

Combinatorics with Computer Science Applications

Open book exam. Thu May 22, 2008, 14:00–17:00, Room 403
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1. **(1 point)** Let n be a nonnegative integer. Give two proofs of the identity

$$\sum_{i=0}^n \binom{n}{i} = 2^n,$$

one algebraic proof (for instance, using Newton's binomial theorem) and one combinatorial proof (for instance, by counting the number of subsets of $\{1, \dots, n\}$ in two ways).

2. **(2 points)** Let k , n , and e be nonnegative integers. An error-correcting code of distance $d = 2e + 1$ is a function $E : \{0, 1\}^k \rightarrow \{0, 1\}^n$, such that any two distinct codewords $E(x)$ and $E(y)$ differ in at least d out of the n positions (i.e. the Hamming distance $d_H(E(x), E(y))$ is at least d for all distinct $x, y \in \{0, 1\}^k$). Prove the “sphere packing bound”:

$$2^k \sum_{i=0}^e \binom{n}{i} \leq 2^n.$$

Hint: for every distinct x and y , the “Hamming balls” of radius e around $E(x)$ and $E(y)$ need to be disjoint. Such a code can correct up to e errors on the codeword.

3. (a) **(1 point)** Let n , m , and d be nonnegative integers. Suppose we have $2n$ binary vectors $u_1, \dots, u_n, v_1, \dots, v_n$ of dimension m (i.e., $u_i \in \{0, 1\}^m$ and $v_i \in \{0, 1\}^m$ for every $i \in [n]$) with the following inner-product properties over the field of real numbers:

$$\begin{aligned} \langle u_i, v_i \rangle &= 0 \text{ for every } i \in [n] \\ \langle u_i, v_j \rangle &\in \{1, \dots, d\} \text{ for every } i \neq j \end{aligned}$$

Prove that $n \leq \sum_{i=0}^d \binom{m}{i}$.

- (b) **(1 point)** Suppose we have $2n$ subsets $A_1, \dots, A_n, B_1, \dots, B_n$ of an m -element universe with the following intersection properties:

$$\begin{aligned} A_i \cap B_i &= \emptyset \text{ for every } i \in [n] \\ |A_i \cap B_j| &\in \{1, \dots, d\} \text{ for every } i \neq j \end{aligned}$$

Prove that $n \leq \sum_{i=0}^d \binom{m}{i}$.

4. **(2 points)** Let n and m be nonnegative integers. Let $G = (V, E)$ be an undirected graph on $|V| = n$ vertices with $|E| = m$ edges. A *bipartition* of the graph divides V into two disjoint sets V_1 and V_2 (with $V = V_1 \cup V_2$). Prove that there is a bipartition of G with at least $m/2$ edges going from V_1 -vertices to V_2 -vertices.

Hint: use the probabilistic method.

5. Consider an undirected bipartite graph $G = (V_1 \cup V_2, E)$, where V_1 and V_2 are disjoint sets of n vertices each, and $E \subseteq V_1 \times V_2$ is the set of edges between V_1 -vertices and V_2 -vertices. The *adjacency matrix* of G is the following $n \times n$ matrix A of 0s and 1s: rows are indexed by V_1 , columns by V_2 , and the (i, j) -entry $A_{i,j}$ is 1 iff $(i, j) \in E$.

- (a) **(1 point)** Suppose we take the matrix A and replace each non-zero entry $A_{i,j}$ by a variable $x_{i,j}$ (the 0-entries remain 0), over some field \mathbb{F} of your choice. Call the resulting matrix $A(x)$, where x is the sequence of all variables. The determinant¹ of $A(x)$ is a polynomial in those variables, of total degree at most n . Show that this polynomial is identically equal to 0 iff the graph G does *not* have a perfect matching.
- (b) **(1 point)** Give a randomized algorithm to decide (with error probability less than $1/3$) whether G has a perfect matching, using $O(n^{2.38})$ computational steps.
NB: you're not required to *find* a perfect matching, only to decide if one exists.

¹The *determinant* of an $n \times n$ matrix A is defined as

$$\sum_{\pi \in S_n} \text{sgn}(\pi) \prod_{i=1}^n A_{i, \pi(i)},$$

where the sum is over the set S_n of all $n!$ permutations on n elements, and $\text{sgn}(\pi) \in \{+1, -1\}$ is the signature of the permutation π (it doesn't matter if you don't know what that is). It is known that we can compute the determinant of an $n \times n$ matrix in $O(n^\omega)$ computational steps, for some $\omega \in [2, 2.38]$. Here an addition or multiplication in the field \mathbb{F} counts as one computational step. This ω is the *matrix multiplication exponent*; its precise value is unknown, though it is conjectured to equal 2.