N.S.),* **34(82)** (1990), 357-362.

[11] V. Troitsky, Martingales in Banach lattices, *Positivity,* **9** (2005), 437-456.

Preliminaries - Bands and Principal Bands

- A non-empty linear subspace B of the Riesz space E is a band if:
  - (i) the order interval $[-|f|, |f|]$ is in B for each $f \in B$;
  - (ii) for each $D \subset B$ with $\sup D \in E$ we have $\sup D \in B$.
- A principal band is a band generated by a single element.
- If $e \in E_+$ and the band generated by e is E, then e is called a weak order unit of E and we denote the space of e bounded elements of E by

$$E^e = \{f \in E \ : \ |f| \le ke \text{ for some } k \in \mathbb{R}_+\}.$$

Preliminaries - Band Projections

- In a Dedekind complete Riesz space with weak order unit every band is a principal band and, for each band B and $u \in E_+$,
$$P_B u := \sup\{v \,:\, 0 \le v \le u, v \in B\}$$
exists.
- The above map P_B can be extended to E by setting $P_B u = P_B u^+ - P_B u^-$ for $u \in E$.
- With this extension, P_B is a positive linear projection which commutes with the operations of supremum and infimum in that $P(u \vee v) = Pu \vee Pv$ and $P(u \wedge v) = Pu \wedge Pv$.
- Moreover $0 \le P_B u \le u$ for all $u \in E_+$ and the range of P_B is B.

Preliminaries - Order Continuity

Let $T : E \to F$ be an operator where E and F are Riesz spaces.

- We say that T is a positive operator if T maps the positive cone of E to the positive cone of F, denoted $T \geq 0$.
- Here a set D in E is said to be downwards directed if for $f, g \in D$ there exists $h \in D$ with $h \leq f \wedge g$. In this case we write $D \downarrow$ or $f \downarrow_{f \in D}$. If, in addition, $g = \inf D$ in E, we write $D \downarrow g$ or $f \downarrow_{f \in D} g$.
- Let T be a positive operator between E and F. We say that T is order continuous if for each directed set $D \subset E$ with $f \downarrow_{f \in D} 0$ in E we have that $Tf \downarrow_{f \in D} 0$.
- Band projections are order continuous.

Riesz space Conditional Expectation Operators

- Let E be a Dedekind complete Riesz space with weak order unit, e. We say that T is a conditional expectation operator in E if T is all of the following
  - positive
  - order continuous
  - a projection
  - maps weak order units to weak order units
  - has range, $\mathcal{R}(T)$, a Dedekind complete Riesz subspace of E.
- If T is a conditional expectation operator on E, as T is a projection it is easy to verify that at least one of the weak order units of E is invariant under T.

f-algebras

- To access the averaging properties of conditional expectation operators a multiplicative structure is needed.
- In the Riesz space setting the most natural multiplicative structure is that of an f-algebra. This gives a multiplicative structure that is compatible with the order and additive structures on the space.
- The space E^e, where e is a weak order unit of E and E is Dedekind complete, has a natural f-algebra structure generated by setting $(Pe)\cdot(Qe) = PQe = (Qe)\cdot(Pe)$ for band projections P and Q.
- Using Freudenthal's Theorem this multilpication can be extended to the whole of E^e and in fact to the universal completion E^u.
- Here e becomes the multiplicative unit.
- This multiplication is associative, distributive and is positive in the sense that if $x, y \in E_+$ then $xy \geq 0$.

Averaging Operators

- If T is a conditional expectation operator on the Dedekind complete Riesz space E with weak order unit $e = Te$, then restricting our attention to the f-algebra E^e T is an averaging operator, i.e. $T(fg) = fTg$ for $f, g \in E^e$ and $f \in R(T)$.
- In fact E is an E^e module which allows the extension of the averaging property, above, to $f, g \in E$ with at least one of them in E^e.
- f-algebras and the averaging properties of conditional expecation operators have been well studied. [12] [13] [14] [15]

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[12] K. Boulabiar, G. Buskes, A. Triki, Results in f-algebras, *Positivity, Trends in Mathematics* (2007), 73-96.

[13] G. Buskes, A. van Rooij, Almost f-algebras: Commutativity and the Cuachy-Schwartz inequality, *Positivity,* **4** (2000), 227-231.

