Exam Applied Analysis

Monday 17 December 2007, 10:00-13:00u

- Write your name, studentnumber and major on the first sheet of paper.
- When you use a theorem explain specifically why it applies.

Good luck!

1.) Consider the following system

$$\begin{aligned} \dot{x} &= y - y^3 \\ \dot{y} &= -x - y^2 \end{aligned}$$

where the dot means differentiation with respect to t.

(a) Determine all the fixed points and characterise these.

(b) Determine the unstable and stable subspaces, E^u and E^s, of all the saddle-points and nodes.

(c) Determine the nullclines and sketch these in the (x, y)-plane.

(d) Show that the system is invariant under the transformation $(t \to -t, y \to -y)$.

Note that this implies that every trajectory leads to another one by a reflection in the x-axis and time-reversal. Also, the character of the fixed point at the origin for the full nonlinear system remains the same as found in (a).

(e) Complete the sketch of the phase-plane (using the result from (d)) including stable and unstable manifolds and the flow close to the fixed points.

(f) Determine all possible heteroclinic and homoclinic orbits. Sketch them in the phase-plane in colour and give which fixed points they connect.

(g) Sketch $x(t)$ versus t for all possible heteroclinic and homoclinic orbits (as found in (f)).

!! Continued on the other side !!

2.) Consider the following system

$$\begin{aligned}\dot{x} &= ax(1-2b) + y - ax(x^2+y^2)\\ \dot{y} &= -x + ay - ay(x^2+y^2).\end{aligned}$$

where a and b are parameters that satisfy $0 < a \leq 1$, $0 \leq b < \frac{1}{2}$.
Note that the parameters a and b are such that the origin is the only fixed point.

(a) Rewrite the system in polar coordinates.

(b) Set $b = 0$. Prove that the system has one stable limit cycle. Make a sketch of the phase-plane.

(c) Prove that for $0 < b < \frac{1}{2}$, there exists at least one limit cycle.

3.) Consider

$$\dot{x} = \mu x - \frac{x^3}{1+x^2}$$

where $\mu \in \mathbb{R}$.

(a) Determine all the critical points and determine for which μ they are stable resp. unstable.

(b) For which points (x_c, μ_c) does a bifurcation take place ? Sketch the bifurcation diagram and state the type of bifurcation.

4.) Consider the following equation

$$x^2 + (1 - \varepsilon - \varepsilon^2)x + \varepsilon - 2e^{\varepsilon^2} = 0$$

where $0 < \varepsilon \ll 1$. Determine a two-term approximation of all the roots of the equation.