Exam for Topics in Geometry 2

Let X be a normal curve over a field k. For a closed point P on X and a nonzero function $f \in K(X)$ we define the *order* $\mathrm{ord}_P(f)$ to be the unique integer n for which we can write $f = u\pi^n$ where u is a unit in $\mathcal{O}_{X,P}$ and π is a uniformiser in $\mathcal{O}_{X,P}$, which exists because X is normal. Here we identify $K(X)$ with the field of fractions of $\mathcal{O}_{X,P}$ in the natural way.

We define a *divisor* D on X to be a finite formal sum

$$D = \sum_i n_i P_i$$

with $n_i \in \mathbb{Z}$ and the P_i distinct closed points of X. For P a closed point of X, we set

$$\mathrm{ord}_P(D) = \begin{cases} n_i & \text{if } P = P_i \text{ for some } i, \\ 0 & \text{otherwise.} \end{cases}$$

To a divisor $D = \sum n_i P_i$ we associate a sheaf $\mathcal{O}_X(D)$ on X which is a subsheaf of the constant sheaf $K(X)_X$ and whose sections over the open set U are given by

$$\mathcal{O}_X(D)(U) = \{0\} \cup \{f \in K(X)^* : \mathrm{ord}_P(f) + \mathrm{ord}_P(D) \geq 0 \quad \text{for all closed points } P \in U\}.$$

For example, if $D = 0$ then $\mathcal{O}_X(D)$ is the usual structure sheaf $\mathcal{O}_X$.

Exercise 1. For X a projective normal curve over k, show that $\Gamma(X, \mathcal{O}_X(D))$ is finite dimensional as a vector space over k for all D. You may use here that $k \to \Gamma(X, \mathcal{O}_X)$ is a finite extension of fields.

From now on we suppose that $\mathrm{char}(k) \neq 2$ and that k is algebraically closed. Let $f \in k[x]$ be a polynomial of degree at least 5 without repeated roots. Let X be the normalisation of $\mathbb{P}^1_k$ in the extension of $k(x)$ defined by $y^2 = f$. We will say that "X is the hyperelliptic curve defined by $y^2 = f$."

Exercise 2. Let X be the hyperelliptic curve defined by $y^2 = f$ and let P be a point $(\alpha, 0)$ on X where α is a root of f. Show that there exists a coordinate transformation such that in the new coordinates X is the hyperelliptic curve defined by $v^2 = g(u)$ where $g \in k[u]$ has odd degree $2d + 1$ and such that P is the point at infinity, i.e. P is not in the affine u, v-piece.

Exercise 3. With the same assumptions and notations as in exercise 2, compute $\mathrm{ord}_P(a(u) + b(u)v)$ in terms of the polynomials $a(u)$ and $b(u)$.

Exercise 4. Again using the assumptions and notations from exercise 2, compute a basis for the k-vector space $\Gamma(X, \mathcal{O}_X(nP))$ for any $n \in \mathbb{Z}$.

Exercise 5. Let X be the hyperelliptic curve defined by $y^2 = f(x)$. Now, let $P = (x, y)$ be a point in the affine x, y-piece of X with $y \neq 0$. Show that there is a coordinate transformation such that in the new coordinates X is the hyperelliptic curve defined by $v^2 = g(u)$ where $g(u) \in k[u]$ is a monic polynomial of even degree $2d$ and such that P is one of the two points at infinity. In the sequel, we denote the other point at infinity by Q.

Exercise 6. Using the assumptions and notations of exercise 5, determine a basis for the k-vector space $\Gamma(X, \mathcal{O}_X(nP + nQ))$ for any $n \in \mathbb{Z}$.

Exercise 7. Again, notations and assumptions are as in exercise 5. Let w, t be the coordinates of the other affine piece, i.e.

$$w = \frac{1}{u}, \quad t = \frac{v}{u^d}, \quad t^2 = w^{2d} g\left(\frac{1}{w}\right).$$

Consider the completion $\widehat{\mathcal{O}}_{X,Q}$ of $\mathcal{O}_{X,Q}$, that is

$$\widehat{\mathcal{O}}_{X,Q} = \varprojlim \mathcal{O}_{X,Q}/m_P^n.$$

Prove that $\widehat{\mathcal{O}}_{X,Q} \cong k[[w]]$ and calculate a few terms of t as a power series in w.

Exercise 8. Notations and assumptions are as in exercises 5 and 7. Suppose that n is a nonnegative integer. Consider the natural map

$$\phi : \Gamma(X, \mathcal{O}_X(nP + nQ)) \to k((w)),$$

where the Laurent series ring $k((w))$ is identified with the field of fractions of $\widehat{\mathcal{O}}_{X,Q}$. Show that

$$\Gamma(X, \mathcal{O}_X(nP)) = \phi^{-1}\left(k[[w]]\right)$$

and use this to compute $\dim_k \Gamma(X, \mathcal{O}_X(nP))$.