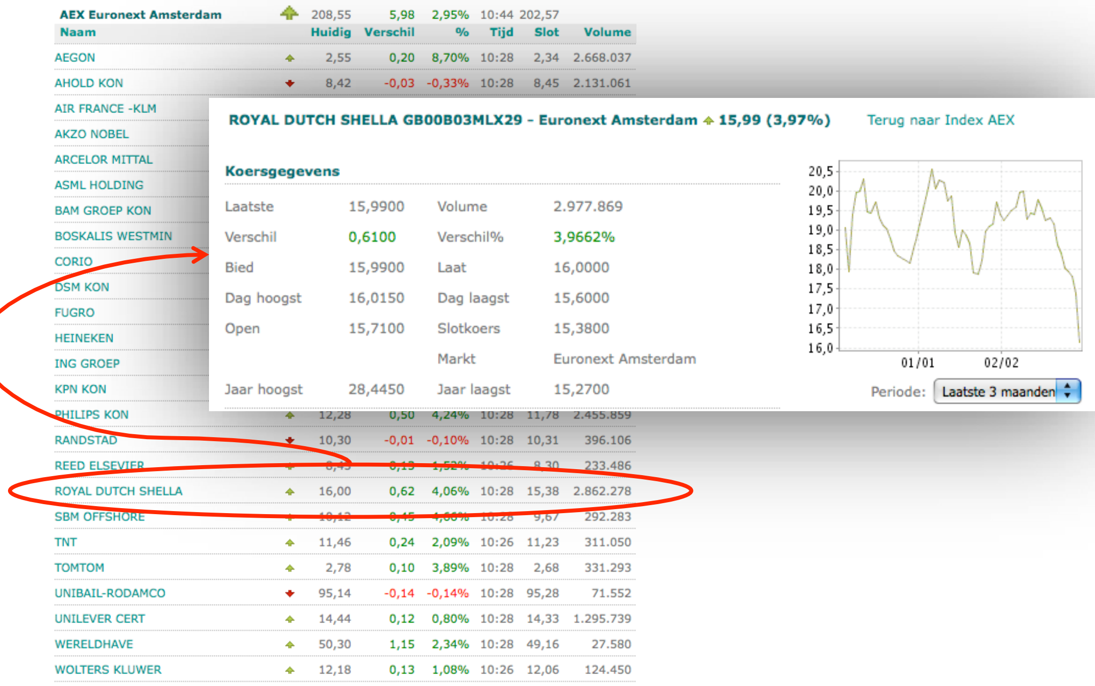
Probabilistic Methods in Option Pricing

Stieltjes Afternoon VU – March 5th 2009 – Jasper Anderluh

Overview

- (Very brief) Introduction to financial mathematics and the link with probability theory.
  - Focus on (equity) derivative pricing.
- Introduction of the Parisian option contract.
- Pricing the Parisian contract with Laplace transforms
  - A side result : an expression for the probability of an event related to excursions of Brownian motion.

Introduction to financial markets - Stocks



AEX Euronext Amsterdam		208,55	5,98	2,95%	10:44	202,57	
Naam		**Huidig**	**Verschil**	**%**	**Tijd**	**Slot**	**Volume**
AEGON	▲	2,55	0,20	8,70%	10:28	2,34	2.668.037
AHOLD KON	▼	8,42	-0,03	-0,33%	10:28	8,45	2.131.061
AIR FRANCE -KLM							
AKZO NOBEL							
ARCELOR MITTAL							
ASML HOLDING							
BAM GROEP KON							
BOSKALIS WESTMIN							
CORIO							
DSM KON							
FUGRO							
HEINEKEN							
ING GROEP							
KPN KON							
PHILIPS KON	▲	12,28	0,50	4,24%	10:28	11,78	2.455.859
RANDSTAD	▼	10,30	-0,01	-0,10%	10:28	10,31	396.106
REED ELSEVIER	▲	8,43	0,13	1,52%	10:26	8,30	233.486
ROYAL DUTCH SHELLA	▲	16,00	0,62	4,06%	10:28	15,38	2.862.278
SBM OFFSHORE	▲	10,12	0,45	4,66%	10:28	9,67	292.283
TNT	▲	11,46	0,24	2,09%	10:26	11,23	311.050
TOMTOM	▲	2,78	0,10	3,89%	10:28	2,68	331.293
UNIBAIL-RODAMCO	▼	95,14	-0,14	-0,14%	10:28	95,28	71.552
UNILEVER CERT	▲	14,44	0,12	0,80%	10:28	14,33	1.295.739
WERELDHAVE	▲	50,30	1,15	2,34%	10:28	49,16	27.580
WOLTERS KLUWER	▲	12,18	0,13	1,08%	10:26	12,06	124.450

