Probability: Coupling Theory

Teacher: F. den Hollander
Written examination: Monday 21 January 2013, 10:00–13:00.

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation. Formulas alone are not enough. Formulate your answers clearly and carefully.
- The use of textbooks, lecture notes or handwritten notes is not allowed.
- The questions below are weighted as follows: (1) 3, 10, 2; (2) 5, 5; (3) 6, 3, 6; (4) 3, 3, 3, 6; (5) 5, 10; (6) 3, 4, 8; (7) 3, 3, 3, 3, 3. *Total:* 100. Pass: ≥ 55; no pass: ≤ 54.

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(1) (a) Give the definition of a coupling of two probability measures $\mathbb{P}$ and $\mathbb{P}'$ on a measurable space $(E, \mathcal{E})$.

(b) State the standard coupling inequality and give its proof.

(c) What is a maximal coupling?

(2) (a) Let U, V be two random variables with probability density functions on $\mathbb{R}$ given by

$$f_U(x) = x^{-2}\, 1_{[1,\infty)}(x), \qquad f_V(x) = 3x^{-4}\, 1_{[1,\infty)}(x).$$

Give a coupling of U and V such that they are ordered.

(b) Let U, V be two random variables with probability mass functions on $\mathbb{Z}$ given by

$$f_U(x) = \tfrac{1}{3}\, 1_{\{-1,0,1\}}(x), \qquad f_V(x) = \tfrac{1}{5}\, 1_{\{-2,-1,0,1,2\}}(x).$$

Give a maximal coupling of U and V.

(3) (a) Describe the Ornstein coupling for two simple random walks on $\mathbb{Z}^d$.

(b) What is a harmonic function on $\mathbb{Z}^d$?

(c) Prove that bounded harmonic functions on $\mathbb{Z}^d$ are constant.

(4) (a) Give the definition of a random card shuffle.

(b) Give the definition of a sequence of threshold times for random card shuffles.

(c) What is a strong uniform time for random card shuffles?

(d) Exhibit an explicit construction of a strong uniform time for the top-to-random shuffle and compute its expectation.

(5) (a) Formulate the standard convergence theorem for a Markov chain on a countable state space.

(b) Prove this theorem with the help of coupling when the state space is finite.

(6) (a) What is a partial ordering?

(b) Give two examples of partial orderings on $\{0,1\}^{\mathbb{Z}^d}$.

(c) Formulate the FKG inequality, and apply it to percolation on $\mathbb{Z}^d$ to show that

$$\mathbb{P}(O[A \cup B]) \geq \mathbb{P}(O[A])\mathbb{P}(O[B]), \qquad A, B \subset \mathbb{Z}^d,$$

where $O[A]$ is the event that all sites in A are occupied (= carry 1).

(7) (a) Give the definition of a spin-flip system.

(b) When is such a system attractive?

(c) What important property do attractive spin-flip systems have?

(d) Give the transition rates of the contact process on $\mathbb{Z}^d$.

(e) Explain in what sense the contact process has a phase transition.