College Meetkunde/Geometry, 3de en 4de jaar, najaar/Fall 2002, 12 weken/weeks.

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The writing of this syllabus is in progress, as one can see from the title above. In the 'studiegids' it is advertised as follows.

> In this course we will study various kinds of geometrical objects, such as differentiable varieties, analytic varieties and algebraic varieties. These objects occur in many areas of mathematics, ranging from physics (general relativity, conformal field theory, mechanics) and analysis (Lie groups, differential equations) to algebra and number theory (arithmetic algebraic geometry). The course starts with definitions and examples of such varieties and the right kind of maps between them. Then we study tangent bundles, vector bundles, tensor constructions, differential forms, integration. Two goals will be to establish the basic properties of de Rham cohomology, and to give the description of what a space-time is in general relativity. The aim of the course is to provide the students with the fundamentals of geometry, and to prepare for more advanced algebraic geometry. Each week exercises will be handed out, and one hour of the course will be used to discuss them. As the teacher (Edixhoven) is new in Leiden, he does not know too much what kind of audience to expect. So the course will be adapted to its public, meaning that more details will always be given if needed, and that the goals can be adjusted if necessary.

The syllabus is based on one that was used in Rennes in 1996, and which can be downloaded in various formats (ps, ps.gz, pdf) at:

`http://www.maths.univ-rennes1.fr/~edix/cours/dea9697.html`

together with a syllabus by Looijenga that is used in Utrecht:

`http://www.math.ruu.nl/people/siersma/difvar.html`

Regularly, new sections will be handed out, including exercises. These sections will also be put on the webpages for this course. It is probably a good idea to print out Looijenga's syllabus, but not the old Rennes syllabus by Edixhoven (unless one does not care about wasting paper).

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1 Differentiable varieties

We are going to study various kinds of geometrical objects, such as differentiable varieties, analytic varieties and algebraic varieties. These objects occur in many areas of mathematics, ranging from physics (general relativity, conformal field theory, mechanics) to algebra and number theory (algebraic geometry) and analysis (Lie groups, differential equations). Students taking this course are invited to look for further details or other points of view in text books. Some books I can think of at this moment are: Spivak's series of books (differential geometry), the book by Bott and Tu (differential forms in algebraic topology), Lang's book (differentiable manifolds) and Bourbaki's book (variétés différentielles et analytiques: fascicule de résultats). A good idea is to look in the library under the appropriate AMS subject classifications (differential geometry is 53, algebraic geometry is 14, Lie groups is 22, analytic geometry: 32, geometry: 51, manifolds: 57 and 58).

We begin with differentiable varieties, also called manifolds. These are usually defined in terms of charts. Intuitively, they are objects that "locally" look like $\mathbb{R}^n$ for some n. We begin by making that precise.

1.1 Definition. *Let X be a set. An atlas for X then consists of the following data: a set I, for each i in I a subset X_i of X, an integer $n_i \geq 0$, an open subset U_i of $\mathbb{R}^{n_i}$ and a bijection $\phi_i\colon U_i \to X_i$. These data are required to satisfy the following conditions. Firstly, the X_i cover X, that is, $\cup_i X_i = X$. Secondly, the charts ϕ_i are compatible, in the sense that we will now explain. For i and j in I let $X_{i,j}$ be $X_i \cap X_j$, and let $U_{i,j}$ be $\phi_i^{-1}X_{i,j}$. Then ϕ_i induces a bijection, still denoted ϕ_i, from $U_{i,j}$ to $X_{i,j}$. Saying that ϕ_i and ϕ_j are compatible means that $U_{i,j}$ is open in U_i, $U_{j,i}$ open in U_j, and the bijection $\phi_j^{-1} \circ \phi_i\colon U_{i,j} \to U_{j,i}$ is differentiable.*

Some remarks are in order here. First of all, the differentiability of $\phi_j^{-1} \circ \phi_i$ in the definition can mean various things. When we just say differentiable, we mean in fact infinitely differentiable, that is, the n_j $\mathbb{R}$-valued functions making up $\phi_j^{-1} \circ \phi_i$ are C^∞-functions on $U_{i,j}$. But we could also consider functions of class C^k for some $k \geq 0$; in that case we will say that the atlas is C^k. Note that the terminology "differentiable" is misplaced in the case $k = 0$; in that case we speak of a topological atlas. The second remark concerns the integers n_i.