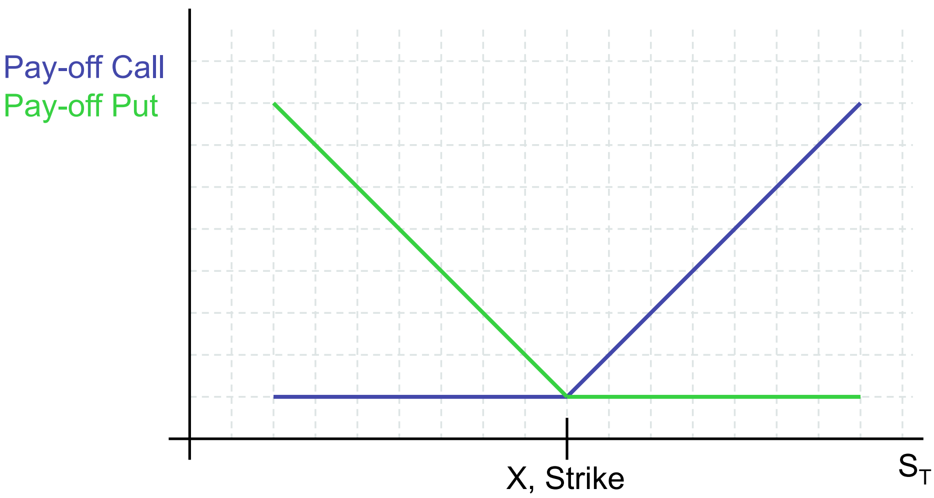
ROYAL DUTCH SHELLA GB00B03MLX29 - Euronext Amsterdam ▲ 15,99 (3,97%) — Terug naar Index AEX

Koersgegevens

Laatste	15,9900	Volume	2.977.869
Verschil	0,6100	Verschil%	3,9662%
Bied	15,9900	Laat	16,0000
Dag hoogst	16,0150	Dag laagst	15,6000
Open	15,7100	Slotkoers	15,3800
		Markt	Euronext Amsterdam
Jaar hoogst	28,4450	Jaar laagst	15,2700

Introduction to financial markets – Options (1)

⇨ The most extensively traded options plain vanilla call and put contracts.



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Introduction to financial markets – Options (2)

▾ Verberg de opties van december 2009

Calls							Puts					
Laatste	Verschil	Vol	Bied	Laat	Uitstaand	Uitoefen	Laatste	Verschil	Vol	Bied	Laat	Uitstaand
3,00	-	-	3,25	3,35	-	14,00	1,90	-	200	1,75	1,85	-
2,70	0,30	225	2,65	2,75	9.334	15,00	2,35	-0,15	735	2,20	2,30	23.542
						15,97						
2,15	0,25	10	2,10	2,20	851	16,00	2,70	-0,30	90	2,65	2,75	10.133
1,45	0,00	-	1,70	1,75	3.154	17,00	3,30	-0,30	91	3,20	3,30	9.602
1,35	0,20	222	1,30	1,35	17.811	18,00	3,85	-0,40	48	3,85	3,95	28.307
0,85	0,00	-	1,00	1,05	11.121	19,00	4,95	0,00	-	4,50	4,60	14.533
0,80	0,15	52	0,75	0,80	62.088	20,00	5,30	-0,45	24	5,25	5,35	48.545
0,60	0,10	130	0,55	0,60	4.504	21,00	6,60	0,00	-	6,05	6,15	512
0,45	0,10	16	0,40	0,45	14.031	22,00	7,45	0,00	-	6,90	7,00	3.220

⇒ Is Royal Dutch moving up or down this day?

⇒ How is it possible that there is a quotation for every expiry and strike?

