Homological Algebra
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Universiteit Leiden

Franck Doray
doray@math.leidenuniv.nl

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Duration: 4 hours

Neither books, nor written notes are permitted. Please do not use pencils!

Exercise 1 (Yoneda's lemma)**.** Let $\mathcal{C}$ be a category. The category of functors from $\mathcal{C}^{\mathrm{op}}$ to $\mathfrak{Set}$ will be denoted $\mathcal{C}^\vee = \mathfrak{Funct}(\mathcal{C}^{\mathrm{op}}, \mathfrak{Set})$. Denote by h the following functor:

$$\begin{array}{rccl} h\colon & \mathcal{C} & \to & \mathcal{C}^\vee \\ & X & \mapsto & \mathrm{Hom}_{\mathcal{C}}(-, X) \end{array}$$

1. Let X be an object of $\mathcal{C}$ and F be a functor $\mathcal{C}^{\mathrm{op}} \to \mathfrak{Set}$. Construct a bijective map

$$\phi\colon \mathrm{Hom}_{\mathcal{C}^\vee}(h(X), F) \to F(X),$$

and its inverse:

$$F(X) \to \mathrm{Hom}_{\mathcal{C}^\vee}(h(X), F).$$

2. Prove that the functor $h\colon \mathcal{C} \to \mathcal{C}^\vee$ is fully faithful.
3. Show that a morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ is an isomorphism if and only if the the induced morphism

$$\mathrm{Hom}_{\mathcal{C}}(W, X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathcal{C}}(W, Y)$$

is a bijective map for any $W \in \mathcal{C}$.

Exercise 2 (Category $\mathfrak{Ring}$)**.** Denote by $\mathfrak{Ring}$ the category of rings.

1. Show that the natural morphism $\mathbf{Z} \to \mathbf{Q}$ is an epimorphism in $\mathfrak{Ring}$.
2. Prove that a ring monomorphism $A \to B$ is an injection.

Exercise 3 (A bit of analysis)**.** Let $I =]0, 1[\subset \mathbf{R}$. Recall that $\mathcal{C}^\infty(I)$ denotes the $\mathbf{R}$-algebra of $\mathbf{R}$-valued $\mathcal{C}^\infty$ functions on I, and $\mathcal{C}^\infty_K(I)$ is the ideal of functions with compact support.

1. Compute the cohomology groups of the complex

$$0 \longrightarrow \mathcal{C}^\infty(I) \xrightarrow{d} \mathcal{C}^\infty(I) \longrightarrow 0$$

where $d = \frac{d}{dx}$ is the usual derivation.

2. Compute the cohomology groups of the complex

$$0 \longrightarrow \mathcal{C}^\infty_K(I) \xrightarrow{d} \mathcal{C}^\infty_K(I) \longrightarrow 0 .$$

Exercise 4 (Filtrant categories)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. For any $Y \in \mathcal{D}$ denote by $\mathcal{C}_Y$ the category whose objects are pairs (X, f) where X is an object of $\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{D}}(FX, Y)$, and a morphism between (X, f) and (X', f') consists of a morphism $\phi \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ such that the following diagram

$$\begin{tikzcd} FX \arrow[r, "F\phi"] \arrow[d, "f"'] & FX' \arrow[dl, "f'"] \\ Y & \end{tikzcd}$$

commutes, i.e. $f' \circ F\phi = f$. We say that F has the property $\mathcal{R}$ if for any $Y \in \mathcal{D}$, the category $\mathcal{C}_Y$ is filtrant[1].

1. Suppose that $\mathcal{C}$ admits finite inductive limits. Show that if F is right exact[2] then it has the property $\mathcal{R}$.
2. Suppose that F has a right adjoint Prove that the functor F has the property $\mathcal{R}$.

Exercise 5 (Ext of Abelian groups)**.** 1. Let X be a $\mathbf{Z}$-module. Show that for any injective $\mathbf{Z}$-module I and any injective morphism $X \to I$, the quotient module I/X is injective[3].

2. Prove that for any two $\mathbf{Z}$-modules M and N, and any integer $i \geqslant 2$, we have

$$\mathrm{Ext}^i_{\mathbf{Z}}(M, N) = 0.$$

Exercise 6 (Some Ext's)**.** 1. Let A be a commutative ring, and M be an A-module.

a. Show that for any $i > 0$, we have

$$\mathrm{Ext}^i_A(A, M) = 0.$$

b. Let $x \in A$ be an element of A, which is not a zero divisor. Compute $\mathrm{Ext}^i_A(A/(x), M)$ for all $i \geqslant 0$.

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[1]A category is filtrant if it is non-empty if for any two objects X and Y there exists a Z with morphisms $X \to Z$ and $Y \to Z$ and if finally for any two parallel morphisms $X \overset{g}{\underset{f}{\rightrightarrows}} Y$ there exist a morphism $h\colon Y \to Z$ such that $hf = hg$.

[2]A functor $\mathcal{C} \to \mathcal{D}$ where $\mathcal{C}$ is a category admitting finite inductive limits is right exact if for any inductive system $(X_i)_{i \in \mathcal{I}}$ in $\mathcal{C}$ indexed by a finite category $\mathcal{I}$, the limit of the induced inductive system $(F(X_i))_{i \in \mathcal{I}}$ in $\mathcal{D}$ is represented by $F(\varinjlim_{\mathcal{I}} X_i)$.

[3]One may use Baer's lemma: an R-module J is injective if and only if for any ideal $\mathfrak{a} \subset R$ the canonical morphism

$$\mathrm{Hom}_R(R, J) \to \mathrm{Hom}_R(\mathfrak{a}, J)$$

is surjective.

c. Let n, m be two integers $\geqslant 1$. Compute $\mathrm{Ext}^i_{\mathbf{Z}}(\mathbf{Z}/n\mathbf{Z}, \mathbf{Z}/m\mathbf{Z})$ for all $i \geqslant 0$.

2. Set $A = k[x_1, x_2]$ where k is a commutative ring. Consider the following A-modules:

$$\begin{cases} M = A/(x_1^2 A + x_1 x_2 A), \\ M'' = A/(x_1). \end{cases}$$

a. Compute the homology groups of the following complex $K_\bullet$:

$$0 \longrightarrow A \xrightarrow{a \mapsto (ax_2, -ax_1)} A \oplus A \xrightarrow{(a,b) \mapsto ax_1 + bx_2} A \longrightarrow 0\ .$$

b. Compute for every i the groups $\mathrm{Ext}^i_A(k, A)$ and $\mathrm{Ext}^i_A(M'', A)$. Deduce $\mathrm{Ext}^i_A(M, A)$ for any $i \geqslant 0$.