Course: Probability

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(written examination)

Tuesday 27 January 2009, 10:00–13:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation. A simple "yes" or "no" is not enough.
- The use of textbooks is not allowed.
- The questions below are weighted as follows: (1) 6, 2, 2; (2) 7, 3; (3) 5, 15; (4) 4, 4, 4, 4, 4; (5) 10, 7, 3; (6) 2, 2, 8, 8. Total: 100 points.

(1) (a) Give the definition of a (time-homogeneous) Markov chain $X = (X_n)_{n\in\mathbb{N}_0}$ on a countable state space S.

(b) When is X irreducible?

(c) When is X aperiodic?

(2) (a) Explain how a (time-homogeneous) Markov chain X on a countable state space S can be simulated with the help of a sequence $U = (U_n)_{n\in\mathbb{N}_0}$ of i.i.d. random variables that are uniformly distributed on the unit interval $[0, 1]$.

(b) Is the way to perform the simulation unique?

(3) (a) State the convergence theorem for a (time-homogeneous) Markov chain X on a finite state space S.

(b) Give a brief sketch of how this theorem is proved with the help of the technique of coupling.

(4) Consider a chessboard with a single knight that makes random allowed moves. Let X_n denote the position of the knight at time $n \in \mathbb{N}_0$.

(a) What is the state space S of $X = (X_n)_{n\in\mathbb{N}_0}$?

(b) Explain why X is a Markov chain.

(c) Is X irreducible?

(d) Is X aperiodic?

(e) Suppose that X_0 is the upper left corner of the chessboard. What is the distribution of X_n in the limit as $n \to \infty$?

(5) Given is a finite connected graph $G = (V, E)$, with vertex set V and edge set E. Suppose that each vertex can have one of two different colors, B (= black) or W (= white), and that only colorings are allowed where no two vertices connected by an edge have the color B. Then the set of allowed colorings is

$$S = \Big\{ s = (s_v)_{v\in V} \in \{B, W\}^V \colon (s_{v_1}, s_{v_2}) \neq (B, B) \ \forall\, (v_1, v_2) \in E \Big\}.$$

Suppose that we want to simulate the distribution π on S given by $\pi(s) = \lambda^{n(s)}/Z_\lambda$, $s \in S$, where $\lambda \in (0, \infty)$, $n(s) = \sum_{v\in V} 1_{\{s_v = B\}}$, and Z_λ is the normalizing constant.

(a) How can the simulation be done with the help of the Gibbs sampler? Give the transition probabilities of the associated Markov chain.

(b) Explain why the Gibbs sampler is irreducible and aperiodic.

(c) What happens when $\lambda \downarrow 0$, respectively, $\lambda \uparrow \infty$?

(6) Let $X = (X_n)_{n\in\mathbb{N}_0}$ be the birth-death chain on state space $\mathbb{N}_0$ with transition probabilities

$$P_{ij} = \begin{cases} p_i & \text{if } j = i + 1, \\ 1 - p_i & \text{if } j = i - 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $p = (p_i)_{i\in\mathbb{N}_0}$ are the birth probabilities (with $p_i \in (0, 1)$ for $i \in \mathbb{N}$ and $p_0 = 1$).

(a) Is X irreducible?

(b) Is X aperiodic?

(c) Give the necessary and sufficient condition on p for X to have a stationary distribution π.

(d) Compute π (hint: X is reversible).