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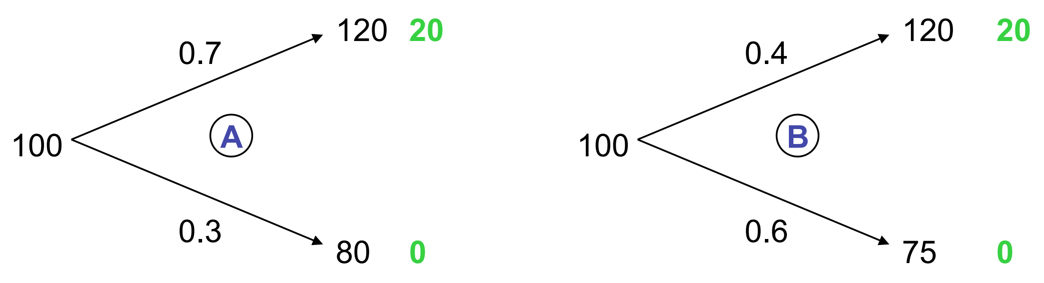
Option Pricing (1)

⇒ Suppose we can model the behavior of a stock as follows,



⇒ And we are interested in a call 100 option

⇒ What is the relation between the call option prices?

- $C_A > C_b$
- $C_A = C_b$
- $C_A < C_b$

Option Pricing (2)

So, in the solution of this very simple example the following elements of derivative pricing arise:

- No-arbitrage assumption. If there is another price in the market than the computed price, it is possible to earn money without taking risk.
- Completeness. If the model satisfies the no-arbitrage assumption and there exists a replication portfolio (or hedge strategy), than the price of the derivative is unique.
- Equivalent Martingale Measure or Risk-Neutral Measure. The probabilities did not appear to be of any interest to the solution of the problem. In fact, the solution could be rewritten such that the option price is an expectation under an equivalent measure. Under this measure, the discounted stock price is a martingale.

Option Pricing (3)

In the Black-Scholes model this is all transferred to continuous time in the following way. Let $(\Omega, \mathbb{P}, \mathcal{F})$ be a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and W_t a standard Brownian motion w.r.t. this filtration.

- Stock price process: $dS_t = \mu S_t dt + \sigma S_t dW_t$
- Risk-free process: $dB_t = rB_t dt$
- Trading (self-fin) : $H_t = H_0 + \int_0^t \phi_u \, \mathrm{d}S_u$
- Risk Neutral $\mathbb{Q}$: $dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$

Option Pricing (4)

- The price of an option with pay-off function $\Phi(S_T)$ is given by:

$$V_\Phi = e^{-rT}\mathbb{E}_\mathbb{Q}\left[\Phi(S_T)\right]$$

- For a call option this would be:

$$V_{\text{call}} = e^{-rT}\mathbb{E}_\mathbb{Q}\left[(S_T - X)^+\right]$$

- Replicating Portfolio : $\Delta = \dfrac{dV_{\text{call}}(s_0)}{ds_o}$

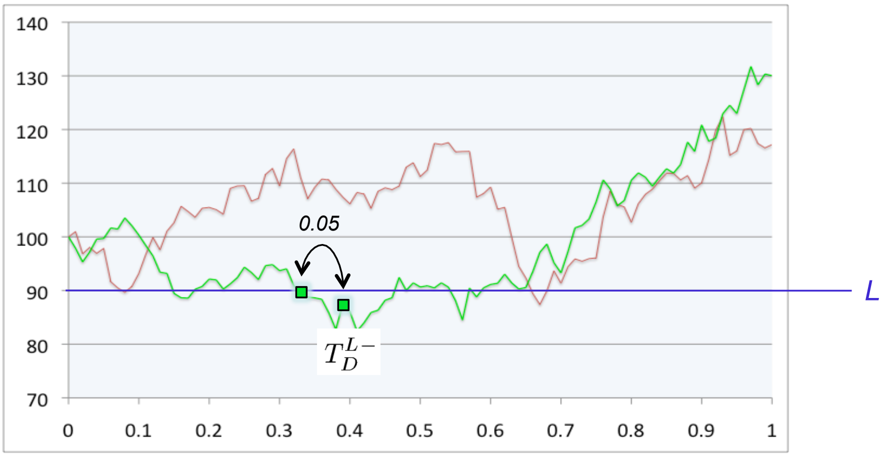
Introduction and Motivation (1)

- Suppose $\{S_t\}_{t\geq 0}$ is a stock price process on some probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$.

