Examination for the course on

Random Walks

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Monday 23 June 2014, 14:00–17:00

- Write your name and student identification number on each piece of paper you hand in.
- All answers must come with a full explanation.
- The use of notes or diktaat is not allowed.
- There are 10 questions. The total number of points is 100 (per question indicated in boldface). A score of ≥ 55 points is sufficient.

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(1) [**5**] Consider simple random walk on $\mathbb{Z}$. Given two stopping times T_1 and T_2, is the minimum $T = \min\{T_1, T_2\}$ again a stopping time? Prove your answer!

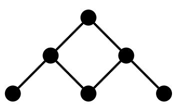
(2) [**10**] Consider a random walk on the square lattice $\mathbb{Z}^2$ with "diagonal jumps", i.e., the jump probabilities are

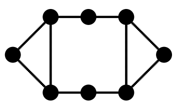
$$P(X_1 = x) = \begin{cases} \frac{1}{4}, & \text{if } x \in \big\{(1,1), (-1,1), (1,-1), (-1,-1)\big\}, \\ 0, & \text{otherwise.} \end{cases}$$

Compute the covariance matrix $(\mathrm{Cov}(X_1^{(i)}, X_1^{(j)}))_{i,j=1,2}$, where $X_1^{(i)}$ denotes the i-th component of X_1. State the central limit theorem for the partial sums $S_n = \sum_{i=1}^n X_i$, $n \in \mathbb{N}$.

(3) [**10**] In the game *double or loose* you bet 1 euro per round. The bet is either doubled or lost, both with 50% chance. The strategy of a gambler is to continue playing until either a total of 10 euro is won (the gambler leaves the game happy) or four times in a row a loss is suffered (the gambler leaves frustrated). Is the expected payoff of this strategy positive, zero or negative? Prove your answer!

(4) Compute the effective resistance between a and b of the following two networks of unit resistances:

(a) [**5**] a  b

(b) [**5**] a  b

(5) Given is a finite connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and two vertices $a, b \in \mathcal{V}$.

(a) **[5]** What is a unit potential from a to b?

(b) **[5]** What is a unit flow from a to b?

(6) Let c_n denote the number of self-avoiding walks of length $n \in \mathbb{N}$ on the triangular lattice (i.e., the two-dimensional lattice where unit triangles are packed together).

(a) **[5]** What inequality is satisfied by $n \mapsto c_n$, and why does this inequality imply the existence of the so-called connective constant μ?

(b) **[5]** Compute c_3.

(c) **[5]** Show that $2^n \leq c_n \leq 6 \times 5^{n-1}$, $n \in \mathbb{N}$ and use this to obtain bounds on μ.

(7) (a) **[5]** Give a description of the path space $\mathcal{W}_n$ of the pinned polymer of length $n \in \mathbb{N}$. The path measure with interaction strength $\zeta \in \mathbb{R}$ is

$$\bar{P}_n^{\zeta}(w) = \frac{1}{Z_n^{\zeta}} e^{\zeta \sum_{i=1}^{n} 1_{\{w_i=0\}}} \bar{P}_n(w), \qquad w \in \mathcal{W}_n.$$

Explain what this path measure models.

(b) **[5]** Give the definition of the free energy $\zeta \mapsto f(\zeta)$, and explain why this quantity is capable of detecting a phase transition.

(c) **[5]** Give the formula that expresses $f(\zeta)$ in terms of the generating function for the probability distribution of the first return time to the origin of one-dimensional simple random walk.

(d) **[Bonus]** Explain how this formula is derived.

(8) Let $(W_t)_{t\geq 0}$ and $(\tilde{W}_t)_{t\geq 0}$ be independent standard Brownian motions. Put $X_t = \alpha W_t + \beta \tilde{W}_t$, where and $\alpha, \beta \in \mathbb{R}$ are such that $\alpha^2 + \beta^2 = 1$.

(a) **[5]** Show that $(X_t)_{t\geq 0}$ is standard Brownian motion as well.

(b) **[5]** Compute the correlation coefficient $\rho(X_t, W_t)$.

(9) **[5]** Let $(W(t))_{t\geq 0}$ be standard Brownian motion, and let $0 < t_1 < t_1 + t_2 < t_1 + t_2 + t_3$. Compute

$$E\big[W(t_1)W(t_1+t_2)W(t_1+t_2+t_3)\big].$$

(10) Suppose that the current price of a stock is $S_0 = 50$ euro, and that at the end of a period of time its price must be either $S_1 = 25$ or $S_1 = 100$ euro. A call option on the stock is available with a striking price of $K = 50$ euro, expiring at the end of the period. It is also possible to borrow and lend at a 25% rate of interest.

(a) **[5]** Compute the arbitrage-free price of the call option.

(b) **[5]** Suppose that you can buy such an option on the market for €15. What should you do?