Examination for the course on

Probability

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Tuesday 12 January 2010, 10:00–13:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with an explanation. A simple "yes" or "no" is not enough.
- The use of notes and/or textbooks is not allowed.
- There are 15 questions, each weighting 5, 10 or 15 points. The weights are indicated in boldface. The total number of points is 135. *Final score*: 1 for free, 1 added per 15 points; ≥ 68 points is sufficient, ≤ 67 points is insufficient.

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(1) **[5]** Let U and V be $\mathbb{R}^2$-valued random variables with probability density functions

$$f_U(x_1, x_2) = e^{-x_1-x_2}\, 1_{[0,\infty)^2}, \qquad f_V(x_1, x_2) = \tfrac{1}{4}\, e^{-|x_1|-|x_2|}.$$

Can U and V be coupled such that they are component-wise ordered?

(2) For which values of $\mu_1, \mu_2 \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in [0, \infty)$ can two $\mathbb{R}$-valued random variables with normal distributions $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$ be coupled such that they are ordered?

(a) **[3]** $\mu_1 = \mu_2$, $\sigma_1 \neq \sigma_2$.

(b) **[3]** $\mu_1 \neq \mu_2$, $\sigma_1 = \sigma_2$.

(c) **[4]** $\mu_1 \neq \mu_2$, $\sigma_1 \neq \sigma_2$. (*question is tricky*)

(3) (a) **[3]** Formulate the standard convergence theorem for a (time-homogeneous) finite-state Markov chain.

(b) **[7]** Sketch the proof of this theorem with the help of coupling.

(c) **[5]** State the standard coupling inequality for two copies $X = (X_n)_{n\in\mathbb{N}_0}$ and $Y = (Y_n)_{n\in\mathbb{N}_0}$ of the same finite-state Markov chain with different initial distributions μ_X and μ_Y.

(4) (a) **[4]** Describe a successful coupling of two copies of simple random walk on $\mathbb{Z}$ starting from $x, y \in \mathbb{Z}$ with $x - y \in 2\mathbb{Z}$.

(b) **[4]** How can this coupling be extended to $\mathbb{Z}^d$ with $d \geq 2$?

(c) **[2]** What is known about the rate of coupling?

(5) Let $(Y_m)_{m=1}^n$ be i.i.d. random variables with

$$P(Y_1 = 1) = p, \quad P(Y_1 = 0) = 1 - p, \qquad p \in [0, 1].$$

Let $X = \sum_{m=1}^n Y_m$ and $\lambda = np$.

(a) **[8]** Prove with the help of the Chen-Stein coupling method that

$$\|P(X \in \cdot\,) - p_\lambda(\cdot)\|_{\text{tv}} \leq 2p\lambda,$$

where p_λ is the Poisson distribution with mean λ.

(b) **[2]** What does this inequality imply when $n \to \infty$ and $p \downarrow 0$ such that $\lambda = np$ is fixed?

(6) Let P, P' be two probability measures on a countable space E with a partial order $\preceq$. Let K, K' be two Markov transition kernels on $E \times E$.

(a) **[3]** When does P' stochastically dominate P?

(b) **[2]** Give an example for $E = \{0,1\}^2$.

(c) **[3]** When does K' stochastically dominate K?

(d) **[2]** Give an example for $E = \{0,1\}^2$.

(7) (a) **[6]** Formulate the Fortuin-Kasteleyn-Ginibre correlation inequality for monotone functions on partially ordered sets.

(b) **[4]** Use this inequality to show that

$$P_p(a \leftrightarrow b,\, c \leftrightarrow d) \geq P_p(a \leftrightarrow b)\, P_p(c \leftrightarrow d) \qquad \forall\, a,b,c,d \in \mathbb{Z}^2,\ p \in [0,1],$$

where P_p is the probability law of ordinary site percolation on $\mathbb{Z}^2$, and $a \leftrightarrow b$ means that a and b are connected by an open lattice path.

(8) (a) **[2]** Give the definition of a random shuffle of a deck of N cards.

(b) **[3]** When is a sequence of times $(t_N)_{N\in\mathbb{N}}$ called a sequence of threshold times for a given random shuffle?

(9) (a) **[3]** What is the "top to random" shuffle?

(b) **[4]** How does t_N grow with N for this shuffle? Give a heuristic explanation why.

(c) **[3]** What information does the growth of t_N provide?

(10) (a) **[3]** Describe how two ordinary percolation models on $\mathbb{Z}^d$, $d \geq 1$, with parameters $p < p'$ can be coupled such that their sets of open bonds are ordered?

(b) **[5]** Explain why this coupling implies that the percolation transition occurs at a unique critical threshold p_c.

(c) **[2]** Does the coupling provide any information on the value of p_c?

(11) (a) **[2]** Give the definition of invasion bond percolation on $\mathbb{Z}^d$, $d \geq 1$.

(b) **[3]** What is known about the sequence of weights of the bonds that are successively invaded? (*question is tricky*)

(12) Let $X = (X_t)_{t\geq 0}$ be a spin-flip system on $\mathbb{Z}^d$, $d \geq 1$, with local flip rates

$$c(x,\eta), \qquad x \in \mathbb{Z}^d,\ \eta \in \{-1,1\}^{\mathbb{Z}^d}.$$

(a) **[5]** What inequalities must these rates satisfy in order for X to be attractive?

(b) **[5]** When is an attractive X ergodic, i.e., $\lim_{t\to\infty} \mu P_t = \nu$ weakly for all initial distributions μ, with $P = (P_t)_{t\geq 0}$ the semigroup of X and ν the same invariant distribution for all μ?

(13) **[5]** Give three examples of attractive spin-flip systems by specifying their local flip rates.

(14) (a) **[5]** Describe the two phases of the Contact Process on $\mathbb{Z}^d$, $d \geq 1$.

(b) **[5]** Prove that the critical threshold λ_d satisfies $d\lambda_d \geq 1$. (*question is tricky*)

(15) (a) **[2]** What is a Brownian motion on $\mathbb{R}$?

(b) **[3]** How does this process arise as a scaling limit of simple random walk on $\mathbb{Z}$?

(c) **[3]** Give the definition of an Itô-diffusion on $\mathbb{R}$.

(d) **[2]** Can two regular recurrent Itô-diffusions on $\mathbb{R}$ be successfully coupled?