1.2 Definition. *Let $k \geq 0$ be an integer or ∞. A variety or manifold of class C^k is a set X equipped with a C^k-atlas. Notation: (X, I, n, U, ϕ).*

For X a C^k-variety and x in X, all n_i for i such that X_i contains x are equal; this integer is called the dimension of X at x; we denote it by $\dim_X(x)$, so that we can view $\dim_X$ as a $\mathbb{Z}$-valued function on X. (For $k > 0$ the equality of the n_i is easy to prove (consider derivatives and use linear algebra); for $k = 0$ one needs some algebraic topology.) Most of the time we will just consider the C^∞ case. As usual, defining the objects to study is not too interesting; we should also say what maps between them we want to consider. For example, we want to say what it means that two manifolds are isomorphic.

1.3 Definition. *Let (X, I, n, U, ϕ) and (Y, J, m, V, ψ) be manifolds. Let f be a map from X to Y. Let x be in X. Then f is called differentiable at x if for every (i, j) such that $x \in X_i$ and $f(x) \in Y_j$ the subset $\phi_i^{-1}((f^{-1}Y_j) \cap X_i) \subset \mathbb{R}^{n_i}$ contains an open neighborhood of $\phi_i^{-1}(x)$ and the map $\psi_j^{-1} f \phi_i$ from $\phi_i^{-1}((f^{-1}Y_j) \cap X_i) \subset \mathbb{R}^{n_i}$ to $\mathbb{R}^{m_j}$ is differentiable at $\phi_i^{-1}(x)$. The map f is called differentiable, or a morphism of manifolds, if it is differentiable at all x in X.*

Note that this definition does not change if we require the openness and differentiability at x only for one pair (i,j). If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms, then $g\circ f\colon X \to Z$ is also a morphism. So we have the category of manifolds: we have objects, morphisms, composition of morphisms, the composition is associative and each object X has an identity morphism id_X. A morphism $f\colon X \to Y$ is called an isomorphism if and only if there exists a morphism $g\colon Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. Equivalently: a map $f\colon X \to Y$ is an isomorphism if and only it is bijective and f and f^{-1} are differentiable. Let us look at some examples of manifolds.

Let $n \geq 0$. Then $\mathbb{R}^n$ with the atlas consisting of the chart $\mathrm{id}_{\mathbb{R}^n}$ is a manifold, that we will denote by $\mathbb{R}^n$. In the same way, every open subset of some $\mathbb{R}^n$ becomes a manifold. If X and Y are manifolds (we have already dropped the atlas from the notation), then $X \times Y$ is easily equipped with an atlas (take $K := I \times J$, $W_{i,j} := U_i \times V_j$, etc.). We leave it as an exercise to the reader to verify that the two projections pr_X and pr_Y from $X \times Y$ to X and Y are differentiable, and that $(X \times Y, \mathrm{pr}_X, \mathrm{pr}_Y)$ has the following universal property: for Z a manifold and morphisms $f\colon Z \to X$ and $g\colon Z \to Y$ there exists a unique morphism $h\colon Z \to X \times Y$ such that $f = \mathrm{pr}_X h$ and $g = \mathrm{pr}_Y h$. Let V be a finite dimensional $\mathbb{R}$-vector space, say of dimension n. Then an isomorphism of $\mathbb{R}$-vector spaces $\phi\colon \mathbb{R}^n \to V$ makes V into a manifold. It is clear that two such isomorphisms $\phi \neq \phi'$ give different atlases but that id_V is an isomorphism between the manifolds.

For $n \geq 0$ the subset $\mathrm{GL}_n(\mathbb{R})$ of $\mathrm{M}_n(\mathbb{R})$ consisting of invertible n by n matrices with coefficients in $\mathbb{R}$ is an open subset (it is $\det^{-1}(\mathbb{R} - \{0\})$). It is easy to check that the maps $m\colon \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ and $i\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ given by $m(x,y) = xy$ and $i(x) = x^{-1}$ are differentiable (for i, use the formula for x^{-1} in terms of the matrix of cofactors and $\det(x)$). In general, a group G equipped with an atlas such that the multiplication and inversion are differentiable is called a Lie group. We will see more examples soon.

