Homological Algebra
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Duration: 3 hours

Neither books, nor written notes are permitted. Please do not use pencils!

Exercise 1. For simplicity, we consider the Abelian category $\mathfrak{Ab}$ of $\mathbf{Z}$-modules. Recall that the category of cochain complexes in $\mathfrak{Ab}$ is denoted $\mathfrak{Ch}^\bullet(\mathfrak{Ab})$. Let $f\colon X^\bullet \to Y^\bullet$ be a morphism in $\mathfrak{Ch}^\bullet(\mathfrak{Ab})$. Suppose that f is homotopic to 0. Prove that for any $n \in \mathbf{Z}$, the natural morphism:

$$\mathrm{H}^n(f)\colon\ \mathrm{H}^n(X) \to \mathrm{H}^n(Y)$$

is 0.

Exercise 2. A functor $F\colon \mathcal{C} \to \mathcal{C}'$ is said *conservative*, if: $\forall f\colon X \to Y$ in $\mathcal{C}$, $F(f)\colon FX \to FY$ is an isomorphism $\Longrightarrow$ $f\colon X \to Y$ is an isomorphism. Let $F\colon \mathcal{A} \to \mathcal{A}'$ be an additive functor between Abelian categories. Consider the three following assertions:

(i) F is faithful.
(ii) F is conservative.
(iii) $\forall X \in \mathcal{A}$, $FX \simeq 0 \Longrightarrow X \simeq 0$.

1. Prove that $(i) \Longrightarrow (ii)$.
2. Prove that $(ii) \Longrightarrow (iii)$.
3. Suppose that F is exact, prove that $(iii) \Longrightarrow (ii)$.
4. Suppose that F is exact, prove that $(ii) \Longrightarrow (i)$.

Exercise 3. Let $\mathcal{C}$ be a category and consider in $\mathcal{C}$ the two following diagrams:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & U \end{array}$$

and

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ Z & \longrightarrow & V \end{array}$$

Suppose that the first square is Cartesian and the second square is cocartesian. Prove that the second square is also Cartesian.

Exercise 4. Consider the category $\mathfrak{Top}^{\text{Hausdorff}}$ of Hausdorff topological spaces. Show that the canonical morphism $i\colon \mathbf{Q} \to \mathbf{R}$ is an epimorphism but that the morphism of underlying sets is not surjective.

Exercise 5. In the category of commutative rings, give an example of an epimorphism that is not a surjection on the underlying sets.

Exercise 6. Let A be a commutative ring. Consider the following exact sequences in $A - \mathfrak{Mod}$:

$$0 \longrightarrow N_1 \longrightarrow P_1 \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow N_2 \longrightarrow P_2 \longrightarrow M \longrightarrow 0$$

where P_1 and P_2 are projective. Show that $N_1 \oplus P_2$, $N_2 \oplus P_1$ and $\operatorname{Ker}(P_1 \oplus P_2 \to M)$ are isomorphic where the morphism $P_1 \oplus P_2 \to M$ is the canonical one deduced from $P_1 \to M$ and $P_2 \to M$.

Exercise 7 (Ext of Abelian groups)**.** 1. Let X be a $\mathbf{Z}$-module. Show that for any injective $\mathbf{Z}$-module I and any injective morphism $X \to I$, the quotient module I/X is injective[1].

2. Fix an Abelian group A and consider the left exact functor $F\colon \mathfrak{Ab} \to \mathfrak{Ab}$ defined as $F(X) = \operatorname{Hom}_{\mathfrak{Ab}}(A, X)$. Prove that for any Abelian group X, and any integer $i \geqslant 2$, we have

$$R^i F(X) = 0.$$

[1] One may use Baer's lemma: an R-module J is injective if and only if for any ideal $\mathfrak{a} \subset R$ the canonical morphism

$$\operatorname{Hom}_R(R, J) \to \operatorname{Hom}_R(\mathfrak{a}, J)$$

is surjective.