MEASURE THEORY, FALL 2010

Exam 05/01/2011 (max 10 points)

(1) [2 points] Subset $A \subseteq \mathbb{R}^2$ is called <u>symmetric</u> if for any $(x_1, x_2) \in A$, $(-x_1, -x_2)$ is also an element of A. Show that collection of all symmetric subsets of $\mathbb{R}^2$ is a σ-algebra.

(2) [2 points] A complex number $z \in \mathbb{C}$ is called <u>algebraic</u> if it is a root of a non-zero polynomial with integer coefficients, i.e.,

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0,$$

for some $n \in \mathbb{N}$, and $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, $a_n \neq 0$. Represent the set of all algebraic numbers as a subset of $\mathbb{R}^2$ as follows:

$$\mathcal{A} = \{(\operatorname{Re} z, \operatorname{Im} z) : \quad z \in \mathbb{C} \text{ is algebraic}\} \subseteq \mathbb{R}^2.$$

Compute $m^{(2)}(\mathcal{A})$ – the 2-dimensional Lebesgue measure of $\mathcal{A}$.

(3) [2 points] Let $(\Omega, \mathcal{A}, \mu)$ be a measure space, show that $\mathcal{L}_\infty(\Omega, \mathcal{A}, \mu) \subset \mathcal{L}_1(\Omega, \mathcal{A}, \mu)$ if and only if $\mu(\Omega) < \infty$. Prove that if $\mu(\Omega) = 1$, then

$$||f||_\infty = \lim_{p \to \infty} ||f||_p.$$

(4) [2 points] For $n \in \mathbb{N}$ and $k = 1, \ldots, n$, define $f_{k,n} : [0,1] \to \mathbb{R}$ as follows:

$$f_{k,n}(x) = \begin{cases} 1, & x \in \left[\frac{k-1}{n}, \frac{k}{n}\right], \\ 0, & \text{otherwise.} \end{cases}$$

From the triangular array $\{f_{k,n} : 1 \leq k \leq n, n \in \mathbb{N}\}$, form a sequence of functions as follows

$$\{g_1, g_2, g_3, g_4, g_5, g_6, \ldots\} := \{f_{1,1}, f_{1,2}, f_{2,2}, f_{1,3}, f_{2,3}, f_{3,3}, \ldots\}.$$

With respect to the Lebesgue measure m on $[0,1]$, does the sequence $\{g_m\}$ converge stochastically (in measure), almost surely, in $\mathcal{L}_p(m)$, $p \in [1, +\infty]$?

(5) [2 points] Denote by m the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Prove that if functions $f_1, f_2 : \mathbb{R} \to \overline{\mathbb{R}}$ are integrable ($f_1, f_2 \in \mathcal{L}_1(m)$), then

- for Lebesgue almost all x, the function
$$y \mapsto f_1(y) f_2(x - y)$$
is integrable;
- the convolution of f_1, f_2, given by
$$F(x) = \int_{\mathbb{R}} f_1(y) f_2(x - y) m(dy),$$
is an integrable function (i.e., $F \in \mathcal{L}_1(m)$).

You can assume without proof that the function

$$h(x, y) = f_1(y) f_2(x - y) : \mathbb{R} \times \mathbb{R} \to \overline{\mathbb{R}}$$

is measurable.