HOMOLOGICAL ALGEBRA - FINAL EXAM

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Solution of Exercise 1. 1. Given an $\alpha \in \mathrm{Hom}_{\mathcal{C}^\vee}(h^\vee(X), F)$, we have $\alpha_X\colon \mathrm{Hom}_{\mathcal{C}}(X, X) \to F(X)$. Hence we define a morphism of sets ϕ :

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}^\vee}(h^\vee(X), F) & \to & F(X) \\ \alpha & \mapsto & \alpha_X(\mathrm{Id}_X). \end{array}$$

Conversely, let $a \in F(X)$. For any object $Y \in \mathcal{C}$, define a morphism α_Y:

$$\begin{array}{rrcl} \alpha_Y\colon & \mathrm{Hom}_{\mathcal{C}}(Y, X) & \to & F(Y) \\ & f & \mapsto & F(f)(a) \end{array}$$

The collection of $(\alpha_Y)_Y$ defines a natural transformation $\mathrm{Hom}_{\mathcal{C}}(-, X) \to F(-)$ and hence we have a morphism of sets ψ :

$$\begin{array}{rrcl} \psi\colon & F(X) & \to & \mathrm{Hom}_{\mathcal{C}^\vee}(h^\vee_{\mathcal{C}}(X), F) \\ & a & \mapsto & (\alpha_Y)_Y \end{array}$$

It is straightforward to check that $\phi \circ \psi = \mathrm{Id}_{FX}$ and $\psi \circ \phi = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}^\vee}(h^\vee_{\mathcal{C}}(X), F)}$.

2. Set $F = h(Y)$, we have, thanks to the question 1. a bijection:

$$\mathrm{Hom}_{\mathcal{C}^\vee}(h(X), h(Y)) \to h(Y)(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y).$$

This means that h is full faithful.

3. If $f\colon X \to Y$ is an isomorphism, it is obvious that for any W the induced map:

$$\mathrm{Hom}_{\mathcal{C}}(W, X) \to \mathrm{Hom}_{\mathcal{C}}(W, Y)$$

is an isomorphism. Conversely, the morphism $f\colon X \to Y$ gives a morphism $h(f)\colon h(X) \to h(Y)$ which is an isomorphism by hypotheses. Denote by G the inverse of $h(f)$. Since the functor $h\colon \mathcal{C} \to \mathcal{C}^\vee$ is full faithfull, there exists a unique morphism $g\colon Y \to X$, such that $h(g) = G$. Moreover we have

$$\begin{array}{rcl} g \circ f = \mathrm{Id}_X & \iff & h(g \circ f) = \mathrm{Id}_{h(X)}, \\ & \iff & h(g)h(f) = \mathrm{Id}_{h(X)}. \end{array}$$

Since $h(g)$ is an inverse of $h(f)$ we get $g \circ f = \mathrm{Id}_X$ and likewise $f \circ g = \mathrm{Id}_Y$.

Solution of Exercise 2 (Category $\mathfrak{Ring}$)**.** 1. Let $\phi, \psi\colon \mathbf{Q} \to A$ be two ring homomorphisms such that $\phi i = \psi i$ where i is the natural morphism $\mathbf{Z} \to \mathbf{Q}$. Let $n \geqslant 1$ be an integer. We must have $n\phi(1/n) =$

$\phi(1) = 1 = n\psi(1/n)$, hence $\phi(1/n) = \psi(1/n) = 1/n \in A$. Thus ϕ and ψ agree and i is an epimorphism.

2. Let $f\colon A \to B$ be a monomorphism. Suppose that the map of the underlying sets is not an injection. There exist two different elements in A a_1 and a_2 such that $f(a_1) = f(a_2)$. Define two morphisms ϕ_i for $i = 1, 2$

$$\begin{array}{rcl} \mathbf{Z}[T] & \to & A \\ T & \mapsto & a_i \end{array}$$

By defintion the two morphisms are different and $f\phi_1 = f\phi_2$. Hence f is not a monomorphism. Contradiction.

Solution of Exercise 3 (A bit of analysis)**.** 1. The first group is the kernel of the morphism $d\colon C^\infty(I) \to C^\infty(I)$, the constant functions. So it is isomorphic to $\mathbf{R}$. The second group is the cokernel of d. But if f is a $\mathcal{C}^\infty$ function on I, it admits a primitive. Therefore the cokernel of d is trivial.

2. The first group, as above, is the kernel of d. Since the only constant function on I with compact support is the 0 function, we get that the first cohomology group is 0. If f has a compact support then $d(f)$ as well and $\int_I df = 0$. The converse is true, if h is a function on I with compact support such that $\int_I h = 0$ then there exists a function H with compact support with $H' = h$. Just take $H(x) = \int_{t=x_0}^{t=x} h(t)df$ for x_0 sufficiently closed to 0. Hence this shows that the second cohomology group is isomorphic to $\mathbf{R}$.

