Mathematisch Instituut
Universiteit Leiden

January 21, 2013
14:00 – 17:00

Percolation theory

Examination

Question 1. Consider percolation on the hypercubic lattice $(\mathbb{Z}^d, E^d)$. The state space for this model is $\Omega = \{0,1\}^{E^d}$.

1. Define the terms *increasing event* and *decreasing event*.
2. Let $x, y \in \mathbb{Z}^d$. Decide for each of the following events whether it is (a) increasing / (b) decreasing / (c) an intersection of an increasing and a decreasing event / (d) neither of the former three:
$$A_1 = \Omega, \qquad A_2 = \{0 \leftrightarrow x\}\Box\{0 \leftrightarrow y\}, \qquad A_3 = \{|\mathscr{C}(x)| = 17\},$$
$$A_4 = \{\text{there exist (at least) two (different) infinite clusters}\}.$$
Motivate your decisions. In case of (c), specify which increasing or decreasing event can be used to get the right intersection.
3. For an increasing event A, prove that $\mathbb{P}_p(A)$ is an increasing function of p.

Question 2. For bond percolation on the square lattice $(\mathbb{Z}^2, E^2)$, show that
$$\lim_{n\to\infty} \mathbb{P}_p\big((0,0) \leftrightarrow (n,0)\big) = \theta(p)^2.$$

Question 3. Consider site percolation on the triangular lattice $\mathbb{T}$, end let $\Lambda(n)$ be the ball of radius n centered at the origin 0. Use the RSW theorem to show that there exists constants $C, \alpha > 0$ such that for any $k, n \in \mathbb{N}$ with $0 < k \le n$ the following hold:
$$\mathbb{P}_{1/2}\big(\Lambda(k) \leftrightarrow \partial\Lambda(n)\big) \le C\left(\frac{n}{k}\right)^{-\alpha}.$$

(You get part of the points for this question if you prove the special case $k = 1$.)