Commutative Algebra, June 20, 2013

Each problem is worth some number of points, as indicated.
Your *highest four* scores (out of five) count.

1. (2 points)
The tensor product $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a vector space over $\mathbb{Q}$ by multiplication in the left factor, i.e., $\lambda(x \otimes y) = (\lambda x) \otimes y$ for $\lambda, x, y \in \mathbb{Q}$. What is the dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ as vector space over $\mathbb{Q}$?

2. (2.5 points)
Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. Let

$$0 \to M' \to M \to M'' \to 0$$

be a short exact sequence of finitely generated A-modules.

(a) Set $r = \dim_k(M' \otimes_A k)$ and $s = \dim_k(M'' \otimes_A k)$. Prove that M can be generated as A-module by $r + s$ elements.

(b) Suppose that the multiplication map $\mathfrak{m} \otimes_A M'' \to M''$ sending $t \otimes x$ to tx is injective. Prove that the induced sequence

$$0 \to M' \otimes_A k \to M \otimes_A k \to M'' \otimes_A k \to 0$$

is exact.

3. (2 points)
Give an example of the following.

(a) A local ring A with maximal ideal $\mathfrak{m}$ and a nonzero A-module M such that $M = \mathfrak{m}M$.

(b) A reduced ring R for which the zero ideal $(0) \subset R$ has exactly three associated primes $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$, such that for each $i \in \{1, 2, 3\}$, the dimension of the ring $R/\mathfrak{p}_i$ equals i.

4. (2.5 points)
Set $R = \mathbb{Q}[x]$ and let $\mathfrak{m} \subset R$ be the ideal generated by x. Let $\hat{R}$ be the completion of R at $\mathfrak{m}$.

(a) Show that $\hat{R}$ contains an element f that satisfies $f \equiv 1 \pmod{\mathfrak{m}\hat{R}}$ and $f^2 = 1 - x^2$.

(b) Set $S = \mathbb{Q}[x, y]$ and $T = S/I$ with $I = (x^2 + y^2 - 1) \subset S$. Let $\mathfrak{n} \subset T$ be the ideal generated by x and $y - 1$. Let $\hat{T}$ be the completion of T at $\mathfrak{n}$. Show that the inclusion $R \hookrightarrow T$ induces an isomorphism $\hat{R} \to \hat{T}$.

5. (3 points)
Recall that a *Dedekind domain* is an integral domain that is Noetherian, integrally closed, of dimension 1. Note that a domain R has dimension 1 if and only if R is not a field and every nonzero prime ideal of R is maximal.

(a) Let R be a Dedekind domain. Let $I \subset R$ be an ideal and suppose that $0 \neq a \in I$. Prove that there exists $b \in I$ such that I is generated by a and b. In particular, every ideal in a Dedekind domain can be generated by 2 elements.

Let R be $\mathbb{Z}[\sqrt[3]{2}]$. Let S be the ring $\mathbb{Z} + 2R$, and let $I \subset S$ be the ideal $2R$.

(b) Prove that I is a maximal ideal of S.

(c) Prove that I/I^2 has dimension 3 as a vector space over S/I.

(d) Conclude that S is not a Dedekind domain.