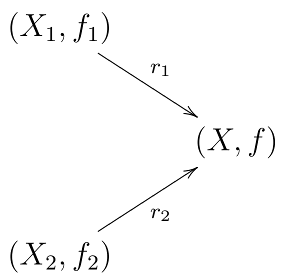
Solution of Exercise 4 (Filtrant categories)**.** 1. Let (X_1, f_1) and (X_2, f_2) be two objects of $\mathcal{C}_Y$. Denote by X the coproduct of X_1 and X_2 in $\mathcal{C}$ and by $r_i\colon X_i \to X$ for $i = 1, 2$ the two canonical morphisms. Since F is right exact, the coproduct of FX_1 and FX_2 exists in $\mathcal{D}$ and is FX. By construction of the coproduct we get morphisms

$$\begin{cases} \rho_1\colon FX_1 \to FX, \\ \rho_2\colon FX_2 \to FX \end{cases}$$

Furthermore, the two morphisms $f_1\colon FX_1 \to Y$ and $f_2\colon FX_2 \to Y$ yield a morphism $FX \to Y$ such that

$$\begin{cases} f\rho_1 = f_1, \\ f\rho_2 = f_2. \end{cases}$$

Thus we have the following diagram in $\mathcal{C}_Y$:



Let $(X_1, f_1) \underset{\psi}{\overset{\phi}{\rightrightarrows}} (X_2, f_2)$ be two parallel morphisms in $\mathcal{C}_Y$. Since $\mathcal{C}$ admits finite inductive limits, the cokernel of

$$X_1 \underset{\psi}{\overset{\phi}{\rightrightarrows}} X_2$$

exists, denote it by C and by $p_i\colon X_i \to C$ the canonical morphisms associated to it. As above, we know that the cokernel of:

$$FX_1 \underset{F\psi}{\overset{F\phi}{\rightrightarrows}} FX_2$$

exists and is FC. The two morphisms f_1 and f_2 yields a morphism $f\colon FC \to Y$ and we get as above the following diagram in $\mathcal{C}_Y$:

$$(X_1, f_1) \underset{\psi}{\overset{\phi}{\rightrightarrows}} (X_2, f_2) \xrightarrow{p_2} (X, f)$$

such that $p_2\phi = p_2\psi$.

2. Denote by G a right adjoint of F, We have a natural isomorphism of bifunctors:

$$\theta_{X,Y}\colon \operatorname{Hom}_{\mathcal{D}}(FX, Y) \to \operatorname{Hom}_{\mathcal{C}}(X, GY).$$

Let $Y \in \mathcal{D}$. The category $\mathcal{C}_Y$ is non-empty since $(GY, \theta^{-1}_{GY,Y}(\mathrm{Id}_{GY}))$ is an object of it. Given two objects (X_1, f_1) and (X_2, f_2) of $\mathcal{C}_Y$. we have in $\mathcal{C}_Y$:

$$\begin{array}{ccc} (X_1, f_1) & & \\ & \searrow^{f_1} & \\ & & (GY, \theta^{-1}_{GY,Y}(\mathrm{Id}_{GY})) \\ & \nearrow_{f_2} & \\ (X_2, f_2) & & \end{array}$$

Likewise given two parallel morphisms in $\mathcal{C}_Y$:

$$(X_1, f_1) \underset{\psi}{\overset{\phi}{\rightrightarrows}} (X_2, f_2)$$

we have a morphism $\theta^{-1}_{X_2,Y}(f_2)\colon (X_2, f_2) \to (GY, \theta^{-1}_{GY,Y}(\mathrm{Id}_{GY}))$ and we compute:

$$\theta^{-1}_{X_2,Y}(f_2)\phi = \theta^{-1}_{X_2,Y}(f_2)\psi.$$

Solution of Exercise 5 (Ext of Abelian groups). 1. Let $n\mathbf{Z}$ be an ideal of $\mathbf{Z}$. Suppose we have a map $f\colon n\mathbf{Z} \to I/X$. We have $f(n) = i + X$ for an $i \in I$. Since I is injective the morphism :

$$\begin{array}{ccc} n\mathbf{Z} & \to & I \\ n & \mapsto & i \end{array}$$

extends to a morphism $\mathbf{Z} \to I$. Hence there exists a $j \in I$ such that $nj = i$. Thus we can extend f :

$$\begin{array}{ccc} \mathbf{Z} & \to & I/X \\ 1 & \mapsto & j + X \end{array}$$

2. Since $\mathbf{Z}$ has enough injectives, there exists an injective $\mathbf{Z}$-module I and a monomorphism

$$N \to I.$$

Thanks to the fist question, the complex

$$I \longrightarrow I/N \longrightarrow 0 \longrightarrow$$

is an injective resolution of N. Therefore $\mathrm{Ext}^i_A(M, N)$ is 0 for $i \geqslant 2$.