Our next example is in a sense more interesting, because it is not isomorphic to an open subset of $\mathbb{R}^n$ for any n. We consider the circle S^1 in $\mathbb{R}^2$: it is the set of (x,y) such that $x^2 + y^2 = 1$. One way to make an atlas is the following. The projection on the first coordinate gives a bijection from $\{(x,y) \in S^1 \mid y > 0\}$ to the open interval $]-1,1[$; the same holds for $\{(x,y) \in S^1 \mid y < 0\}$. We also have the projections on the second coordinate from the sets $\{(x,y) \in S^1 \mid x > 0\}$ and $\{(x,y) \in S^1 \mid x < 0\}$. The four inverses of these maps form an atlas. Another atlas is obtained by restricting the map $(\sin,\cos)$ from $\mathbb{R}$ to $\mathbb{R}^2$ to suitable subsets of $\mathbb{R}$. Yet a third atlas is given by projection from points of S^1. For t in $\mathbb{R}$ consider the line through $(t,0)$ and $(0,1)$. This line intersects S^1 in $(0,1)$ and a unique other point: $(2t/(t^2+1), (t^2-1)/(t^2+1))$. This map gives a bijection from $\mathbb{R}$ to $S^1 - \{(0,1)\}$. Considering lines through $(0,-1)$ gives a second chart. The map id_{S^1} is an isomorphism between all three atlases for S^1 that we have just seen. It is interesting to note what kind of functions we get from the charts and transition maps (i.e., the $\phi_j^{-1}\phi_i$) in these three cases. In the first case the charts and the transition maps are algebraic functions (they are built up from rational functions and square roots). In the second case the charts are given by the transcendental functions sin and cos, but the transition maps are just translations in $\mathbb{R}$. In the third case all functions are rational functions.

We could treat the n-sphere S^n in a similar way (it is defined as the subset of $(x_0,\ldots,x_n)$ in $\mathbb{R}^{n+1}$ such that $x_0^2 + \cdots + x_n^2 = 1$). In particular, the first and third methods we used for $n = 1$ are easily adapted (not the second method, as far as I can see; it has to do with the fact that S^n is simply connected for $n \geq 2$ (well, of course, one has the usual spherical coordinates r, θ and ϕ, but it is not as nice)). But it is more useful to develop a systematic way to make subsets of $\mathbb{R}^n$ that are defined by suitable equations into manifolds. In order to do this we need the implicit function theorem. We will state this theorem in a quite general context, so that it will suffice for the whole course.

1.4 Theorem. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let X, Y and Z be normed K-vector spaces, with Y complete. Let U be an open subset of $X \times Y$, $k \geq 1$ and $f\colon U \to Z$ a C^k-map. Let (x, y) be in U such that the derivative $(D_2 f)(x, y)\colon Y \to Z$ of f with respect to the second variable is an isomorphism of topological vector spaces (i.e., it is bijective and its inverse is continuous). Then there exist open neighborhoods V of x in X and W of y in Y such that $V \times W \subset U$ and for every v in V there exists a unique w in W with $f(v, w) = f(x, y)$. The map $g\colon V \to Y$ thus defined is C^k. Moreover, if X and Y are finite dimensional and f analytic, then g is analytic.*

For a proof the reader is referred to the standard text books, or to course notes from analysis courses. In the complex case, i.e., $K = \mathbb{C}$, we say that $f\colon U \to Z$ is C^k if it is so when we view X, Y and Z as $\mathbb{R}$-vector spaces. When we want to talk about differentiability in the complex sense, we will allways explicitly say so. The reason for this terminology is that a function $f\colon U \to \mathbb{C}$, with $U \subset \mathbb{C}^n$ open, is analytic if and only if it is C^1 in the complex sense.

