MEASURE THEORY AND INTEGRATION, FALL 2011

Exam 13/01/2012 (max 10 points)

(1) [2 points] Denote by λ the Lebesgue measure on $\mathbb{R}$, and by λ^* the corresponding outer measure. For a set $E \subset \mathbb{R}$ and $a > 0$, let

$$aE = \{ax : \quad x \in E\}.$$

Prove that $\lambda^*(aE) = a\lambda^*(E)$. Prove that if E is Lebesgue measurable then aE is also Lebesgue measurable.

(2) [2 points] Let $\alpha > 2$. A point $x \in [0,1]$ is called α-good, if the inequality

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^\alpha}$$

holds for **infinitely many** pairs $(p, q) \in \mathbb{N}^2$. Such α-points are well approximated by rationals. Let E_α be the set of α-good points in $[0,1]$. Show that the Lebesgue measure of E_α is zero.

(3) [2 points] Evaluate the following limits or prove that the limits does not exist:

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin(\frac{x}{n})}{x(1+x^2)} dx,$$

$$\lim_{N\to\infty} \sum_{n=0}^{N} \int_0^{\pi/2} (1 - \sqrt{\sin(x)})^n \cos(x) dx.$$

Justify your answer.

(4) [2 points] Prove that

$$\int_1^\infty \frac{\sqrt[3]{1+x}}{x^2} dx \leq \sqrt[3]{6}$$

Show that the inequality is in fact strict.

(5) [2 points] Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = 2(x - y)e^{-(x-y)^2}$$

if $x > 0$, and zero otherwise. Evaluate the integral

$$\int_{\mathbb{R}^2} |f(x, y)| dx dy$$

Justify your answer. You can use without a proof that

$$\int_{-\infty}^{\infty} e^{-z^2} dz = \sqrt{\pi}.$$