[14] P.G. Dodds, C.B. Huijsmans, B. de Pagter, Characterizations of conditional expectation-type operators, *Pacific J. Math.,* **141** (1990), 55-77.

[15] J.J. Grobler, B. de Pagter, Operators representable as multiplication-conditional expectation operators, *J. Operator Theory,* **48** (2002), 15-40.

T-universal completeness

- Let E be a Dedekind complete Riesz space with weak order unit and T be a strictly positive conditional expectation on E. The space E is universally complete with respect to T, i.e. T-universally complete, if for each increasing net (f_α) in E_+ with (Tf_α) order bounded in E^u, we have that (f_α) is order convergent in E.
- If E is a Dedekind complete Riesz space and T is a strictly positive conditional expectation operator on E, then E has a T-universal completion which is the natural domain of T, denoted $\mathrm{dom}(T)$ in the universal completion, E^u, of E.
- Here $\mathrm{dom}(T) = D - D$ and $Tx := Tx^+ - Tx^-$ for $x \in \mathrm{dom}(T)$ where

$$D = \{x \in E^u_+ | \exists (x_\alpha) \subset E_+, x_\alpha \uparrow x, (Tx_\alpha) \text{ order bounded in } E^u\},$$

and $Tx := \sup_\alpha Tx_\alpha$, for $x \in D$, where (x_α) is an increasing net in E_+ with $(x_\alpha) \subset E_+$, (Tx_α) order bounded in E^u.

Martingales in Riesz spaces

- Let (T_i) be a sequence of conditional expectations on E indexed by either $\mathbb{N}$ or $\mathbb{Z}$, we say that (T_i) is a filtration on E if
$$T_iT_j = T_i = T_jT_i, \quad \text{for all} \quad i \leq j.$$
- If (T_i) is a filtration and T is a conditional expectation with $T_iT = T = TT_i$ for all i, then we say that the filtration is compatible with T.
- Given a conditional expectation T, the sequence (T_i) of conditional expectations in E compatible with T being a filtration is equivalent to $\mathcal{R}(T_i) \subset \mathcal{R}(T_j)$ for $i \leq j$.
- If (T_i) is a filtration on E and (f_i) is a sequence in E, we say that (f_i) is adapted to the filtration (T_i) if $f_i \in \mathcal{R}(T_i)$ for all i in the index set.
- The double sequence (f_i, T_i) is called a martingale if (f_i) is adapted to the filtration (T_i) and in addition
$$f_i = T_if_j, \quad \text{for} \quad i \leq j.$$

Martingale difference sequences in Riesz spaces

- The double sequence (g_i, T_i) is called a martingale difference sequence if (g_i) is adapted to the filtration (T_i) and
$$T_i g_{i+1} = 0.$$
- We observe that if (f_i) is adapted to the filtration (T_i) then $(f_i - T_{i-1}f_i, T_i)$ is a martingale difference sequence.
- Conversely, if (g_i, T_i) is a martingale difference sequence, then (s_n, T_n) is a martingale, where
$$s_n = \sum_{i=1}^{n} g_i, \quad n \geq 1,$$
and the martingale difference sequence generated from (s_n, T_n) is precisely (g_n, T_n).

Conditional Independence

Let E be a Dedekind complete Riesz space with conditional expectation T and weak order unit $e = Te$.

- Let P and Q be band projections on E. We say that P and Q are T-conditionally independent if

$$TPTQe = TPQe = TQTPe. \tag{1}$$

- We say that two Riesz subspaces E_1 and E_2 of E containing $\mathcal{R}(T)$, are T-conditionally independent if all band projections $P_i, i = 1, 2,$ in E with $P_ie \in E_i, i = 1, 2,$ are T-conditionally independent.
- Let $P_i, i = 1, 2,$ be band projections on E. Then $P_i, i = 1, 2,$ are T-conditionally independent if and only if the closed Riesz subspaces $E_i = \langle P_ie, \mathcal{R}(T)\rangle, i = 1, 2,$ are T-conditionally independent.

