Commutative Algebra: Final Exam Resit

Part 1: Answer each of the following three questions. You may not refer to your notes or homework solutions during the exam, but you may cite any results from the lecture that you can recall from memory. You may also cite the results of earlier parts in each problem, even if your solutions to those parts are incomplete. Each question is worth 5 points.

1. Let k be a field with $\mathrm{char}(k) \neq 2$, and consider the ring of formal power series $k[[x]]$, i.e. the completion of $k[x]$ at the ideal (x).

   (a) Show that x does not have a square root in $k[[x]]$.

   (b) Show that $1 + x$ does have a square root in $k[[x]]$. (If you wish, you may use any version of Hensel's Lemma that you can state correctly.)

2. Recall that an ideal $\mathfrak{q}$ is called *primary* if it is proper and if $ab \in \mathfrak{q} \implies a \in \mathfrak{q}$ or $b \in \sqrt{\mathfrak{q}}$. If $\mathfrak{q}$ is primary, then its radical $\mathfrak{p} = \sqrt{\mathfrak{q}}$ is automatically a prime ideal, and $\mathfrak{q}$ is said to be $\mathfrak{p}$-primary.

   (a) Let R be a ring, $\mathfrak{p} \subset R$ a prime ideal, and $f : R \to R_{\mathfrak{p}}$ the canonical homomorphism sending $r \mapsto r/1$. Let $\mathfrak{q} \subset R$ be an ideal such that $\sqrt{\mathfrak{q}} = \mathfrak{p}$. Show that $\mathfrak{q}$ is $\mathfrak{p}$-primary if and only if $\mathfrak{q} = f^{-1}(\mathfrak{q}A_{\mathfrak{p}})$.

   (b) Let k be a field and $R = k[x, y, z]/(xz - y^2)$. Let $\mathfrak{p} = (x, y) \subset R$ and $\mathfrak{q} = \mathfrak{p}^2$. Show that $\mathfrak{p}$ is prime and that $\sqrt{\mathfrak{q}} = \mathfrak{p}$, but that $\mathfrak{q}$ is not $\mathfrak{p}$-primary.

3. Let R be a ring. Recall that an element $e \in R$ is called *idempotent* if $e^2 = e$. We say that an ideal I of R is *idempotent* if $I^2 = I$.

   (a) Let $I \hookrightarrow R$ be an ideal such that R/I is a flat R-module. Show that the R-module homomorphism

   $$I/I^2 \cong I \otimes_R R/I \to R \otimes_R R/I \cong R/I$$

   is both injective and the zero map, and hence that I is idempotent.

(b) Let I be an arbitrary ideal. Show that if $e \in I$ satisfies $(1-e)I = 0$, then $e^2 = e$ and $I = (e)$, so I is idempotent.

(c) Let I be a finitely generated ideal. Show that if I is idempotent, then there is an element $e \in I$ such that $(1-e)I = 0$, and conclude that I is generated by a single idempotent element.

Part 2: Multiple choice. Each of the following statements either

- is true as stated,
- requires the hypothesis "R is local" to be true, or
- has counterexamples even assuming R is local.

Circle the appropriate category for each statement, and **turn in this sheet with your solutions to Part 1**. Note: you do not need to provide proofs or counterexamples. Each correct answer is worth one point.

1. Let $I \subset R$ be a proper ideal. If M is a finitely generated R-module such that $IM = M$, then $M = 0$.

True **True if *R* is local** **False**

2. Let $\mathfrak{m} \subset R$ be a maximal ideal. The completion $\hat{R}_{\mathfrak{m}}$ is a local ring.

True **True if *R* is local** **False**

3. Let M be a finitely-generated R-module. If M is flat, then M is free.

True **True if *R* is local** **False**

4. If R is Noetherian and $\mathfrak{m} \subset R$ is a maximal ideal of height d, then $\mathfrak{m}$ can be generated by d elements of R.

True **True if *R* is local** **False**

5. Let M be an R-module. If the localization $M_{\mathfrak{m}}$ is the zero module for each maximal ideal $\mathfrak{m} \subset R$, then $M = 0$.

True **True if *R* is local** **False**