MEASURE THEORY, FALL 2010

Exam 25/02/2011 (max 10 points)

(1) [2 points] Suppose Ω is a non-empty set, and $\mathcal{A}$ is a collection of subsets Ω such that

- $\Omega \in \mathcal{A}$,
- $A, B \in \mathcal{A}$ implies that $A \cap B^c \in \mathcal{A}$.

Show that $\mathcal{A}$ is an algebra.

(2) [2 points]

- Suppose $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing. Prove that f is Borel measurable.
- Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable, i.e., $f'(x)$ is defined for every $x \in \mathbb{R}$. Prove that f' is a Borel measurable function.

(3) [2 points] State the dominated convergence theoreom, and use this theorem to find

$$\lim_{n\to\infty} \int_1^\infty f_n(x)dx,$$

where

$$f_n(x) = \frac{\sqrt{x}\log(nx)\sin(nx)}{1+nx^3}, \quad n \in \mathbb{N}.$$

(4) [2 points] Suppose $(\Omega, \mathcal{A}, \mu)$ is a measure space, and real numbers $p, q, s > 1$ satisfy

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1,$$

Show that for any $f \in \mathcal{L}_p(\Omega, \mu)$, $g \in \mathcal{L}_q(\Omega, \mu)$, $h \in \mathcal{L}_s(\Omega, \mu)$, the function $f \cdot g \cdot h$ is integrable.

(5) [2 points] Let m be the Lebesgue measure on $[0, 1]$. Suppose $K(x, y) : [0, 1] \times [0, 1] \to \mathbb{R}$ is a square-integrable function:

$$\int_0^1 \int_0^1 |K(x,y)|^2 m(dx)m(dy) < \infty.$$

The corresponding Hilbert-Schmidt integral operator A_K is given by

$$(A_K f)(x) = \int_0^1 K(x,y)f(y)m(dy), \quad x \in [0,1].$$

Prove that for a square-integrable kernel K, the operator A_K maps $\mathcal{L}_2([0, 1], m)$ into itself, i.e., $A_K f \in \mathcal{L}_2([0, 1], m)$ for every $f \in \mathcal{L}_2([0, 1], m)$. Show also that the norm of A_K:

$$||A_K|| := \sup\{||A_K f||_2 : \quad f \in \mathcal{L}_2([0,1], m), \ ||f||_2 \le 1\}$$

is bounded.