Hilbert modular forms and local Langlands.

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Abstract

The automorphic representation associated to an eigenform will be described, as well as its local factors at the places of F. The statement that the local factor of the automorphic representation determines the local Galois representation (after F-semisimplification) will be explained, and an explicit description will be given in at least the case of a principal series local representation.

The main reference used in this text is Saito's preprint, of which we try to follow the notation. We have $\mathbb{Q} \to F$ totally real, $n := \dim_{\mathbb{Q}}(F)$, $I := \mathrm{Hom}(F, \mathbb{R})$ and we fix a bijection $\{1, \ldots, n\} \tilde{\longrightarrow} I$.

1 Hilbert modular forms, before adèles

We begin by recalling some parts of the 2 hours by van der Geer in this seminar. We have $\mathbb{H}$ the complex upper half plane, with its action by $\mathrm{SL}_2(\mathbb{R})$. Then on $\mathbb{H}^n$ we have an action by $\mathrm{SL}_2(\mathbb{R})^n$, and $\mathrm{SL}_2(O_F)$ embeds as a discrete subgroup into $SL_2(\mathbb{R})^n$ using all n disctinct embeddings of F into $\mathbb{R}$. Hence any congruence subgrioup Γ of $\mathrm{SL}_2(O_F)$ acts on $\mathbb{H}^n$. Let $k = (k_1, \ldots, k_n)$ be in $\mathbb{Z}^n$. Then a *Hilbert modular form* on Γ, of weight k, is a holomorphic function:

$$f \colon \mathbb{H}^n \longrightarrow \mathbb{C},$$

such that, for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ and all z in $\mathbb{H}^n$ we have:

$$f(\gamma z) = (cz + d)^k f(z),$$

where $(cz + d)^k = (c_1 z_1 + d_1)^{k_1} \cdots (c_n z_n + d_n)^{k_n}$. If $F = \mathbb{Q}$, one asks moreover f to be "holomorphic at the cusps". Equivalently: f gives a section of some holomorphic line bundle on a compactification $\overline{\Gamma \backslash \mathbb{H}^n}$ (and compactifying is not necessary for $n > 1$).

2 Adèlic description

We let $G := \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_{2,F}$, the $\mathbb{Q}$-group scheme obtained by restricting GL_2 from F to $\mathbb{Q}$. This means that for every $\mathbb{Q}$-algebra A we have $G(A) = \mathrm{GL}_2(A \otimes_{\mathbb{Q}} F)$. In particular, we have:

$$G(\mathbb{R}) = GL_2(\mathbb{R} \otimes F) = \mathrm{GL}_2(\mathbb{R}^n) = \mathrm{GL}_2(\mathbb{R})^n.$$

We let $X := \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$, the "double half plane", with its natural $\mathrm{GL}_2(\mathbb{R})$-action. Then we have the space of *cuspidal* weight k forms (over $\mathbb{C}$):

$$S_{\mathbb{C}}^{(k)} = \{f\colon X^n \times G(\mathbb{A}_{\mathrm{fin}}) \longrightarrow \mathbb{C} \mid \text{certain conditions}\},$$

where $\mathbb{A}_{\mathrm{fin}}$ denotes the topological $\mathbb{Q}$-algebra of finite adèles of $\mathbb{Q}$; for the precise conditions the reader is referred to the notes by van der Geer, or the preprint of Saito.

The space $S_{\mathbb{C}}^{(k)}$ has a natural action by $G(\mathbb{A}_{\mathrm{fin}})$ given by right translations. One of the conditions on elements f of $S_{\mathbb{C}}^{(k)}$ is that the stabiliser of f in $G(\mathbb{A}_{\mathrm{fin}})$ is open. Representations with this property are called *smooth*. It follows that:

$$S_{\mathbb{C}}^{(k)} = \bigcup_K (S_{\mathbb{C}}^{(k)})^K,$$

where K runs over the open compact subgroups of $G(\mathbb{A}_{\mathrm{fin}})$. Each space $(S_{\mathbb{C}}^{(k)})^K$ can also be seen as:

$$(S_{\mathbb{C}}^{(k)})^K = \mathrm{H}^0(\overline{G(\mathbb{Q})\backslash(X^n \times G(\mathbb{A}_{\mathrm{fin}})/K)}, \mathcal{L}^{(k)}(-\mathrm{cusps})),$$

where $\mathcal{L}^{(k)}$ is a holomorphic line bundle and $\mathcal{L}^{(k)}(-\mathrm{cusps})$ its coherent subsheaf of sections vanishing at the cusps. In particular, the spaces $(S_{\mathbb{C}}^{(k)})^K$ are finite dimensional.

Sometimes it is a good idea to treat the Archimedean places of $\mathbb{Q}$ in the same way as the finite ones. Then one writes:

$$S_{\mathbb{C}}^{(k)} = \{f\colon G(\mathbb{A}) \longrightarrow \mathbb{C} \mid \text{more conditions}\},$$

using the fact that $X^n = G(\mathbb{R})/K_\infty$, with $K_\infty = (\mathbb{C}^\times)^n$. The conditions are then that the elements should be invariant under right translation by K_∞, and that the induced function on $X^n \times G(\mathbb{A}_{\mathrm{fin}})$ satisfies the conditions above.

3 Hecke operators

A general question is the following: what operators does the $G(\mathbb{A}_{\mathrm{fin}})$-action on $S_{\mathbb{C}}^{(k)}$ induce on the spaces $(S_{\mathbb{C}}^{(k)})^K$?

Let us first discuss what happens for finite groups. Let G be a finite group, and H and K subgroups. Then we have the functor taking H-invariants:

$$(\cdot)^H \colon \mathbb{Z}[G]-\mathrm{Mod} \longrightarrow \mathbb{Z}-\mathrm{Mod}, \quad V \mapsto V^H = \mathrm{Hom}_H(1, V_H) = \mathrm{Hom}_G(1_H^G, V).$$

The identities show that $(\cdot)^H$ is represented by the induced representation $1_H^G = \mathbb{Z}[G/H]$, the free $\mathbb{Z}$-module on the set G/H. By Yoneda's lemma, we then have:

$$\mathrm{Hom}((\cdot)^H, (\cdot)^K) = \mathrm{Hom}_G(1_K^G, 1_H^G) = (1_H^G)^K = \mathbb{Z}[K\backslash G/H].$$

Concretely, the map from V^H to V^K given by an element KgH of $K\backslash G/H$ is the map $v \mapsto \sum_i g_i v$, where KgH is the disjoint union of the $g_i H$.

In our case, of compact open subgroups K and H of $G(\mathbb{A}_{\mathrm{fin}})$ and smooth representations, this theory works in exactly the same way; indeed, each KgH decomposes as a finite union of $g_i H$, as KgH is open and compact.

Let us state a few simple basic properties of compact open subgroups of $G(\mathbb{A}_{\mathrm{fin}})$.

Up to conjugacy, there is exactly one maximal open compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$: $K_0 = \prod_v \mathrm{GL}_2(O_{F_v})$, where v runs through the set of finite places of F. (The proof of this uses the facts that K_0 is the stabiliser of the standard lattice $\hat{O}^2$ of $\mathbb{A}_{\mathrm{fin}}^2$, that each compact subgroup of $G(\mathbb{A}_{\mathrm{fin}})$ stabilises some lattice, and that $G(\mathbb{A}_{\mathrm{fin}})$ acts transitively on the set of all lattices). If K and K' are open compact, then $K \cap K'$ is open in each of K and K', and hence of finite index. Hence:

$$K = \prod_{\text{a.a. } v} \mathrm{GL}_2(O_{F_v}) \times R$$

where the first product is over almost all v and where R is open compact in the product of the remaining $\mathrm{GL}_2(O_{F_v})$.