- And that we are interested in a standard barrier option with Barrier L=90.

Introduction and Motivation (2)

- Suppose that we request the stock price process to *consecutively* stay below the barrier for time D=0.05.



- Then, it is *only* the green path that triggers the option at the ***Parisian*** stopping time.

Introduction and Motivation (3)

- We use the following notation for the Parisian stopping time, which represents

  "The first time the stock price process S spends a consecutive time period of length D below barrier L."

barrier → T_D^{L-} ← above / below (-/+)

length

- This represents a single-sided Parisian stopping time. We introduce the double-sided Parisian stopping time:

$$T_{D_1,D_2}^{L_1-,L_2+} := T_{D_1}^{L_1-} \wedge T_{D_2}^{L_2+}$$

Introduction and Motivation (5)

⟹ Now we can construct a variety of Parisian options, e.g.

Double Parisian In Call

$$V_{DPIC} = e^{-rT}\mathbb{E}\left[\mathbb{I}_{\left\{T^{L_1-,L_2+}_{D_1,D_2} \leq T\right\}} (S_T - K)^+\right]$$

Double Parisian Out Put

$$V_{DPOP} = e^{-rT}\mathbb{E}\left[\mathbb{I}_{\left\{T^{L_1-,L_2+}_{D_1,D_2} > T\right\}} (K - S_T)^+\right]$$

Parisian Down and In Call

$$V_{PDIC} = e^{-rT}\mathbb{E}\left[\mathbb{I}_{\left\{T^{L_1-,L_2+}_{D_1,D_2} \leq T\right\}} (S_T - K)^+\right], \qquad D_2 > T$$

Introduction and Motivation (6)

⟹ Parisian Options are not exchange traded. Suggested to use instead of barrier options for illiquid stocks.

⟹ Applications of Parisian Options :

- Convertible Bonds (Screw clauses)
- Default Risk of Life Insurance Companies
- Real Options

⟹ Double-Sided Parisian options :

- Might have applications in some of the areas mentioned above
- Are the most general type of Parisian options. Every contract can be derived from the Double-Sided Parisian In Call.

Literature and Problem

- There is a lot of literature on the single-sided Parisian options. Here some pioneering work:

  Laplace Transform

  (1997) Chesney, Jeanblanc and Yor,

  "Brownian Excursions and Parisian Barrier Options"

- PDE approach

  (1999) Haber, Schonbucher and Wilmott,

  "Pricing Parisian Options"

- Binomial Tree

  (2002) Costabile

  "A combinatorial approach for pricing Parisian options"

Theory (1)

- We assume the stock price process S to be the standard GBM. Like Carr/Madan, we write the pricing formula

$$\boxed{V_{DPIC} = e^{-rT}\mathbb{E}\left[\mathbb{I}_{\left\{T_{D_1,D_2}^{L_1-,L_2+} \leq T\right\}}(S_T - K)^+\right]}$$

- In terms of probabilities

$$\boxed{V_{DPIC} = S_0 P_{r+\sigma^2}(T) - Ke^{-rT}P_r(T)}$$

- where

$$\boxed{P_\mu(T) = \mathbb{P}_\mu\left[S_T > K; T_{D_1,D_2}^{L_1-,L_2+}(S) \leq T\right], \qquad \mu \in \mathbb{R}}$$

Drift of GBM

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Theory (2)

- This probability is in terms of GBM. We translate it to a – due to Girsanov - "psuedo" probability in terms of standard Brownian motion:

$$P_r(T) = e^{-\frac{1}{2}m^2T}\mathbb{E}\left[e^{mW_T}1_{\{W_T>k\}}1_{\{\tau\leq T\}}\right]$$

Where tau is shorthand notation for the Parisian stopping time and *m* is the constant for the Girsanov transformation (zero for r =1/2 σ^2)

- There is no explicit formula for $P_r(T)$, so we calculate its Fourier Transform:

$$\phi(v) = \int_0^\infty e^{ivT}\underbrace{e^{-aT}}_{\text{damping}}P_r(T)\,\mathrm{d}T.$$

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Theory (3)

