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Midterm Exam : 05 November 2007

Duration : 2 hours

No books, written notes are permitted. Please do not use pencils!

Exercise 1

Let A be a commutative ring and $\mathfrak{p}$ be a prime ideal of A. Set $S = A \backslash \mathfrak{p}$.

1. Show that $(S, \leq)$ is a filtrant pre-ordered set if we define $\leq$ as follows :
$$f \leq g \Longleftrightarrow \exists h \notin \mathfrak{p},\ fh = g.$$
2. Let M be an A-module. For $a \in A$, define $M_a = M[\frac{1}{a}]$, and set $M_{\mathfrak{p}} = S^{-1}M$. Prove that the natural morphism
$$\varinjlim_{t \notin \mathfrak{p}} M_t \to M_{\mathfrak{p}}.$$
is an isormophism of A-modules.
3. Show that for any short exact sequence of A-modules
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
and any $t \in S$, the sequence
$$0 \longrightarrow M'[\tfrac{1}{t}] \longrightarrow M[\tfrac{1}{t}] \longrightarrow M''[\tfrac{1}{t}] \longrightarrow 0$$
is exact.
4. Deduce that for any short exact sequence of A-modules
$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$
the sequence of A-modules obtained after applying the functor $- \otimes_A A_{\mathfrak{p}}$:
$$0 \longrightarrow M' \otimes_A A_{\mathfrak{p}} \longrightarrow M \otimes_A A_{\mathfrak{p}} \longrightarrow M'' \otimes_A A_{\mathfrak{p}} \longrightarrow 0$$
is exact.

Exercise 2

Let $\mathcal{C}$ be a category with an initial object denoted $\alpha_{\mathcal{C}}$ and a terminal object denoted $\omega_{\mathcal{C}}$. Let $\mathcal{I}$ be the empty category. Let $F \colon \mathcal{I} \to \mathcal{C}$ and $G \colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}$ be two functors. Show that :

(1) $$\varinjlim_{\mathcal{I}} F = \alpha_{\mathcal{C}},$$

(2) $$\varprojlim_{\mathcal{I}} G = \omega_{\mathcal{C}}.$$

Exercise 3

Let $\mathcal{C}$ be a category. The category $\mathcal{C}$ is *cartesian* if the following holds :

(1) $\mathcal{C}$ has a final object, denoted $\omega_{\mathcal{C}}$.
(2) For all X and Y in $\mathcal{C}$, the product $X \prod Y$ exists.

A category is said *cocartesian* if its opposite category is cartesian, and *bicartesian* if it is cartesian and cocartesian.

1. Show that $\mathcal{C}$ is cocartesian if and only if the following holds
   (a) $\mathcal{C}$ has an initial object, denoted $\alpha_{\mathcal{C}}$.
   (b) For all X and Y in $\mathcal{C}$, the coproduct $X \coprod Y$ exists.
2. Suppose $\mathcal{C}$ is cartesian. Prove that for any $X \in \mathcal{C}$, the objects $X \prod \omega_{\mathcal{C}}$, $\omega_{\mathcal{C}} \prod X$ and X are isomorphic.
3. Suppose that $\mathcal{C}$ is a cartesian category. Let $f : X \to Y$ and $a : A \to B$ be morphisms. Prove that there is a unique morphism $\phi : X \prod A \to Y \prod B$ such that :

$$\begin{cases} f\pi_X & = & \pi_Y \phi, \\ a\pi_A & = & \pi_B \phi, \end{cases}$$

   where the $\pi's$ are the canonical morphisms

$$\begin{cases} \pi_X : X \prod Y \to X, \\ \pi_Y : X \prod Y \to Y, \\ \pi_A : A \prod B \to A, \\ \pi_B : A \prod B \to B. \end{cases}$$

   The morphism ϕ will be denoted $f \times a$.
4. Let $\mathcal{C}$ be a cartesian category. To simplify the notations, for A and B of $\mathcal{C}$ we will write $A \times B$ instead of $A \prod B$ and, if $A \coprod B$ exists it will be denoted $A + B$. For two objects A and B of $\mathcal{C}$, denote by $\mathfrak{Exp}(\mathcal{C})_{A,B}$ the category whose objects are diagrams (in $\mathcal{C}$) :

$$X \times A \xrightarrow{f} B.$$

   and a morphism between $X \times A \xrightarrow{f} B$ and $Y \times A \xrightarrow{g} B$ consists of a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that the diagram :

$$\begin{array}{ccc} X \times A & & \\ & \searrow^{f} & \\ \Big\downarrow{\scriptstyle \phi \times \mathrm{Id}_A} & & B \\ & \nearrow_{g} & \\ Y \times A & & \end{array}$$

   commutes. Let A and B be two objects of $\mathcal{C}$. If the category $\mathfrak{Exp}(\mathcal{C})_{A,B}$ has a final object, we will denote it by $B^A \times A \xrightarrow{e} B$, where $B^A \in \mathcal{C}$ and $e \in \mathrm{Hom}_{\mathcal{C}}(B^A \times A, B)$.

   Show that $\mathfrak{Set}$ is a bicartesian category. Prove that if A, B are two sets, the category $\mathfrak{Exp}(\mathfrak{Set})_{A,B}$ has a final object (construct B^A and the morphism e).
5. From now on $\mathcal{C}$ is a bicartesian category such that for any $A, B \in \mathcal{C}$, the category $\mathfrak{Exp}(\mathcal{C})_{A,B}$ admits a final object.

a. Show that for any object $X, Y, Z \in \mathcal{C}$, we have an isomorphism :

$$\mathrm{Hom}_{\mathcal{C}}(X \times Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Z^Y).$$

b. From now on, we fix three objects A, B and C of $\mathcal{C}$. Prove that there is a canonical morphism :

$$\phi\colon (A \times C) + (B \times C) \to (A + B) \times C.$$

c. Show that we have a canonical morphism :

$$\bar{\psi}\colon A + B \to ((A \times C) + (B \times C))^C.$$

And show that it correponds uniquely to a morphism ψ :

$$\psi\colon (A + B) \times C \to (A \times C) + (B \times C).$$

d. Prove that $\psi \circ \phi = \mathrm{Id}_{(A\times C)+(B\times C)}$.

e. Prove that $\phi \circ \psi = \mathrm{Id}_{(A+B)\times C}$.