It follows that on $(S_{\mathbb{C}}^{(k)})^K$ we have, for the almost all v above, standard Hecke operators T_v and R_v, given by the double K-cosets of $\left(\begin{smallmatrix} t_v^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} t_v^{-1} & 0 \\ 0 & t_v^{-1} \end{smallmatrix}\right)$, where t_v is a uniformiser of O_{F_v}.

4 Representation theory

We just list some facts.

The representation $S_{\mathbb{C}}^{(k)}$ is completely reducible, i.e., the direct sum of irreducible representations, and each irreducible representation occurs at most once. The irreducible representations that occur are called *cuspidal automorphic representations* of weight k of $\mathrm{GL}_2(\mathbb{A}_{F,\mathrm{fin}})$. They are infinite dimensional.

Each irreducible representation V in $S_{\mathbb{C}}^{(k)}$ contains a certain well-defined one-dimensional subspace of the form $(S_{\mathbb{C}}^{(k)})^{K_1(n_f)}$ with the subgroup $K_1(n_f)$ defined as in van der Geer's lecture, with n_f a non-zero ideal of O_F, maximal for the property that $(S_{\mathbb{C}}^{(k)})^{K_1(n_f)}$ is non-zero. This subspace contains a unique element f whose first coefficient $c(1,f)$ equals 1; this f is called the (normalised) *newform* of V and n_f is called the level of f or also the conductor of V. The representation V is then simply the $\mathbb{C}$-span of the set of gf, with g in $G(\mathbb{A}_{\mathrm{fin}})$. We now write the decomposition of $S_{\mathbb{C}}^{(k)}$ as:

$$S_{\mathbb{C}}^{(k)} = \oplus_f \pi_f,$$

where the sum runs over the set of normalised newforms of weight k.

Each π_f itself can be decomposed as a so-called *restricted tensor product* over the finite places of F, corresponding to the fact that $\mathbb{A}_{F,\mathrm{fin}}$ is the restricted product of the F_v over the finite places v of F. Let us make this precise. For each finite set S of finite places of F we consider the subspace of $S_{\mathbb{C}}^{(k)}$ of elements invariant under the subgroup $K_1(n_f)^S := \prod_{w \notin S} K_1(n_f)_w$. This subspace is a representation of $\prod_{v\in S} \mathrm{GL}_2(F_v)$, which turns out to be irreducible, and is therefore given as a tensor product:

$$\otimes_{v\in S}\pi_{f,v}, \quad \pi_{f,v} = V^{K_1(n_f)^v}.$$

Then we have:

$$\pi_f = \otimes'_v \pi_{f,v},$$

where the notation $\otimes'_v$ means the direct limit of the finite tensor products, for the maps given by the inclusions into π_f as subspaces of invariants for the corresponding subgroups.

5 Analogy with GL_1

In this section F is not necessarily totally real, as class field theory works for arbitrary number fields. So we let

$$\chi\colon F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$$

be a continuous character, decomposed as $\chi = \prod_v \chi_v$, with the condition that on the infinite places χ_v is given by some algebraic formulas: on $F_v^\times = \mathbb{R}^\times > 0$ it should send t to t^n for some integer n, and for $F_v^\times = \mathbb{C}^\times$ it should send z to $z^n \overline{z}^m$ for some integers n and m. Such characters are called *algebraic Hecke characters* (unless I'm mistaken).

Now CFT (class field theory) gives a surjection:

$$F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F),$$

where $F \to F^{\mathrm{ab}}$ is the maximal abelian subextension of $F \to \overline{\mathbb{Q}}$. Under this map, the connected components of the $F_v^\times$ at the infinite places are sent to 1, hence χ cannot directly correspond to a representation of $\mathrm{Gal}(F^{\mathrm{ab}}/F)$.

The image of χ restricted to the finite adèles $\mathbb{A}_{F,\mathrm{fin}}$ is contained in a number field L, and for every finite place λ of L we get, by changing χ at the $v|l\infty$, a continuous character χ^λ of $F^\times\backslash\mathbb{A}_F^\times$ to $L_\lambda^\times$ that factors through $\mathrm{Gal}(F^{\mathrm{ab}}/F)$. The way to change χ at the $v|l\infty$ is (probably) to choose an isomorphism between $\overline{\mathbb{Q}}_l$ and $\mathbb{C}$, in order to get a bijection between the embeddings of F into $\overline{\mathbb{Q}}_l$ and into $\mathbb{C}$. For details the reader is referred to Serre's book "Abelian l-adic representations and elliptic curves".