Let us now consider the following situation. We have positive integers n and m, we have an open subset U of $\mathbb{R}^n$ and a C^k map f from U to $\mathbb{R}^m$, for some $k \geq 1$. Let X be the set of zeroes of f: $X := \{x \in U \mid f(x) = 0\}$. We want to equip X with an atlas, in some natural way (for example, the charts should be C^k-maps to $\mathbb{R}^n$). It turns out that at least some conditions have to be satisfied for this to be possible. For example, consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$. Then X is the union of the two coordinate axes; consequently, X, with its induced topology, cannot be a C^0-manifold, since no neighborhood of $(0, 0)$ is homeomorphic to an open interval in $\mathbb{R}$. Note that $(0, 0)$ is special, since both partial derivatives of f vanish at that point, i.e., f has derivative zero at $(0, 0)$. So, in the situation above, we assume that for all x in X the derivative $(Df)x$ is surjective (i.e., f is a submersion at all x in X). Let now x be in X. Let V be the kernel of the linear map $(Df)x\colon \mathbb{R}^n \to \mathbb{R}^m$. Let $V^\perp$ be the orthogonal of V (for the standard inner product on $\mathbb{R}^n$). We view $\mathbb{R}^n$ as the product $V \times V^\perp$, and hence U as an open subset of $V \times V^\perp$. In this situation we can apply Theorem 1.4, since $(D_2 f)x$ is an isomorphism from $V^\perp$ to $\mathbb{R}^m$. We get an open subset $V' \subset V$ and a C^k-map $g\colon V' \to V^\perp$ such that x is in $V' \times V^\perp$ and for all v in V' we have $f(v, g(v)) = 0$. Hence the map $\phi\colon V' \to X$ defined by $\phi(v) = (v, g(v))$ is a chart at x. We can obviously cover X with such charts (for example, take one chart for each x). These charts are compatible because ϕ^{-1} is just the orthogonal projection on V. Note that $\dim_X(x) = \dim V = n - m$.

1.5 Exercise. Suppose that we have $U \subset \mathbb{R}^n$ open, and C^k-maps $f\colon U \to \mathbb{R}^m$ and $f'\colon U \to \mathbb{R}^{m'}$, defining the same X and both submersions at all x in X. Then show that any two atlases obtained from the construction above are such that id_X is an isomorphism between them.

We can now easily give some more examples of Lie groups: the classical matrix groups.

1.6 Example. Let $n \geq 1$. The group special linear group $\mathrm{SL}_n(\mathbb{R})$ is defined as the kernel of the morphism of groups $\det\colon \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$. We have to show that for all x in $\mathrm{SL}_n(\mathbb{R})$ the derivative $(D\det)x$ is non-zero. So we have to "compute" $\det(x + \varepsilon y)$ for small ε in $\mathbb{R}$ and any y in $\mathrm{M}_n(\mathbb{R})$. We have:

$$\det(x + \varepsilon y) = \det(x(1 + \varepsilon x^{-1} y)) = \det(x)\det(1 + \varepsilon x^{-1} y) = 1 + \varepsilon \mathrm{tr}(x^{-1} y) + O(\varepsilon^2), \tag{1.6.1}$$

with $\mathrm{tr}(x^{-1}y)$ the trace of $x^{-1}y$. It follows that $((D\det)x)y = \mathrm{tr}(x^{-1}y)$. This cannot be zero for all y, since x^{-1} is invertible. (We note that the "standard" inner product on $\mathrm{M}_n(\mathbb{R})$ is given by $(a,b) \mapsto \mathrm{tr}(a^t\, b)$.)

1.7 Example. Let again $n \geq 1$. The orthogonal group $\mathrm{O}_n(\mathbb{R})$ is the subgroup of x in $\mathrm{GL}_n(\mathbb{R})$ that preserve the standard scalar product of $\mathbb{R}^n$, i.e., the x such that $x^t\, x = 1$. The special orthogonal group is the subgroup $\mathrm{SO}_n(\mathbb{R})$ of x in $\mathrm{O}_n(\mathbb{R})$ with $\det(x) = 1$. We consider the map $f\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})^+$ given by $f(x) = x^t\, x - 1$, where $\mathrm{M}_n(\mathbb{R})^+$ denotes the set of real symmetric n by n matrices. We have to show that this map is submersive at all x in $\mathrm{O}_n(\mathbb{R})$. For x in $\mathrm{O}_n(\mathbb{R})$, y in $\mathrm{M}_n(\mathbb{R})$ and small ε in $\mathbb{R}$ we have (with $z := x^{-1}y$):

$$\begin{aligned} f(x+\varepsilon y) &= f(x(1+\varepsilon z)) = (1+\varepsilon z)^t x^t\, x(1+\varepsilon z) - 1 = (1+\varepsilon z^t)(1+\varepsilon z) - 1 = \\ &= \varepsilon(z^t + z) + O(\varepsilon^2). \end{aligned} \tag{1.7.1}$$

