Commutative Algebra
Final Exam

August 26th, 2013

The use of any electronic devices during the exam is forbidden. You may consult books, lecture notes, and any notes of your own.

Problem 1. Let R be a commutative ring. For any prime ideal $\mathfrak{p}$ of R, we denote the localization of R at $\mathfrak{p}$ by $R_{\mathfrak{p}}$, and we the denote the residue field of $R_{\mathfrak{p}}$ by $\kappa(\mathfrak{p})$. If $I \subset R$ is an ideal, then we denote the localization of I at $\mathfrak{p}$ by $I_{\mathfrak{p}}$. Let I and J be ideals of R.

(a) Suppose that $I_{\mathfrak{p}} = J_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subset R$. Prove: $I = J$.

Assume that R is Noetherian.

(b) Suppose that $I_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}) = 0$ for all prime ideals $\mathfrak{p} \subset R$. Prove: $I = 0$.

Problem 2. Are the following assertions true or false? Give a full proof in each case. (We recall the following definitions. Let R be a commutative ring and $\mathfrak{p}$ a prime ideal of R. Then an ideal $\mathfrak{q}$ of R is called primary if $xy \in \mathfrak{q}$ implies that either $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some positive integer n. A primary ideal $\mathfrak{q}$ is called $\mathfrak{p}$-primary if the radical of $\mathfrak{q}$ is $\mathfrak{p}$.)

(a) Let R be a commutative ring, let $\mathfrak{p}$ be a prime ideal, and let n be a positive integer. Then the radical of $\mathfrak{p}^n$ is $\mathfrak{p}$.

(b) Let R be a commutative ring, let I be an ideal, and let n be a positive integer. Then the radical of I^n is I.

(c) Let R be a commutative ring, let $\mathfrak{p}$ be a prime ideal, and let $\mathfrak{q}_1$ and $\mathfrak{q}_2$ be $\mathfrak{p}$-primary ideals. Then $\mathfrak{q}_1 + \mathfrak{q}_2$ is a $\mathfrak{p}$-primary ideal.

(d) Let R be a commutative ring, let $\mathfrak{p}$ be a prime ideal, and let $\mathfrak{q}_1$ and $\mathfrak{q}_2$ be $\mathfrak{p}$-primary ideals. Then $\mathfrak{q}_1\mathfrak{q}_2$ is a $\mathfrak{p}$-primary ideal.

(e) A quotient of a principal ideal domain is a principal ideal domain.

(f) A quotient of a unique factorization domain is a unique factorization domain.

Problem 3. Let A be a $\mathbf{Z}$-module and write A_{tors} for its torsion submodule.

(a) Prove that the sequence

$$A \to A \otimes_{\mathbf{Z}} \mathbf{Q} \to A \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z} \to 0$$

is exact, where the first map sends a to $a \otimes 1$ for all $a \in A$.

(b) Let $A_{(0)}$ be the localization of A at the prime ideal (0) of $\mathbf{Z}$. Give an isomorphism between $A \otimes_{\mathbf{Z}} \mathbf{Q}$ and $A_{(0)}$.

(c) Let a be an element of A and let n be a positive integer. Prove that $a \otimes (1/n)$ is the zero element of $A \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z}$ if and only if a is contained in $nA + A_{\mathrm{tors}}$.

Problem 4. Let $(R, \mathfrak{m})$ be a local artinian ring. Let M be an R-module such that $\mathfrak{m}M = M$.

(a) Prove: for all positive integers i we have the equality $\mathfrak{m}^i M = M$.
(b) Prove: $M = 0$.

Problem 5. Recall that a commutative ring R is called semi-local if it has only finitely many maximal ideals. Also recall that an integral domain R is called a Dedekind domain if it is Noetherian and of dimension one, and every localization of R at a non-zero prime ideal is a discrete valuation ring.

In this exercise we will prove that a semi-local Dedekind domain is a principal ideal domain. Let R be a semi-local Dedekind domain, and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be its maximal ideals. Let $I \subset R$ be a non-zero ideal.

(a) Prove: for all positive integers $e_1, \ldots, e_n$, the canonical map

$$R \to \bigoplus_{i=1}^{n} R/\mathfrak{m}_i^{e_i}$$

is surjective.
(b) Show that there exist integers $f_1, \ldots, f_n$ such that $I = \prod_{i=1}^{n} \mathfrak{m}_i^{f_i}$.
(c) Let $f_1, \ldots, f_n$ be as in (b). Show that there exists an element $x \in R$ such that x is contained in $\mathfrak{m}_i^{f_i} - \mathfrak{m}_i^{f_i+1}$ for all i with $1 \leq i \leq n$.
(d) Let $x \in R$ be as in (c). Prove: $I = xR$.