Re-examination for the course on

Random Walks

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Tuesday 8 July 2014, 10:00–13:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with an explanation.
- The use of notes and/or diktaat is not allowed.
- There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) **[5]** Consider simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$ with length $N \in \mathbb{N}$ starting at 0. Which of the following two times is a stopping time? Prove your answer!

$$T_1 = \lfloor \tfrac{1}{2} N \rfloor, \qquad T_2 = \max\{0 \leq k \leq N \colon S_k = 0\}.$$

(2) **[5]** Consider a random walk on the square lattice $\mathbb{Z}^2$ with "diagonal jumps", i.e., the jump probabilities are:

$$P(X_1 = x) = \begin{cases} \frac{1}{4}, & \text{if } x \in \{(0,1),(1,1),(0,-1),(-1,-1)\}, \\ 0, & \text{otherwise.} \end{cases}$$

Compute the covariance matrix $(\mathrm{Cov}(X_1^{(i)}, X_1^{(j)})_{i,j=1,2}$, where $X_1^{(i)}$ is the i-th component of X_1, and state the central limit theorem for the partial sums $S_n = \sum_{i=1}^n X_1$, $n \in \mathbb{N}$.

(3) **[10]** Use the reflection principle to compute the probability $P(S_m \leq 2 \text{ for } 0 \leq m \leq 5)$ for a simple random walk $(S_n)_{n\in\mathbb{N}_0}$ on $\mathbb{Z}$. In other words: What is the probability that a simple random walk does not exceed the value 2 during its first 5 steps?

(4) **[5]** Compute the effective resistance between 0 and 2 of the graph with vertex set $\{0, 1, 2\}$ and edge set as follows: 2 edges between vertices 0 and 1; 4 edges between vertices 1 and 2; 1 edge between vertices 0 and 2.

(5) Given is a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with conductances $C_{xy} \in (0, \infty)$ assigned to each of the edges $xy \in \mathcal{E}$ such that $C_{xy} = C_{yx}$. Pick any two vertices $a, b \in V$ and place a battery across them.

(a) **[5]** Formulate the Dirichlet Principle.

(b) **[5]** Formulate the Thomson Principle.

(c) **[10]** Express the effective resistance of $\mathcal{G}$ in terms of both these variational principles.

(6) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the rooted tree of degree 6 starting at the root.

(a) **[5]** Give a formula for c_n.

(b) **[5]** Show that c_n is at least as large as the number of self-avoiding walks of length n on $\mathbb{Z}^3$.

(c) **[5]** What is c_n on the unrooted tree?

(7) Consider the wetted polymer (= pinned polymer with hard wall constraint) of length $n \in \mathbb{N}$ with interaction strength $\zeta \in \mathbb{R}$.

(a) **[5]** Give the definition of the free energy $\zeta \mapsto f^+(\zeta)$.

(b) **[10]** Explain why $f^+(\zeta) = f(\zeta - \zeta_c^+)$ with $\zeta_c^+ = \log 2$, where $f(\zeta)$ is the free energy of the pinned polymer.

(c) **[Bonus]** Explain how this formula is derived.

(8) **[10]** Let $(W_t)_{t\geq 0}$ be a standard Brownian motion. Put $X_t = W_{2t} - W_t$. Is $(X_t)_{t\geq 0}$ a Brownian motion?

(9) **[5]** Let $(W(t))_{t\geq 0}$ be standard Brownian motion. The Ornstein-Uhlenbeck process $(U_t)_{t\geq 0}$ is defined by

$$U(t) = e^{-t} W(e^{2t}).$$

Provide an explicit expression for $\mathrm{cov}(U(s), U(t))$ for $s, t \geq 0$.

(10) Suppose that the current price of a stock is $S_0 = 50$ euro, and that at the end of a period of time its price must be either $S_1 = 25$ or $S_1 = 100$ euro. A call option on the stock is available with a striking price of $K = 50$ euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) **[5]** Compute the arbitrage-free price of the call option.

(b) **[5]** Suppose that you can buy such an option on the market for €25. What should you do?