It follows that $((Df)x)y = z^t + z$,which clearly shows that $(Df)x$ is surjective. So $\mathrm{O}_n(\mathbb{R})$ is now a manifold. Our computation above also shows that it has everywhere the same dimension, namely $(n^2 - n)/2 = \dim_{\mathbb{R}}(\mathrm{M}_n(\mathbb{R}))^-$, the dimension of the $\mathbb{R}$-vector space of anti-symmetric n by n matrices. For every x in $\mathrm{O}_n(\mathbb{R})$ we have $1 = \det(x^t\, x) = \det(x)^2$, hence $\det(x) = \pm 1$. There are x in $\mathrm{O}_n(\mathbb{R})$ with $\det(x) = -1$, hence we have a short exact sequence:

$$1 \to \mathrm{SO}_n(\mathbb{R}) \to \mathrm{O}_n(\mathbb{R}) \to \{\pm 1\} \to 1. \tag{1.7.2}$$

This sequence is split: send -1 to the diagonal matrix $\mathrm{diag}(-1, 1, \ldots, 1)$, for example. If n is odd, we even have a splitting with image in the center: send -1 to -1; hence for odd n we have an isomorphism of Lie groups from $\mathrm{O}_n(\mathbb{R})$ to $\mathrm{SO}_n(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}$. For even n there is not such a splitting, and one cannot do better than say that $\mathrm{O}_n(\mathbb{R})$ is isomorphic to the semi-direct product $\mathrm{SO}_n(\mathbb{R}) \times_\alpha \mathbb{Z}/2\mathbb{Z}$, with $\alpha\colon \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathrm{SO}_n(\mathbb{R}))$ the morphism of groups that sends -1 to the inner automorphism given by conjugation by $\mathrm{diag}(-1, 1, \ldots, 1)$.

1.8 Example. Again, $n \geq 1$. The symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ that preserves the "standard" alternating bilinear form on $\mathbb{R}^{2n}$ that is given by the matrix $\psi := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, in which the coefficients are n by n matrices. One computes easily that $\mathrm{Sp}_{2n}(\mathbb{R})$ is the subset of $\mathrm{GL}_{2n}(\mathbb{R})$ of the x that satisfy $x^t\psi x = \psi$. So we consider the map f from $\mathrm{GL}_{2n}(\mathbb{R})$ to $M_{2n}(\mathbb{R})^-$ given by $f(x) = x^t\psi x - \psi$. Let x be in $\mathrm{Sp}_{2n}(\mathbb{R})$, y in $M_{2n}(\mathbb{R})$ and ε in $\mathbb{R}$. Put $z := x^{-1}y$. Then we have:

$$f(x+\varepsilon y) = f(x(1+\varepsilon z)) = (1+\varepsilon z^t)\psi(1+\varepsilon z) - \psi = \varepsilon(z^t\psi + \psi z) + O(\varepsilon^2). \tag{1.8.1}$$

This shows that $((Df)x)y = z^t\psi + \psi z$, so we have to show that the map $z \mapsto z^t\psi + \psi z$ from $\mathrm{M}_{2n}(\mathbb{R})$ to $\mathrm{M}_{2n}(\mathbb{R})^-$ is surjective. To do this, we compute its kernel (this is interesting anyway, since this kernel is what is called the Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{R})$). So write z as a two by two matrix of n by n matrices: $z = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then a short computation gives:

$$z^t\psi + \psi z = 0 \iff (c^t = c \text{ and } b^t = b \text{ and } d = -a^t). \tag{1.8.2}$$

It follows that the kernel has dimension $2n^2+n$. The dimension of $\mathrm{M}_{2n}(\mathbb{R})^-$ equals $((2n)^2-(2n))/2$. Linear algebra then implies that our map $z \mapsto z^t\psi + \psi z$ is surjective.

1.9 Exercise. Let $n \geq 1$. Show that the Lie groups $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{R})$ are connected (the last one is more difficult to do). Show also that $\mathrm{Sp}_{2n}(\mathbb{R})$ is contained in $\mathrm{SL}_{2n}(\mathbb{R})$.