6 Galois representations

Let f and π_f be as above. Let $L \subset \overline{\mathbb{Q}} \subset \mathbb{C}$ be the field of definition of the weight k: the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $\mathrm{Hom}(F, \overline{\mathbb{Q}})$, hence on $\mathbb{Z}[\mathrm{Hom}(F, \overline{\mathbb{Q}})]$ of which k is an element, so L, with its embedding into $\overline{\mathbb{Q}}$, corresponds to k.

Then $S_{\mathbb{C}}^{(k)}$ has a natural L-structure, i.e., there exists an L-vector space $S_L^{(k)}$, with an action of $G(\mathbb{A}_{\mathrm{fin}})$, such that:

$$S_{\mathbb{C}}^{(k)} = \mathbb{C} \otimes_L S_L^{(k)}.$$

In order to prove this, one uses the moduli interpretation of the Hilbert modular varieties, and shows that the "geometry giving rise to $S_{\mathbb{C}}^{(k)}$" is defined over L.

Let $L(f)$ be the field of definition of π_f (this is a finite extension of L, determined by Fourier coefficients, or by Hecke eigenvalues if one wants). With these definitions, we have the following theorem, due to "many people".

6.1 Theorem. *Let λ be a finite place of $L(f)$. Then there is a unique continuous representation*

$$\rho_{f,\lambda} \colon \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow \mathrm{GL}_2(L(f)_\lambda),$$

unramified at all v not dividing $n_f l$, and such that for such v the characteric polynomial of the Frobenius $\det(1 - T\rho_{f,\lambda}(\mathrm{Frob}_v))$ is given by a suitable formula in terms of the T_v and R_v-eigenvalues and the weight k of f.

Moreover, for all v not dividing l, the restriction $\rho_{f,\lambda,v}$ of $\rho_{f,\lambda}$ to $\mathrm{Gal}(\overline{F}_v/F_v)$ is, after F-semisimplification (this notion will be explained below), determined by $\pi_{f,v}$ via a suitably normalised local Langlands correspondence:

$$'\rho_{f,\lambda,v}^{\mathrm{F-s.s.}} = \check{\sigma}_h(\pi_{f,v}).$$

Moreover, if f can be realised in the cohomology of a Shimura curve, then the same is true for all v dividing l, but then the definition of $'\rho_{f,\lambda,v}^{\mathrm{F-s.s.}}$ involves Fontaine's D_{pst}-functor.

The very last statement of this theorem will be explained somehow in the lecture by Johan de Jong, later in the seminar. This is the main result of [1].

Let us explain the notion of F-semisimplicity. The representation $\rho_{f,\lambda,v}$ is called F-semisimple if for each character α such that $\alpha\otimes\rho_{f,\lambda,v}$ is unramified, $\alpha\otimes\rho_{f,\lambda,v}(\mathrm{Frob}_v)$ is semisimple. Conjecturally, all representations coming from cohomology of algebraic varieties are F-semisimple, but we only know it in the cases coming from H^0 (finite image) and H^1 (related to abelian varieties). In general, one has a functor that F-semisimplifies.

7 Some examples of the local theory: Galois side

Let λ and v be as above, with v not dividing l. Let:

$$\rho\colon \mathrm{Gal}(\overline{F}_v/F_v) \longrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$$

be continuous, and F-semisimple. Then one has the following classification.

1. ρ is decomposable: $\rho = \alpha \oplus \beta$ with α and β characters of $\mathrm{Gal}(\overline{F}_v/F_v)$.