Solution of Exercise 6 (Some Ext's)**.** 1. a. For a given i, the groups $\mathrm{Ext}^i_A(A, M)$ is the ith left derived functor of

$$\begin{array}{ccc} A - \mathfrak{Mod} & \to & A - \mathfrak{Mod} \\ X & \to & \mathrm{Hom}_A(A, X) = X \end{array}$$

which is an exact functor. Hence $\forall i > 0$, $\mathrm{Ext}^i_A(A, M) = 0$

b. Since x is not a zero divisor, we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{\times x} A \longrightarrow A/(x) \longrightarrow 0\ .$$

If we apply the functor $M \mapsto \mathrm{Hom}_A(-, M)$ we get a long exact sequence:

$$\mathrm{Ext}^i_A(A/x, M) \longrightarrow \mathrm{Ext}^i_A(A, M) \longrightarrow \mathrm{Ext}^i_A(A, M) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^{i+1}(A/x, M) \longrightarrow \mathrm{Ext}^{i+1}_A(A, M) \longrightarrow \mathrm{Ext}^{i+1}_A(A, M)$$

Hence for $i \geqslant 2$, $\mathrm{Ext}^i(A/x, M) = 0$. Of course $\mathrm{Ext}^0(A/x, A) = \mathrm{Hom}_A(A/x, M)$. Furthermore the module $\mathrm{Ext}^1_A(A/(x), M)$ is the cokernel of the morphism

$$\mathrm{Hom}_A(A, M) \xrightarrow{\times x} \mathrm{Hom}_A(A, M)\ .$$

Hence $\mathrm{Ext}^1_A(A/(x), M) = M/xM$.

2. a. All the groups are zero, except H_0, the cokernel of $A \oplus A \to A$ which equals to $A/(x_1, x_2) = k$.

b. The above complex gives a projective resolution of k, hence $\mathrm{Ext}^i_A(k, A) = 0$ for $i \geqslant 3$. Furthermore, the complex $\mathrm{Hom}_A(K_\bullet, A)$ is isomorphic to:

$$0 \longrightarrow A \xrightarrow{a \mapsto (ax_1, ax_2)} A \times A \xrightarrow{(a,b) \mapsto ax_1 - bx_2} A \longrightarrow 0\ .$$

So we can compute :

$$\begin{cases} \mathrm{Ext}^0_A(k,A) = \mathrm{Hom}_A(k,A) = 0, \\ \mathrm{Ext}^1_A(k,A) = 0, \\ \mathrm{Ext}^2_A(k,A) = k, \\ \mathrm{Ext}^i_A(k,A) = 0,\ i \geqslant 3 \end{cases}$$

For M'' the question 1b. gives:

$$\begin{cases} \mathrm{Ext}^0(M'',A) = \mathrm{Hom}_A(M'',A), \\ \mathrm{Ext}^1(M'',A) = A/x_1A = k[x_2], \\ \mathrm{Ext}^i(M'',A) = 0,\ i \geqslant 2. \end{cases}$$

Moreover, we have a short exact sequence:

$$0 \longrightarrow k \xrightarrow{\times x_1} M \longrightarrow M'' \longrightarrow 0 \ .$$

So for any $i \geqslant 0$, we get a long exact sequence :

$$\mathrm{Ext}^i_A(M'',A) \longrightarrow \mathrm{Ext}^i_A(M,A) \longrightarrow \mathrm{Ext}^i_A(k,A) \longrightarrow \mathrm{Ext}^{i+1}_A(M'',A) \longrightarrow \mathrm{Ext}^{i+1}_A(M,A) \longrightarrow \mathrm{Ext}^{i+1}_A(k,A).$$

Hence for $i \geqslant 3$, we have $\mathrm{Ext}^i(M,A) = 0$. For $i = 1$, we have a long exact sequence:

$$0 \longrightarrow \mathrm{Hom}(M'',A) \longrightarrow \mathrm{Hom}(M,A) \longrightarrow 0 \longrightarrow k[x_2] \longrightarrow \mathrm{Ext}^1_A(M,A) \longrightarrow 0$$

So $k[x_2] \to \mathrm{Ext}^1_A(M,A)$ is an isomorphism. Furthemore, for $i = 2$ we have:

$$0 \longrightarrow \mathrm{Ext}^2(M,A) \longrightarrow k \longrightarrow 0 \ .$$

So, $\mathrm{Ext}^2(M,A) \to k$ is an isomorphism. Of course $\mathrm{Ext}^0_A(M,A) = \mathrm{Hom}_A(M,A)$.