1.10 Exercise. Let $n \geq 1$. Show that $\mathrm{GL}_n(\mathbb{C})$ is a manifold (i.e., make it into one, in the right way). The unitary group $\mathrm{U}_n(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of those x that preserve the standard scalar product on $\mathbb{C}^n$ (the one that sends (v, w) to $v_1\overline{w_1} + \cdots + v_n\overline{w_n}$), or, equivalently, the x with $\overline{x}^t x = 1$. Make $\mathrm{U}_n(\mathbb{R})$ into a manifold. Compute its dimension. Show that it is connected and compact. Do the same things for its subgroup $\mathrm{SU}_n(\mathbb{R})$ consisting of those x with $\det(x) = 1$.

We have already quite a few examples at our disposal, and it seems a good moment to do something about the foundations again. The reader is certainly aware that up to now each time we had various atlases on one set X, they had the property that id_X was an isomorphism between them. There should be a much more natural way to express this. In fact, we should replace the atlases by something else, giving us an equivalent category (this will be made precise). For example, when one studies groups, it is very clumsy to deal only with groups in terms of generators and relations. So we look for an object associated to an atlas for X such that two atlases such that id_X is an isomorphism between them give exactly the same object. We do this by considering a topology on X and the notion of differentiable functions on open subsets of X.

Let X be a set equipped with a C^k-atlas (some $k \geq 0$) (I, n, U, ϕ). Then we can define a topology on X by saying that a subset V of X is open if and only if for all i in I the subset $\phi_i^{-1}(V \cap X_i)$ is open in $\mathbb{R}^{n_i}$. (The verification that this works is left to the reader, and also that of the following assertion.) A subset V of X is open if and only if for all x in V there exists an i such that $x \in X_i$ and the subset $\phi_i^{-1}(V \cap X_i)$ of $\mathbb{R}^{n_i}$ contains an open neighborhood of $\phi_i^{-1}(x)$. Suppose now that $V \subset X$ is open. Then a function $f\colon V \to \mathbb{R}$ is called of class C^k if and only if for all i the function $f \circ \phi_i$ on $\phi_i^{-1}(V \cap X_i)$ is C^k. The reader will verify that $f\colon V \to \mathbb{R}$ is C^k if and only if for all x in V there exists i such that $f \circ \phi_i$ on $\phi_i^{-1}(V \cap X_i)$ is C^k. The set of C^k functions on V will be denoted by $C^k_X(V)$; it is clearly an $\mathbb{R}$-algebra, usually of infinite dimension. For an open subset W contained in V we have the restriction map $\mathrm{res}(V, W)\colon C^k_X(V) \to C^k_X(W)$, that sends a function f on V to its restriction $f|_W$ to W. It is clear that for Z an open subset contained in W we have $\mathrm{res}(V, Z) = \mathrm{res}(W, Z)\mathrm{res}(V, W)$. Such a collection of sets $C^k_X(V)$ and maps $\mathrm{res}(V, W)$ is what one calls a presheaf on the topological space X, denoted C^k_X (a concise reference for the notions of presheaf and sheaf is Hartshorne's book "Algebraic Geometry" (GTM 52, Springer)). A very important property of this presheaf C^k_X is the following direct consequence of the local nature of a function being C^k: if we have an open subset V of X, and a covering of V by open subsets V_j with the j in some set J, and for each j an element f_j of $C^k_X(V_j)$ such that for all j and j' we have $\mathrm{res}(V_j, V_j \cap V_{j'})f_j = \mathrm{res}(V_{j'}, V_j \cap V_{j'})f_{j'}$, then there exists a unique f in $C^k_X(V)$ such that $\mathrm{res}(V, V_j)(f) = f_j$ for all j. (I apologize for the long sentence.) In general, a presheaf that satisfies this glueing condition will be called a sheaf. For the moment, the notions of sheaf and presheaf are just convenient for us for notational matters; we won't do anything complicated with sheaves for some time.

Given a set X with an atlas, the object we associate to it is the pair (X, C^k_X), consisting of a topological space and a sheaf of rings on it. Such objects are called ringed spaces, and all geometrical objects we will consider in this course will be ringed spaces. Let us now look at what it means for a map to be a morphism in terms of these sheaves.

