Exam Commutative Algebra

11th of June 2008, 9am - 1pm

Organization:

- Questions 1 and 2 form the **oral part**. You can prepare them between 9am and 10am and then you have to hand them in. Between 10am and 1pm I will see each student during approximately 20 minutes to discuss these questions (in order of handing in).
- Questions 3 and 4 form the **written part**. You have time to prepare them until 1pm. *Please be sufficiently detailed, since these questions will not be discussed orally. If you use a theorem from the book, make a clear reference to it.*
- You can use the following: the book of Eisenbud, *your own* solutions of the homework exercises, *personal handwritten* notes that you made during class or to prepare the exam, all information on the website of the course, pen and paper, your brain, food and drinks.
- Please write your name on every page that you hand in.

Good luck!

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1. Explain the proof of the 'First version of the Principal Ideal Theorem' (Theorem 10.1). *Please copy the proof (and if necessary extra explanations) on the sheet of paper that you hand in, since this is what will be used during the oral examination.*

2. (a) Let $R \subseteq S$ be rings with S integral over R. Show that if $x \in R$ is a unit in S, then it is already a unit in R.

   (b) Let $f : S \to S'$ be a morphism of R-algebras that makes S' integral over S. Let T be an R-algebra. Then

$$f \otimes 1 : S \otimes_R T \to S' \otimes_R T$$

   makes $S' \otimes_R T$ integral over $S \otimes_R T$.

3. (a) (Exercise 3.12.) Let M be a finitely generated module over the Noetherian ring R. Let $U \subset R$ be a multiplicatively closed subset and $0 = \cap_{i=1}^{n} M_i$ be a minimal primary decomposition of 0. Prove the following:

   i. The kernel of the localization map $M \to M[U^{-1}]$ is the intersection of the M_i whose corresponding primes of $\mathrm{Ass}(M)$ do not meet U.

   ii. This kernel may be written as $H_I^0(M)$ for some ideal $I \subset R$.

   (b) Give an example of a module that does not have any associated primes.

4. Let R be a local Noetherian ring of dimension d with maximal ideal M and residue class field k. Prove the following facts:

   (a) R is regular if and only if

$$\mathrm{gr}_M(R) = \frac{R}{M} \oplus \frac{M}{M^2} \oplus \frac{M^2}{M^3} \oplus \cdots$$

   is as a graded ring isomorphic to the polynomial ring $k[x_1, \ldots, x_d]$. *Hint for the 'only if' part*: it is not hard to give a surjective map $\varphi : k[x_1, \ldots, x_d] \to \mathrm{gr}_M(R)$. Show that the kernel contains a nonzero homogeneous polynomial f if it is nontrivial. Now use the technique of Hilbert functions to show that this is in contradiction with $\dim R = d$.

   (b) R is regular if and only if the completion of R with respect to M is regular.