How the Octonions Form a Division Algebra

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During the talk it will be shown that $\mathbb{O}$, that is the *Octonions*, form a *division algebra*.

Definition. *An algebra A is said to be a* division algebra *if for any $a \in A, a \neq 0$, the left multiplication l_a and right multiplication r_a*

$$l_a, r_a : A \longrightarrow A$$

given for any $z \in A$ by, resp.,

$$z \longmapsto az$$

$$z \longmapsto za$$

are bijective.

In the finite dimensional case, this is equivalent to there existing no zero divisors in A.
We then construct $\mathbb{O}$ starting off from the quaternions, $\mathbb{H}$, through the Cayley-Dickson construction. Since the conjugation on $\mathbb{H}$ satisfies some nice properties, it then follows from direct computation that $\mathbb{O}$ is *alternative*.

Definition. *Let A be an algebra. A is said to be* alternative *if*

$$\begin{aligned} (aa)b &= a(ab), \\ (ab)a &= a(ba), \\ (ba)a &= b(aa). \end{aligned}$$

We can now prove that $\mathbb{O}$ is a division algebra using Artin's Lemma in conjunction with the equally nice properties of conjugation on $\mathbb{O}$.

Artin's Lemma *An algebra A is alternative iff every subalgebra generated by two of its elements is associative.*