- This Fourier Transform can be rewritten into,

$$\begin{aligned}\phi(v) &= \int e^{(iv-\alpha)T}\mathbb{E}\left[e^{mW_T}1_{\{W_T>k\}}1_{\{\tau\leq T\}}\left(1_{\{\tau^+<\tau^-\}}+1_{\{\tau^-<\tau^+\}}\right)\right]\,\mathrm{d}T \\ &=: \phi_+(v)+\phi_-(v).\end{aligned}$$

The excursion happens first above the upper barrier

The excursion happens first below the lower barrier

- This, we can rewrite again as

$$\phi_+(v) = \mathbb{E}_+(\tilde{v}_\alpha)\mathbb{E}\left[\int_0^\infty e^{(iv-\alpha)\rho}h\left(\rho, l_2+\sqrt{D_2}N\right)\,\mathrm{d}\rho\right]$$

where

$$\mathbb{E}_+(\lambda) := \mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau}1_{\{\tau^+<\tau^-\}}\right]$$

Theory (4)

- The key problem for the double-sided Parisian is to establish the independence of the pair (W_τ, τ) and to calculate:

$$\mathbb{E}_+(\lambda) := \mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau}1_{\{\tau^+<\tau^-\}}\right]$$

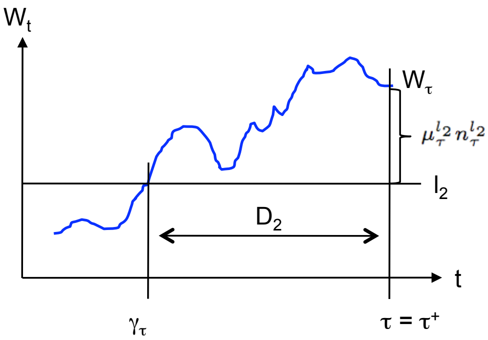
- The idea is to use the results of CJY on the single-sided Parisian option problem, combined with several application of the strong Markov property. Start with the Martingale:

$$\begin{aligned}1 &= \mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda W_\tau}\right]\\ &= \mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda W_\tau}1_{\{\tau^+<\tau^-\}}\right]+\mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda W_\tau}1_{\{\tau^-<\tau^+\}}\right]\end{aligned}$$

Theory (5)

- We proceed by:

$$\mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda W_\tau}1_{\{\tau^+<\tau^-\}}\right]=\mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda\left(\mu_\tau^{l_2}n_\tau^{l_2}+l_2\right)}1_{\{\tau^+<\tau^-\}}\right]$$



$$\mu_t^l = 1_{\{T_l<t\}}\mathrm{sgn}(W_t-l)\sqrt{t-\gamma_t^l}$$

$$n_t^l = \frac{1_{\{T_l<t\}}}{\sqrt{t-\gamma_t^l}}|W_t-l|$$

Theory (6)

- And by similar arguments we find:

$$\mathbb{E}\left[e^{-\frac{1}{2}\lambda^2\tau+\lambda W_\tau}1_{\{\tau^+>\tau^-\}}\right] = e^{\lambda l_1}\Psi(-\lambda\sqrt{D_1})\mathbb{E}_-(\lambda)$$

- Substitution in the martingale equation gives:

$$\mathbb{E}_+(\lambda) = \frac{e^{\lambda l_1}\Psi(-\lambda_1) - e^{-\lambda l_1}\Psi(\lambda_1)}{e^{\lambda(l_1-l_2)}\Psi(-\lambda_1)\Psi(-\lambda_2) - e^{\lambda(l_2-l_1)}\Psi(\lambda_1)\Psi(\lambda_2)}$$

Where Ψ is a special function related to the z-transform of n_t^l

Theory (7)

- A nice theoretical result is obtained by λ to zero:

Corollary 2.3 *The probability that a Brownian motion will spend time D_1 below level l_1 before it spends D_2 above level l_2 is given by the following formula,*

$$\mathbb{P}\left[\tau^- < \tau^+\right] = \frac{l_2\sqrt{\frac{2}{\pi}} + \sqrt{D_2}}{(l_2-l_1)\sqrt{\frac{2}{\pi}} + \sqrt{D_1} + \sqrt{D_2}} \qquad l_1 < 0 < l_2.$$

