SUBJECT: DISTRIBUTIONS

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Introduction

Distributions (also known as generalized functions) are objects which generalize functions. They extend the concept of derivative to all integrable functions and beyond, and are used to formulate generalized solutions of partial differential equations. They are important in physics and engineering where many non-continuous problems naturally lead to differential equations whose solutions are distributions, such as the Dirac delta distribution.

"Generalized functions" were introduced by Sergei Sobolev in 1935. They were independently introduced in late 1940s by Laurent Schwartz, who developed a comprehensive theory of distributions and received the Field medal for that.

Presentation

Let us give few details about distributions. Let U be an open subset of $\mathbf{R}^N$ for a given $N \geqslant 1$. The space of infinitely differentiable functions from U to $\mathbf{C}$ with compact support[1] is denoted $\mathcal{D}(U)$. This space is naturally endowed with a structure of a topological space and its dual i.e. the space of all continuous linear functionals $\mathcal{D}(U) \to \mathbf{C}$ is denoted $\mathcal{D}'(U)$. It is the space of distributions. Given a distribution $T \in \mathcal{D}'(U)$ and a a function $\phi \in \mathcal{D}(U)$, we will write

$$< T, \phi >= T(\phi).$$

Any locally integrable function f on U is naturally a distribution:

$$\begin{array}{rcl} \mathcal{D}(U) & \to & \mathbf{C} \\ \phi & \mapsto & \int_U f\phi dx. \end{array}$$

For instance any continuous function is a distribution. Let us give a non-trivial example of a distribution. We define the Dirac distribution[2]

[1] A function $f\colon U \to \mathbf{C}$ has compact support if there exists a compact subset $K \subset U$ such that $f(x) = 0$ for $x \in U \backslash K$.

[2] often called *Dirac function* although it is not a function.

as:

$$\begin{array}{rrcl} \delta_0: & \mathcal{D}(\mathbf{R}^N) & \rightarrow & \mathbf{C} \\ & \phi & \mapsto & \phi(0) \end{array}$$

If T is a distribution on U, we define its partial derivative with respect to x_i, $\frac{\partial T}{\partial x_i}$ as:

$$< \frac{\partial T}{\partial x_i}, \phi >= - < T, \frac{\partial \phi}{\partial x_i} >, \forall \phi \in \mathcal{D}(U).$$

This implies that in the space of distributions, a function that is not continous has a derivative! Let us give an example. The *Heaviside step function* is defined as:

$$H(x) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0. \end{cases}$$

It is not a continuous function, and hence has no derivative. For any $\phi \in \mathcal{D}(U)$, we have:

$$\begin{aligned} < H', \phi > &= - < H, \phi' >, \\ &= - \int_{\mathbf{R}^+} \phi'(x) dx, \\ &= \phi(0). \end{aligned}$$

Hence in the space of distributions the Heaviside function has a derivative, and it is the Dirac function.

We have the notion of Fourier transformation for distributions (We need for that to introduce a new space of distributions, tempered distributions).

Distribution may help a lot to prove the existence of solution of certain differential equations. A method can consist of looking for solutions in a space of distributions and then prove that these distributions are actually functions. As an example, consider the wave equation: in the space $\mathbf{R}^4 = \mathbf{R} \times \mathbf{R}^3$, the "time" times "the 3-dimensional space", we introduce the following operator

$$\Box E = \frac{\partial^2 E}{\partial t^2} - \sum_{i=1...3} \frac{\partial^2 E}{\partial x_i^2}.$$

We are looking for an elementary solution of the wave equation, ie. a distribution E satisfying:

$$\Box E = \delta(t, x).$$

Once this E is found, we can solve the following Cauchy problem:

$$\begin{cases} \Box u = 0, \\ u(-,0) = f_0, \\ \frac{\partial u}{\partial t}(-,0) = f_1 \end{cases}$$

with f_0 and f_1 $\mathcal{C}^2$ functions. The solution found u will be actually $\mathcal{C}^2$.

What can be done in a Bachelor thesis

There are different possibilities. The first task is to understand the theory of distributions. A background in topology, differential calculus, and Lebesgue integration is required. Thereafter, I can propose to solve the wave equation as written above. Depending on the time and on the student's wishes, we can stress out the "topological part" of the subjet, study the "Sobolev spaces" or go further in "differential equations".

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