Exam Commutative Algebra: Solutions

2. (a) Let $y \in S$ such that $xy = 1$. We need to show that $y \in R$. Assume that y satisfies the integral equation

$$y^n + a_1 y^{n-1} + \cdots + a_n = 0$$

with $a_i \in R$. Multiply this equation with x^{n-1} to find

$$y = -a_1 - \cdots - a_n x^{n-1}.$$

The right hand side belongs to R, so $y \in R$.

(b) It suffices to show that an elementary tensor $s' \otimes t \in S' \otimes_R T$ is integral over $S \otimes_R T$. Assume that s' has integral equation

$$(s')^n + f(a_1)(s')^{n-1} + \cdots + f(a_n) = 0,$$

with $a_i \in S$. Then $s' \otimes t$ satisfies the integral equation

$$(s' \otimes t)^n + (f(a_1) \otimes t)(s' \otimes t)^{n-1} + \cdots + (f(a_n) \otimes t^n) = 0$$

by definition of the addition and multiplication in $S' \otimes_R T$.

3. (a) i. We may assume that the primes in $\mathrm{Ass}(M)$ that do not meet U are $P_1, \ldots, P_t$. By Theorem 3.10d

$$0 = \bigcap_{i=1}^{t} M_i[U^{-1}]$$

is a minimal primary decomposition of 0 in $M[U^{-1}]$. By Corollary 2.6 localization commutes with finite intersections, so $0 = (\cap_{i=1}^t M_i)[U^{-1}]$ in $M[U^{-1}]$. If we take the inverse image under the localization map $\varphi : M \to M[U^{-1}]$, then it suffices to show that

$$\varphi^{-1}\left((\bigcap_{i=1}^{t} M_i)[U^{-1}] \right) = \bigcap_{i=1}^{t} M_i.$$

It is obvious that the right hand side is contained in the left hand side. For the other inclusion: take $m \in M$ and assume that $\varphi(m) = \frac{m}{1}$ equals $\frac{m'}{u}$ for $m' \in \cap_{i=1}^t M_i$, $u \in U$. Then there exists a $u' \in U$ such that $u'um = u'm'$. In particular $u'um \in \cap_{i=1}^t M_i$. The following lemma shows that this implies $m \in \cap_{i=1}^t M_i$.

Lemma. *Let M_i be a P_i-primary submodule of an R-module M and U a multiplicatively closed subset of R not meeting P_i. Then for all $u \in U$ and $m \in M$ we have: $um \in M_i \Rightarrow m \in M_i$.*

Proof. By definition P_i is the only associated prime of M/M_i. Theorem 3.1b implies that the elements of U act as nonzerodivisors on M/M_i. And this is a restatement of what we need to show. □

ii. Let $P_{t+1}, \ldots, P_n$ be the associated primes of M that meet U. Let $I = \cap_{i=t+1}^n P_i$. Then $I \cap U \neq \emptyset$ since we can take $u_i \in P_i \cap U$ for $i = t+1, \ldots, n$ and then $\prod_{i=t+1}^n u_i \in I \cap U$. So for an associated prime of M we have: not containing I and being disjoint from U are equivalent. Proposition 3.13a now shows that $H_I^0(M) = \cap_{i=1}^t M_i$.

(b) Let R be the ring $\frac{k[x_1,x_2,x_3,\ldots]}{(x_1,x_2^2,x_3^3,\ldots)}$ with k a field. Consider R as R-module. Since R has only one prime ideal, namely $M = (x_1, x_2, x_3, \ldots)$, it suffices to show that this is not the annihilator of an element of R. Indeed, a coset in R may be uniquely represented by a polynomial f in variables $x_1, \ldots, x_n$ for some $n \geq 1$ and with $\deg_{x_i}(f) < i$. But then x_{n+1} does not belong to the annihilator of $f + (x_1, x_2^2, x_3^3, \ldots)$.

4. (a) If $\mathrm{gr}_M(R)$ is a polynomial ring in d variables, then $\dim_k \frac{M}{M^2} = d$ and thus by definition R is regular. Conversely, assume that R is regular. Choose regular parameters $y_1, \ldots, y_d$ in M (so these generate the maximal ideal). Define a k-algebra homomorphism

$$\varphi : k[x_1, \ldots, x_d] \to \mathrm{gr}_M(R)$$

by mapping x_i to y_i. This is obviously a surjective morphism of graded rings. Assume that $\ker \varphi$ is nontrivial. Choose a nonzero element $g \in \ker \varphi$. Then each homogeneous piece of g is sent to 0 by φ since it is a morphism of graded rings. So $\ker \varphi$ contains a nonzero (and necessarily nonconstant) homogeneous polynomial f. Let f have degree $e > 0$. The number of monomials of degree n in $k[x_1, \ldots, x_d]$ equals $\binom{n+d-1}{d-1}$. For $n \geq e$ the dimension (as k-vector space) of the homogeneous piece of degree n in the graded ring $\frac{k[x_1,\ldots,x_d]}{(f)}$ equals

$$\binom{n+d-1}{d-1} - \binom{n-e+d-1}{d-1}$$

So

$$\dim_k \frac{M^n}{M^{n+1}} \leq \binom{n+d-1}{d-1} - \binom{n-e+d-1}{d-1}$$

for $n \geq e$. This implies that the Hilbert polynomial of R has degree $\leq d-2$ and thus $\dim R \leq d-1$ (Theorem 12.1) which is a contradiction.

(b) The completion of R with respect to M is also a Noetherian local ring of the same dimension and it has the same associated graded ring as R (we use Theorem 7.1 and Corollary 10.12). So (a) immediately gives the solution.