- And now the barriers to zero gives the probability that a standard Brownian motion makes an excursion of length D_2 above zero before it makes an excursion below zero of length D_1.

$$\frac{\sqrt{D_1}}{\sqrt{D_1}+\sqrt{D_2}}$$

Practical Examples (1)

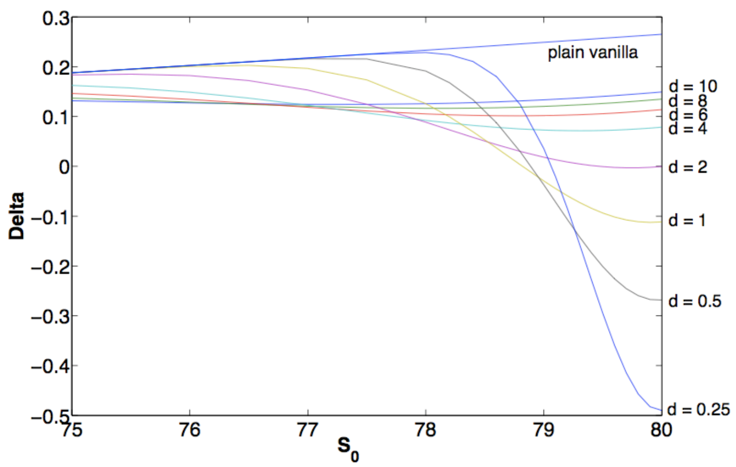
We can use the double-sided Parisian as a general option to compute the following Call contracts.

Contract type	$S_0 = 90$	$S_0 = 95$	$S_0 = 100$	$S_0 = 105$	$S_0 = 110$
Plain-vanilla	6.362	8.768	11.591	14.800	18.349
Double-sided knock-in	6.362	8.767	11.591	14.799	18.349
Double-sided P knock-in (1)	6.236	8.568	11.371	14.608	18.226
Single P up-in (2a)	5.792	8.218	11.113	14.435	18.129
P up-before-down-in (3a)	3.568	6.844	10.284	13.957	17.886
Single P down-in (2b)	2.676	1.742	1.123	0.719	0.457
P down-before-up-in (3b)	2.668	1.723	1.087	0.651	0.339

Here we have $L_1 = 90$, $L_2 = 110$, $K = 100$, $\sigma = 25\%$, $r = 3.5\%$, T=1, D_1=D_2=10/250.

Practical Examples (2)

The following graph shows peculiar delta behavior.



Here we have $L_1 = 80$, $L_2 = 120$, $K = 100$, $\sigma = 25\%$, $r = 3.5\%$, T=1, D_1=D_2=10/250.

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Practical Examples (3)

- Surprising behavior of theta:

	$S_0 = 70$		$S_0 = 74$		$S_0 = 76$		$S_0 = 80$	
d	Par	Plain	Par	Plain	Par	Plain	Par	Plain
10	0.953	0.957	1.474	1.547	1.737	1.923	2.257	2.856
9	0.946	0.949	1.474	1.536	1.740	1.911	2.240	2.841
8	0.939	0.941	1.475	1.525	1.744	1.898	2.222	2.825
7	0.932	0.933	1.476	1.514	1.750	1.885	2.205	2.809
6	0.925	0.925	1.476	1.503	1.759	1.873	2.187	2.794
5	0.917	0.917	1.475	1.492	1.770	1.860	2.170	2.778
4	0.909	0.909	1.472	1.481	1.783	1.848	2.152	2.762
3	0.901	0.901	1.467	1.470	1.797	1.835	2.135	2.747
2	0.893	0.893	1.459	1.459	1.809	1.823	2.117	2.731
1	0.885	0.885	1.448	1.448	1.809	1.810	2.100	2.715

- Here we have $L_1 = 80$, $L_2 = 120$, $K = 100$, $\sigma = 25\%$, $r = 3.5\%$, $T=1$, $D_1=D_2=10/250$.

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Final Remarks

- This talk is only about a very tiny fraction of everything that is of interest in the field of financial mathematics.
- In the field of financial mathematics many mathematical concepts, like equivalent measures, stochastic integrals fit one-to-one to the problem at hand.
- There is an interplay between mathematical results and problems arising in the financial practice.
- What is the impact to the field of the credit/financial crisis?