Uniform Integrability in L^1

- If $(\Omega, \mathcal{A}, \mu)$ is a probability space and $f_\alpha, \alpha \in \Lambda$, is a family in $L^1(\Omega, \mathcal{A}, \mu)$, indexed by Λ, the family is said to be uniformly integrable if for each $\epsilon > 0$ there is $c > 0$ so that

$$\int_{\Omega_\alpha(c)} |f_\alpha| \, d\mu \leq \epsilon, \quad \text{for all} \quad \alpha \in \Lambda,$$

  where

$$\Omega_\alpha(c) = \{x \in \Omega : |f_\alpha(x)| > c\}.$$

- This concept can be extended to the Riesz space setting as T-uniformity, see the definition below, where T is a conditional expectation operator.

T-Uniformity

- Let E be a Dedekind complete Riesz space with conditional expectation operator T and weak order unit $e = Te$. Let $f_\alpha, \alpha \in \Lambda$, be a family in E, where Λ is some index set. We say that $f_\alpha, \alpha \in \Lambda$, is T-uniform if

$$\sup\{TP_{(|f_\alpha|-ce)^+}|f_\alpha| \,:\, \alpha \in \Lambda\} \to 0 \quad \text{as} \quad c \to \infty. \tag{2}$$

- In the case of the Riesz space being $L^1(\Omega, \mathcal{A}, \mu)$ and T being the expectation operator, the two concepts coincide.

T-uniform families

Lemma

Let E be a Dedekind complete Riesz space with conditional expectation T and let e be a weak order unit which is invariant under T. If $f_\alpha \in E, \alpha \in \Lambda,$ is a T-uniform family, then the set $\{T|f_\alpha| \,:\, \alpha \in \Lambda\}$ is bounded in E.

Proof

Proof: As the sequence $f_\alpha, \alpha \in \Lambda$, is T-uniform

$$J_c := \sup\{TP_{(|f_\alpha|-ce)^+}|f_i| \,:\, \alpha \in \Lambda\} \to 0 \quad \text{as} \quad c \to \infty.$$

In particular this implies that J_c exists for $c > 0$ large and that, for sufficiently large $K > 0$, the set $\{J_c \,:\, c \geq K\}$ is bounded in E. Hence there is $g \in E_+$ so that

$$TP_{(|f_\alpha|-ce)^+}|f_\alpha| \leq g, \quad \text{for all} \quad \alpha \in \Lambda, c \geq K,$$

By the definition of $P_{(|f_\alpha|-ce)^+}$,

$$(I - P_{(|f_\alpha|-ce)^+})|f_\alpha| \leq ce, \quad \text{for} \quad \alpha \in \Lambda, c > 0.$$

Combining the above for $c = K$ gives

$$T|f_\alpha| = TP_{(|f_\alpha|-Ke)^+}|f_\alpha| + T(I - P_{(|f_\alpha|-Ke)^+})|f_\alpha| \leq g + Ke,$$

for all $\alpha \in \Lambda$. □

Mixingales in L^1

In classical probability theory $((f_i)_{i\in\mathbb{N}}, (\mathcal{A}_i)_{i\in\mathbb{Z}})$ is a mixingale in the probability space $(\Omega, \mathcal{A}, \mu)$ if the following hold:

- $(\mathcal{A}_i)_{i\in\mathbb{Z}}$ is an increasing sequence of sub-σ-algebras of $\mathcal{A}$ (i.e. $(\mathcal{A}_i)_{i\in\mathbb{Z}}$ is a filtration);
- $(f_i)_{i\in\mathbb{N}}$ is a sequence in $L^1(\Omega, \mathcal{A}, \mu)$;
- there are sequences $(c_i), (\Phi_i) \subset \mathbb{R}_+$ with $\Phi_i \to 0$ as $i \to \infty$ so that

$$\mathbb{E}[|\mathbb{E}[f_i|\mathcal{A}_{i-m}]|] \leq c_i\Phi_m$$

and

$$\mathbb{E}[|f_i - \mathbb{E}[f_i|\mathcal{A}_{i+m}]|] \leq c_i\Phi_{m+1}.$$

Mixingales in Riesz Spaces

Definition

- Let E be a Dedekind complete Riesz space with conditional expectation operator, T, and weak order unit $e = Te$.
- Let $(T_i)_{i\in\mathbb{Z}}$ be a filtration on E compatible with T.
- Let $(f_i)_{i\in\mathbb{N}}$ be a sequence in E.
- We say that (f_i, T_i) is a mixingale in E compatible with T if there exist $(c_i)_{i\in\mathbb{N}} \subset E_+$ and $(\Phi_m)_{m\in\mathbb{N}} \subset \mathbb{R}_+$ such that $\Phi_m \to 0$ as $m \to \infty$ and for all $i, m \in \mathbb{N}$ we have
  (i) $T|T_{i-m}f_i| \le \Phi_m c_i$,
  (ii) $T|f_i - T_{i+m}f_i| \le \Phi_{m+1} c_i$.

