Exam Mathematics for Statisticians

November 2, 2012, 11 am - 2 pm

You may use a simple calculator. Motivate your answers by calculations or arguments.

Please, write your name and student number on each exam paper.

The grade for the exam is 1+ **the number of points divided by** 8**. Your final grade is** 1/3 **times the average homework grade plus** 2/3 **times the exam grade.**

Problem 1 Check whether the following limits exist. Calculate the limit, if it does.

a) (5) $\lim_{x \to -3} \dfrac{x^2 - 2x - 15}{5x^2 + 5x - 30}$.

b) (5) $\lim_{x \to -\infty} \dfrac{5x^3 + 12x^2 - 13}{3x^3 - 28}$.

Problem 2 (4) Calculate the derivative of the function $f(x) = \cos\left(\sqrt[3]{e^{\ln x^{3/2}}}\right)$.

Problem 3 (12). We would like to estimate $\sqrt{125}$ by using a Taylor polynomial for $f(x) = \sqrt{x}$ about $a = 121$.

a) Determine the second order Taylor polynomial $P_{2,121}(x)$ with base $a = 121$. Calculate $P_{2,121}(125)$ as an estimate of $\sqrt{125}$.

b) Calculate an estimate of the error $|f(125) - P_{2,121}(125)|$.

c) Explain why we have chosen base 121 to estimate $\sqrt{125}$.

Problem 4 (10) Heineken is producing beer bottles. They suspect that the beer bottles systematically contain too little beer. The standard volume should be 30 cl per bottle. Clearly there are always small deviations but the average volume should be 30 cl.

They take a sample of 10 bottles. The measured volumes are:

$$30.1, 29.8, 30.2, 29.7, 29.8, 30.05, 29.6, 29.9, 30, 30.1$$

They assume that the actual contents have a normal distribution with standard deviation 1, and unknown mean μ. The density of the measured sample then equals

$$\frac{1}{2^5\pi^5} e^{-(30.1-\mu)^2/2} \cdot e^{-(29.8-\mu)^2/2} \cdot e^{-(30.2-\mu)^2/2} \cdot e^{-(29.7-\mu)^2/2} \cdot e^{-(29.8-\mu)^2/2} \cdot$$
$$\cdot e^{-(30.05-\mu)^2/2} \cdot e^{-(29.6-\mu)^2} \cdot e^{(29.9-\mu)^2/2} \cdot e^{-(30-\mu)^2/2} \cdot e^{-(30.1-\mu)^2/2}.$$

Calculate the value of μ that maximises this product. Interpret the obtained value.

Problem 5 Determine primitives of the following functions.

a) (8) $\dfrac{e^{1+\sqrt{2x}}}{\sqrt{2x}}$

b) (8) $\dfrac{3x^2}{\sqrt{1+x}}$.

Problem 6 a) (4) Calculate $\sum_{n=1}^{100}(2+3(n-2))$.

b) (6) Are the following series convergent or divergent? You may simply answer by yes or no:

1. $\sum_{n=1}^{\infty}\frac{2n+1}{3n}$;
2. $\sum_{n=1}^{\infty}(\frac{\pi}{4})^n$;
3. $\sum_{n=1}^{\infty}\frac{1}{\ln n}$.

Problem 7 Throw an unbiased dice repeatedly. The random variable X denotes the number of times you have to throw the dice until the first 5 shows up.

a) (2) Calculate the probability $\mathsf{P}\{X=n\}$ that the first 5 shows up the n th time that you throw the dice, $n=1,2,\ldots$.

b) (4) Use the formula for the geometric series on the formulae sheet to derive an expression for $\sum_{n=1}^{\infty} a\cdot n\cdot x^{n-1}$.

c) (4) Calculate the expected number of times $\mathsf{E}X$ you have to throw the dice until the first 5 shows up. If you could not answer a), you may assume that $\mathsf{P}\{X=n\}=(\frac{2}{3})^n/3$.