2. ρ is reducible but indecomposable: $\rho \cong \left(\begin{smallmatrix} \alpha\chi_l & * \\ 0 & \alpha \end{smallmatrix}\right)$, where $\chi_l\colon \mathrm{Gal}(\overline{F}_v/F_v) \to \mathbb{Z}_l^\times$ is the l-adic cyclotomic character.

3. ρ is irreducible and one of the following is true:

   (a) ρ is the induced of a character of $\mathrm{Gal}(\overline{F}_v/K)$ with $F_v \to K$ a quadratic extension.

   (b) wild inertia acts irreducibly. This can only occur if v divides 2, and in this case the representation is called extraordinary. For $F_v = \mathbb{Q}_2$ these have been classified by André Weil in his article "exercices dyadiques".

This classification as stated is not so hard to prove. The special form of the two diagonal characters in part 2 comes from the structure of semi-direct product of the tame quotient of the local Galois group. For the last part, one uses that an irreducible representation over $\mathbb{C}$ of a finite p-group has dimension a power of p.

8 Some examples of the local theory: GL_2 side

On this side, we will only sketch the representations that correspond to the first two cases above on the Galois side. Not surprisingly, these are obtained in terms of characters of $F_v^\times$. Let λ be as above, and let χ_1 and χ_2 be continuous characters from $F_v^\times$ to $\mathbb{C}^\times$. Let B be the Borel subgroup of $\mathrm{GL}_2(F_v)$ consisting of the upper triangular matrices, and let (χ_1, χ_2) be the character of B given by $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \mapsto \chi_1(x)\chi_2(z)$. Then we let $V(\chi_1, \chi_2)$ be the representation of $\mathrm{GL}_2(F_v)$ given by right-translation on the space of functions: $f\colon \mathrm{GL}_2(F_v) \to \mathbb{C}$ whose stabiliser in $\mathrm{GL}_2(F_v)$ is open, and such that for all $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$ in B and all g in $\mathrm{GL}_2(F_v)$ we have:

$$f(\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)g) = \chi_1(x)\chi_2(z)f(g).$$

This representation $V(\chi_1, \chi_2)$ is called the (naive) parabolic induction of the representation (χ_1, χ_2), and is (at least) sometimes denoted $\mathrm{Ind}_B^{\mathrm{GL}_2(F_v)}(\chi_1, \chi_2)$. It satisfies the following adjoint property (Frobenius reciprocity) in the category of smooth representations of $\mathrm{GL}_2(F_v)$:

$$\mathrm{Hom}_{\mathrm{GL}_2(F_v)}(\pi, \mathrm{Ind}_B^{\mathrm{GL}_2(F_v)}(\chi_1, \chi_2)) = \mathrm{Hom}_B(\pi|_B, (\chi_1, \chi_2)).$$

With these notations one has the following (not so hard to prove) theorem.

8.1 Theorem. *1. If $\chi_1\chi_2^{-1} \notin \{1, |\cdot|^2\}$ then $V(\chi_1, \chi_2)$ is irreducible, called* principal series*. Moreover, $V(\chi_1, \chi_2) \cong V(\chi_1', \chi_2')$ if and only if $(\chi_1', \chi_2') = (\chi_1, \chi_2)$ or $(\chi_1', \chi_2') = (\chi_2|\cdot|, \chi_1|\cdot|^{-1})$.*

2. If $\chi_1 = \chi_2$ then we have a short exact sequence:

$$0 \longrightarrow \mathbb{C}\chi_1 \circ \det \longrightarrow V(\chi_1, \chi_2) \longrightarrow \chi_1 \otimes \mathrm{Sp} \longrightarrow 0,$$

where Sp is irreducible and called the special *representation of $\mathrm{GL}_2(F_v)$.*

3. If $\chi_1\chi_2^{-1} = |\cdot|^2$ then we have a short exact sequence:

$$0 \longrightarrow (\chi_1|\cdot|^{-1}) \otimes \mathrm{Sp} \longrightarrow V(\chi_1, \chi_2) \longrightarrow \chi_1|\cdot|^{-1} \longrightarrow 0.$$

To finish, let us show the following result.

