**EXAM COMPLEX ANALYSIS**
**Friday 6 June 2008, 10.00-13.00.**

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State every theorem (including the conditions under which the theorem holds) and result that you use.

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**I.** Let $G$ be the region $\{z \in \mathbf{C} : |z+1| < \sqrt{2},\ |z-1| < \sqrt{2}\}$. Give an analytic isomorphism $k : G \to H$ between $G$ and the upper half plane $H$. (20p)

**II.** Consider the function $f(z) = \sqrt{-1+\sqrt{z}}$.

a. Give the branch points of $f$ in the (finite) complex plane. (4p)

Let $f_1(z)$ be an analytic function in a neighbourhood $G$ of $z = 4$ such that $f_1(z)$ coincides with the branch of $f(z)$ for which $f(4) = i\sqrt{3}$. $f_1$ can be defined by a power series about the point $z = 4$.

b. What is the radius of this power series? Explain. (5p)

c. Indicate a region $G_1$ such that $f_1$ can be analytically continued to a (one-valued) analytic function on $G_1$ and such that $G_1$ is as large as possible. (4p)

d. Let $\gamma$ be the the circle $\{z \in \mathbf{C} : |z| = 4\}$, taken in the counterclockwise direction, with beginning and end point $z = 4$. Continue $f_1$ analytically along $\gamma$. What is the value of the analytic continuation at the end point? (7p)

**III.** Let $G$ be the interior of the triangle with vertices at the points $0, 1$ and $i$ in the complex plane. $g : G \to D$ is an analytic isomorphism between $G$ and the (open) unit disk $D$. Moreover, $g$ extends to a homeomorphism between the closure of $G$ and the closed unit disk.

a. State a theorem from which it follows that such a function $g$ indeed exists. (4p)

b. Prove that $g$ can be analytically continued to an elliptic function on $\mathbf{C}$. (5p)

c. Determine both the periods of this elliptic function and its order. (7p)

**Please turn over for problem IV.**

**IV.** Let $\Lambda$ be the lattice consisting of the points $z = m + n\omega$ where $m, n \in \mathbf{Z}$ and $\omega = \frac{1}{2} + \frac{1}{2}i\sqrt{3}$.

a. State (necessary and sufficient) conditions on the real numbers $a, b, c, d$ such that $\{a + b\omega, c + d\omega\}$ is a basis of $\Lambda$. Explain your answer. (4p)

b. Construct an entire function $h(z)$ that has zeros of multiplicity 1 exactly at the lattice points and such that $h'(0) = 1$. (8p)

(You may use that $\displaystyle\sum_{\Omega\in\Lambda}{}' \frac{1}{|\Omega|^\ell}$ converges for $\ell > 2$).

Let $A_k := \displaystyle\sum_{\Omega\in\Lambda}{}' \frac{1}{\Omega^{2k}}$. As has been shown in the lectures, $g_2 = 60A_2$ and $g_3 = 140A_3$.

c. Prove that $g_2 = 0$ and $g_3 \in \mathbf{R}$. (Hint: first show that $\Lambda = \omega\Lambda$.) (8p)

d. Show that

$$\int_{\sqrt[3]{g_3/4}}^{\infty} \frac{dx}{\sqrt{4x^3 - g_3}} = \frac{1}{2}.$$

(The integral is taken over the real axis and the square root is assumed to be non-negative.) (10p)