Mixingales

- The numbers $\Phi_m, m \in \mathbb{N}$, are referred to as the mixingale numbers. These numbers give a measure of the temporal dependence of the sequence (f_i).
- The constants (c_i) are chosen to index the 'magnitude' of the the random variables (f_i).
- In many applications the sequence (f_i) is adapted to the filtration (T_i).

Means of mixingales

Lemma

Let E be a Dedekind complete Riesz space with conditional expectation operator, T, and weak order unit $e = Te$. Let $(f_i, T_i)_{i\in\mathbb{N}}$ be a mixingale in E compatible with T.

(a) *The sequence (f_i) has T-mean zero, i.e. $Tf_i = 0$ for all $i \in \mathbb{N}$.*

(b) *If in addition $(f_i)_{i\in\mathbb{N}}$ is T-conditionally independent and $\mathcal{R}(T_i) = \langle f_1, \ldots, f_{i-1}, \mathcal{R}(T)\rangle$ then the mixingale numbers may be taken as zero, where $\langle f_1, \ldots, f_{i-1}, \mathcal{R}(T)\rangle$ is the order closed Riesz subspace of E generated by $f_1, \ldots, f_{i-1}$ and $\mathcal{R}(T)$.*

Proof

Proof: **(a)** Here we observe that the index set for the filtration (T_i) is $\mathbb{Z}$, thus

$$\begin{aligned} |Tf_i| &= |TT_{i-m}f_i| \\ &\leq T|T_{i-m}f_i| \\ &\leq c_i\Phi_m \\ &\to 0 \quad \text{as } m \to \infty \end{aligned}$$

giving $Tf_i = 0$ for all $i \geq 0$.
(b) As (f_i) is adapted to the filtration (T_i), $f_i \in \mathcal{R}(T_i)$ for all $i \in \mathbb{N}$ it follows that

$$f_i - T_{i+m}f_i = 0, \quad \text{for all} \quad i, m \in \mathbb{N}.$$

As (f_i) is T-conditionally independent and as (f_i) has T-mean zero (from (a)), we have that

$$T_{i-m}f_i = Tf_i = 0,$$

for $i, m \in \mathbb{N}$. Thus we can choose $\Phi_m = 0$ for all $m \in \mathbb{N}$. □

Lemma to the Weak Law of Large Numbers

Lemma

Let E be a Dedekind complete Riesz space with conditional expectation operator T, weak order unit $e = Te$ and filtration $(T_i)_{i\in\mathbb{N}}$ compatible with T. Let (f_i) be an e-uniformly bounded sequence adapted to the filtration (T_i), and $g_i := f_i - T_{i-1}f_i$, then (g_i, T_i) is a martingale difference sequence with

$$T|\overline{g}_n| \to 0 \quad \text{as } n \to \infty,$$

where

$$\overline{g}_n := \frac{1}{n}\sum_{i=1}^{n} g_i.$$

Weak law of large numbers

Theorem

[Weak Law of Large Numbers] *Let E be a Dedekind complete Riesz space with conditional expectation operator T, weak order unit $e = Te$ and filtration $(T_i)_{i\in\mathbb{Z}}$. Let $(f_i, T_i)_{i\in\mathbb{N}}$ be a T-uniform mixingale with c_i and Φ_i as defined previously.*

(a) *If $\left(\frac{1}{n}\sum_{i=1}^{n} c_i\right)_{n\in\mathbb{N}}$ is bounded in E then*

$$T|\bar{f}_n| = T\left|\frac{1}{n}\sum_{i=1}^{n} f_i\right| \to 0 \quad \text{as } n \to \infty.$$

(b) *If $c_i = T|f_i|$ for each $i \geq 1$ then*

$$T|\bar{f}_n| = T\left|\frac{1}{n}\sum_{i=1}^{n} f_i\right| \to 0 \quad \text{as } n \to \infty.$$

END

Thank you