8.2 Proposition. *Let χ_1 and χ_2 be continuous characters from $F_v^\times$ to $\mathbb{C}^\times$, let $V := V(\chi_1, \chi_2)$ and let $K := \mathrm{GL}_2(O_{F_v})$. Then $V^K \neq 0$ if and only if χ_1 and χ_2 are unramified. Suppose that $V^K \neq 0$, and let t be a uniformiser of O_{F_v}. Recall that we have Hecke operators $R = K\left(\begin{smallmatrix} t_v^{-1} & 0 \\ 0 & t_v^{-1} \end{smallmatrix}\right)K$ and $T = K\left(\begin{smallmatrix} t_v^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)K$. Then V^K is of dimension one, $\mathbb{C}\cdot f$, say, and have $Rf = \chi_1^{-1}(t)\chi_2^{-1}(t)f$, and $Tf = (\chi_1^{-1}(t) + (|\cdot|^{-1}\chi_2^{-1})(t))f$.*

Proof. We recall that V is the space of functions $f\colon \mathrm{GL}_2(F_v) \to \mathbb{C}$ whose stabiliser for right translations is open, and such that for all $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$ in B and all g in $\mathrm{GL}_2(F_v)$ we have:

$$f(\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)g) = \chi_1(x)\chi_2(z)f(g).$$

Hence V^K is the set of such functions f that are invariant under right translations by $\mathrm{GL}_2(O_{F_v})$. Now we can use the well known fact that $\mathrm{GL}_2(F_v) = BK$, a kind of non-archimedean version of Gramm-Schmidt. A very nice way to see that one has this is to note that $B\backslash\mathrm{GL}_2(F_v)$ is the same as $\mathbb{P}^1(F_v)$, compatibly with the action of $\mathrm{GL}_2(F_v)$, and that $\mathbb{P}^1(F_v) = \mathbb{P}^1(O_{F_v})$, so that the action of $\mathrm{GL}_2(O_{F_v})$ on $B\backslash\mathrm{GL}_2(F_v)$ is transitive. Let now $f \in V^K$. Then we have, for all $k \in K$ and all $b = \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \in B$ that $f(bk) = \chi_1(x)\chi_2(z)f(1)$, where 1 denotes the two by two identity matrix. In particular, $\dim(V^K) \leq 11$, and, as the stabiliser in B of the image of 1 in $\mathrm{GL}_2(F_v)/K$ is $B \cap K$, we have $V^K \neq 0$ if and only if χ_1 and χ_2 are trivial on $O_{F_v}^\times$.

Let now $f \in V^K$, and suppose that $f \neq 0$. By the above, we may as well assume that $f(1) = 1$.

As the element $r := \left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ defining R is central, we have $KrK = rK$, and hence:

$$(Rf)1 = f(1{\cdot}r) = \chi_1(t^{-1})\chi_2(t^{-1})f(1), \quad \text{hence} \quad Rf = \chi_1(t^{-1})\chi_2(t^{-1})f.$$

To compute Tf we note that:

$$K\left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)K = \left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)K \coprod \coprod_a \left(\begin{smallmatrix} 1 & at^{-1} \\ 0 & t^{-1} \end{smallmatrix}\right)K$$

where the union is disjpoint (as indicated in fact) and where a runs through a list of representatives of the quotient $O_{F_v} \to k_v$. Therefore:

$$\begin{aligned}(Tf)1 &= f\left(\begin{smallmatrix} t^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right) + \sum_a f\left(\begin{smallmatrix} 1 & at^{-1} \\ 0 & t^{-1} \end{smallmatrix}\right) = \chi_1^{-1}(t)f(1) + \#k_v\chi_2(t^{-1})f(1) \\ &= \left(\chi_1^{-1}(t) + (|\cdot|^{-1}\chi_2^{-1})(t)\right) f(1).\end{aligned}$$

□

References

[1] T. Saito. *Hilbert modular forms and p-adic Hodge theory (preliminary version)*. Preprint, available from the author's home page:
`http://www.ms.u-tokyo.ac.jp/~t-saito/pp.html`