Probability: Coupling Theory

Teacher: F. den Hollander
Written examination: Monday 10 January 2011, 14:00–17:00.

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation. Formulas alone are not enough. Formulate your answers clearly and carefully.
- The use of textbooks, lecture notes or handwritten notes is not allowed.
- The questions below are weighted as follows: (1) 3, 10, 2; (2) 5, 5; (3) 6, 3, 6; (4) 3, 3, 3, 6; (5) 3, 10, 2; (6) 3, 3, 5, 4; (7) 3, 3, 3, 3, 3. *Total:* 100. Pass: ≥ 55; no pass: ≤ 54.

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(1) (a) Give the definition of a coupling of two probability measures $\mathbb{P}$ and $\mathbb{P}'$ on a measurable space $(E, \mathcal{E})$.

(b) State the standard coupling inequality and give its proof.

(c) What is a maximal coupling?

(2) (a) Let U, V be two random variables on $\mathbb{R}$ with probability density functions

$$f_U(x) = 1_{[0,1]}(x), \qquad f_V(x) = \tfrac{1}{2}\, 1_{[0,2]}(x), \qquad x \in \mathbb{R},$$

where 1_S is the indicator function of the set S. Give a coupling of U and V such that they are ordered.

(b) You have two coins, one with success probability $\frac{1}{4}$ and one with success probability 1. You are allowed to throw each of them as often as you want. How do you couple the two coins such that they are ordered?

(3) (a) Describe the Ornstein coupling for two simple random walks on $\mathbb{Z}^d$.

(b) What is a harmonic function on $\mathbb{Z}^d$?

(c) Prove that bounded harmonic functions on $\mathbb{Z}^d$ are constant.

(4) (a) Give the definition of a random card shuffle.

(b) Give the definition of a sequence of threshold times for random card shuffles.

(c) What is a strong uniform time for random card shuffles?

(d) Exhibit an explicit construction of a strong uniform time for the top-to-random shuffle and compute its expectation.

(5) (a) Formulate the standard convergence theorem for a Markov chain on a countable state space.

(b) Prove this theorem with the help of coupling when the state space is finite.

(c) What rate of convergence holds for a finite state space?

(6) (a) What is a partial ordering?

(b) Give two examples of partial orderings on $\mathbb{R}^2$.

(c) What is the standard partial ordering on the space of probability measures on $\mathbb{R}$?

(d) Formulate the Holley inequality and explain why this gives an explicit criterion for partial ordering of probability measures on the space of subsets of a finite set.

(7) (a) Give the definition of a spin-flip system.

(b) When is such a system attractive?

(c) What important property do attractive spin-flip systems have?

(d) Give the transition rates of the contact process on $\mathbb{Z}^d$.

(e) Explain in what sense the contact process has a phase transition.