Commutative Algebra: Final Exam

Part 1: Answer each of the following three questions. You may not refer to your notes or homework solutions during the exam, but you may cite any results from the lecture that you can recall from memory. You may also cite the result of each part (a) when solving the corresponding part (b), even if your solution to part (a) is incomplete. Each question is worth 5 points.

1. Recall that for an ideal I of a ring R, the *radical* of I is $\sqrt{I} = \{r \in R : r^n \in I \text{ for some } n \in \mathbb{N}\}$.

   (a) Show that if I and J are two ideals of R, then $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.

   (b) Now suppose I is an ideal of a Noetherian ring R. Show that $\sqrt{I}$ is an intersection of finitely many prime ideals of R.

2. Let $f : R \to S$ be a ring homomorphism making S an R-algebra. We say that S is *torsion-free* as an R-algebra if whenever r is a non-zerodivisor in R, then $f(r)$ is a non-zerodivisor in S.

   (a) Show that if S is a flat R-algebra, then S is also a torsion-free R-algebra.

   (b) Now let k be a field, $R = k[\varepsilon]/(\varepsilon^2)$, and $S = k$, with $f : R \to S$ the k-algebra map sending $\varepsilon \mapsto 0$. Show that S is torsion-free but not flat as an R-algebra.

3. Let R be a Noetherian ring and $\mathfrak{m} \subset R$ a maximal ideal, so that the quotient $k = R/\mathfrak{m}$ is a field. Recall that the *Hilbert function* of R at $\mathfrak{m}$ is

   $$H_{R,\mathfrak{m}}(n) = \dim_k(k \otimes_R \mathfrak{m}^n) = \dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}).$$

   (a) Let $R = \mathbb{C}[[x, y]]/(xy)$ with $\mathfrak{m} = (x, y) \subset R$ its unique maximal ideal. Show that $H_{\mathfrak{m}}(n) = 2$ for all $n \geq 1$.

   (b) With R and $\mathfrak{m}$ as in part (a), show that $\mathfrak{m}$ has height 1 but that $\mathfrak{m}$ cannot be generated by one element.

Part 2: Multiple choice. Each of the following statements either

- is true as stated,
- requires the hypothesis "R is Noetherian" to be true, or
- has counterexamples even assuming R is Noetherian.

Circle the appropriate category for each statement, and **turn in this sheet with your solutions to Part 1**. Note: you do not need to provide proofs or counterexamples. Each correct answer is worth one point.

1. Every nonzero R-module has an associated prime.

True **True if *R* is Noetherian** **False**

2. If R is local with maximal ideal $\mathfrak{m}$, and M is a finitely generated R-module such that $\mathfrak{m}M = M$, then $M = 0$.

True **True if *R* is Noetherian** **False**

3. Let $S \subset R$ be a multiplicative subset. For any R-module M, its localization $S^{-1}M$ is isomorphic to $S^{-1}R \otimes_R M$.

True **True if *R* is Noetherian** **False**

4. Let $I \subset R$ be an ideal. For any R-module M, its completion $\hat{M}_I$ is isomorphic to $\hat{R}_I \otimes_R M$.

True **True if *R* is Noetherian** **False**

5. Let $\mathfrak{m} \subset R$ be a maximal ideal. The completion $\hat{R}_\mathfrak{m}$ is a local ring with maximal ideal $\mathfrak{m}\hat{R}_\mathfrak{m}$.

True **True if *R* is Noetherian** **False**