Written examination for the course

Probability: Large Deviations

Teacher: F. den Hollander

Wednesday 04 January 2012, 14:00–17:00, room 174, Snellius

- Write your name and student identification number on each piece of paper you hand in.
- The use of notes and/or textbooks is not allowed.
- The weight of each question is indicated in boldface. The total number of points is 100. *Final score*: ≥ 58 points is sufficient, ≤ 57 points is insufficient.

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(1) (a) **[5]** Formulate Cramér's Theorem for large deviations of the empirical average of i.i.d. real-valued random variables $(X_i)_{i\in\mathbb{N}}$.

(b) **[5]** Show that the associated rate function is non-negative and convex.

(c) **[5]** Compute the rate function when the law of X_1 is the Poisson distribution $P(X_1 = k) = e^{-\lambda}\lambda^k/k!$, $k \in \mathbb{N}_0$, with $\lambda \in (0,\infty)$ a parameter.

(2) (a) **[5]** Formulate Sanov's Theorem for large deviations of the empirical distribution of i.i.d. real-valued random variables $(X_i)_{i\in\mathbb{N}}$ taking values in a finite set Γ.

(b) **[5]** Show that the associated rate function is non-negative and convex.

(c) **[5]** Compute the rate function when the law of X_1 is the uniform distribution on Γ.

(3) (a) **[5]** Give the definition of the Large Deviation Principle for a sequence of probability measures $(P_n)_{n\in\mathbb{N}}$ on a Polish space $\mathcal{X}$.

(b) **[5]** Formulate Varadhan's lemma.

(c) **[10]** Formulate the Contraction Principle and give its proof.

(4) (a) **[10]** Consider a Markov chain $(X_i)_{i\in\mathbb{N}}$ on a finite state space Γ with a strictly positive transition kernel $(P_{st})_{s,t\in\Gamma}$ starting from its stationary distribution $(\pi_s)_{s\in\Gamma}$. Give a sketch of how the Large Deviation Principle for its empirical pair distribution is derived from that of an auxiliary i.i.d. sequence (i.e., from Sanov's Theorem for pairs) via a change-of-measure argument.

(b) **[5]** Write down the associated rate function and show that it is non-negative and convex.

(5) (a) **[5]** Formulate the Gärtner-Ellis Theorem on $\mathbb{R}^d$, $d \geq 1$.

(b) **[5]** Show why Cramér's Theorem is an immediate corollary of the Gärtner-Ellis Theorem.

(6) (a) **[5]** Explain what is the hypothesis testing problem in statistics. In what way is the Neyman-Pearson test optimal?

(b) **[10]** Explain how Cramér's Theorem can be used to find the accuracy of the Neyman-Pearson test?

(7) (a) **[5]** What is the Parabolic Anderson Model?

(b) **[5]** What can be proved about the scaling of the first moment of the solution of the Parabolic Anderson Model for the special case where the disorder has a double-exponential distribution?