Final Exam

Commutative Algebra

June 14, 2012

Exercise 1.

Let R be a commutative ring and I and J ideals in R such that every prime ideal of R contains either I or J, but not both. Show that:

1. $R = I + J$;
2. every element of IJ is nilpotent.

Exercise 2.

Let R be a commutative ring with two ideals I and J.

1. Show that there is an exact sequence

$$0 \to (I \cap J)/(IJ) \to I \otimes_R (R/J) \to R/J \to (R/I) \otimes_R (R/J) \to 0.$$

2. Give an example of a ring R and an ideal I such that R/I is not flat over R.

Exercise 3.

Let R be a commutative ring. You may use that the nilradical of R is the intersection of all its prime ideals.

1. Let $\mathfrak{p}$ be a minimal prime ideal of R (a minimal element in the set of prime ideals of R ordered by inclusion). Show that any element in $\mathfrak{p}R_\mathfrak{p}$ (where $R_\mathfrak{p}$ is the usual localisation at $\mathfrak{p}$) is nilpotent.

2. Show that any element $\mathfrak{p}$ is a zero divisor in R.

3. Assume that R is reduced (its only nilpotent element is 0). Show that every zero divisor of R is contained in some minimal prime ideal of R.

Exercise 4.

Let A be a ring that is complete with respect to an ideal $\mathfrak{a}$. Let (a_n) be a sequence in A. Prove that $\sum_{n=0}^{\infty} a_n$ converges if and only if $a_n \to 0$ for $n \to \infty$.

Exercise 5.

Let A be the localisation of $\mathbb{Z}[x, y]$ at the maximal ideal $(3, x, y)$ and let $B = A/(-y^2+x^3-x)$.

1. Show that $\dim(A) \geq 3$ by giving a chain of prime ideals.
2. Give $\dim(A)$. You may use that $\dim(A) = \delta(A)$.
3. Is A regular?
4. Give $\dim(B)$.
5. Is B regular?