Course: Probability

Teacher: F. den Hollander

(written examination: 2nd chance)

Wednesday 25 March 2009, 10:00–13:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation. A simple "yes" or "no" is not enough.
- The use of textbooks is not allowed.
- The questions below are weighted as follows: (1) 4, 6; (2) 4, 4, 5, 3, 4, 5; (3) 15, 5; (4) 5, 10, 5; (5) 2, 8, 2, 2, 8, 3. Total: 100.

(1) A (time-homogeneous) Markov chain X on a countable state space S is characterized by two ingredients: an initial distribution μ_0 and a transition matrix P.

(a) What is the formula for the distribution μ_n at time $n \in \mathbb{N}$?

(b) Derive this formula.

(2) (a) What is a stationary distribution π for a (time-homogeneous) Markov chain X on a countable state space S?

(b) If S is finite, then does such a π always exist?

(c) If S is finite and π exists, then is it always unique?

(d) Give a formula (without proof) that relates π to the average return time of X to the states of S.

(e) When is X reversible?

(f) Give a formula for π when X is reversible.

(3) Given is a finite connected graph $G = (V, E)$, with vertex set V and edge set E. Suppose that each vertex can have one of two different colors, B (= black) or W (= white), and that only colorings are allowed

where no two vertices connected by an edge have the color B. Then the set of allowed colorings is

$$S = \Big\{ s = (s_v)_{v \in V} \in \{B, W\}^V \colon (s_{v_1}, s_{v_2}) \neq (B, B) \ \forall\, (v_1, v_2) \in E \Big\}.$$

Suppose that we want to simulate the distribution π on S given by $\pi(s) = \lambda^{n(s)}/Z_\lambda$, $s \in S$, where $\lambda \in (0, \infty)$, $n(s) = \sum_{v \in V} 1_{\{s_v = B\}}$, and Z_λ is the normalizing constant.

(a) How can the simulation be done with the help of the Propp-Wilson algorithm? Give a brief sketch of the main ingredients in this algorithm.

(b) Does the sandwiching technique apply to this model?

(4) Given are a finite set S and a function $f \colon S \to \mathbb{R}$ that assumes a unique minimum on S.

(a) Give the formula for the Boltzmann distribution $\pi_{f,T}$ on S, with energy function f and temperature $T \in (0, \infty)$.

(b) Describe the simulated annealing algorithm to find $\min_{s \in S} f(s)$.

(c) What are the restrictions on the choice of T so that the algorithm works effectively? Give an intuitive explanation.

(5) The Ehrenfest urn model is defined as follows. Fix $N \in \mathbb{N}$. Consider two urns containing a total of N balls, labelled $1, \ldots, N$. At each unit of time draw a label at random and move the ball carrying that label to the other urn. Let X_n denote the number of balls in the first urn at time $n \in \mathbb{N}_0$.

(a) Explain why $X = (X_n)_{n \in \mathbb{N}_0}$ is a Markov chain.

(b) Compute the transition probabilities of X.

(c) Is X irreducible?

(d) Is X aperiodic?

(e) Compute the stationary distribution π of X (hint: X is reversible).

(f) Give an intuitive explanation why π is "sharply peaked" around $\frac{1}{2}N$ when N is large.