1.11 Proposition. *Let $k \geq 0$. Let X and Y be C^k-manifolds, and let $f\colon X \to Y$ be a map (of sets). Then f is a morphism of C^k-manifolds if and only if f is continuous and for each open U in Y and $g \in C^k_Y(U)$ the function gf is in $C^k_X(f^{-1}U)$.*

Proof. Suppose that f is a morphism of C^k-manifolds, i.e., f is C^k. Then it follows directly from the definitions, and the fact that compositions of C^k-maps are C^k, that f is continuous and that for each open U in Y and $g \in C^k_Y(U)$ the function gf is in $C^k_X(f^{-1}U)$.

Suppose now that f is continuous and that for each open U in Y and g in $C^k_Y(U)$ the function gf is in $C^k_X(f^{-1}U)$. Then $X_i \cap f^{-1}Y_j$ is open in X_i, hence $\phi_i^{-1}(X_i \cap f^{-1}Y_j)$ is open in U_i, hence in $\mathbb{R}^{n_i}$. We have to show that the map $\psi_j^{-1} f \phi_i$ from $\phi_i^{-1}(X_i \cap f^{-1}Y_j)$ to $V_j \subset \mathbb{R}^{m_j}$ is C^k. It is equivalent to show that the m_j coordinate functions $x_k \psi_j^{-1} f \phi_i$ of this map are C^k. Now $x_k \psi_j^{-1}$ is in $C^k_Y(Y_j)$, hence $x_k \psi_j^{-1} f$ is in $C^k_X(f^{-1}Y_j)$. It follows that $x_k \psi_j^{-1} f \phi_i$ is differentiable. $\square$

We are now ready to formulate a new, improved definition of the category of C^k-manifolds. Note that a morphism of C^k-manifolds $f\colon X \to Y$ induces, for every open U in Y, a morphism of $\mathbb{R}$-algebras $f^*(U)\colon C^k_Y(U) \to C^k_X(f^{-1}U)$. In the language of sheaves, this is just a morphism of sheaves from C^k_Y to $f_* C^k_X$. For $f\colon X \to Y$ a morphism of topological spaces and F a sheaf on X, f_*F is the sheaf defined by $(f_*F)(U) = F(f^{-1}U)$. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair (f, ϕ) with f a continuous map from X to Y and ϕ a morphism of sheaves from $\mathcal{O}_Y$ to $f_*\mathcal{O}_X$. Let (f, ϕ) be a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ and (g, γ) a morphism from $(Y, \mathcal{O}_Y)$ to $(Z, \mathcal{O}_Z)$. Then $k := gf\colon X \to Z$ is continuous, and for every open $U \subset Z$ we have a morphism of rings $\kappa(U)$ from $\mathcal{O}_Z(U)$ to $\mathcal{O}_X(k^{-1}U)$ obtained by composing $\gamma(U)\colon \mathcal{O}_Z(U) \to \mathcal{O}_Y(g^{-1}U)$ and $\phi(g^{-1}U)\colon \mathcal{O}_Y(g^{-1}U) \to \mathcal{O}_X(f^{-1}g^{-1}U)$. In the case where X, Y and Z are obtained from manifolds as above, these maps are just the maps that do pullback of functions. Anyway, one easily verifies that κ is a morphism of sheaves (i.e., the $\kappa(U)$ are compatible with the restriction maps) so that (k, κ) is a morphism of ringed spaces. This composition of morphisms gives us a category: the category of ringed spaces. So now we also have the notion of isomorphisms between ringed spaces.

But in fact all this is not exactly what we need at this moment. Our ringed spaces (X, C^k_X) are somehow special: the sheaf C^k_X is a sheaf of $\mathbb{R}$-valued functions. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are topological spaces with sheaves of $\mathbb{R}$-valued functions, we define a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ to be a continuous map $f\colon X \to Y$ such that for all $U \subset Y$ open and all g in $\mathcal{O}_Y(Y)$ the function $f^*g := gf$ is in $\mathcal{O}_X(f^{-1}U)$. One checks immediately that this also gives us a category (let us call it the category of topological spaces with a sheaf of $\mathbb{R}$-valued functions), hence also a notion of isomorphism. For $(X, \mathcal{O}_X)$ a topological space with a sheaf of $\mathbb{R}$-valued functions, and $U \subset X$ open, we have the ringed space $(U, \mathcal{O}_X|_U)$, with $\mathcal{O}_X|_U(V) = \mathcal{O}_X(V)$ for all open $V \subset U$; this topological space with a sheaf of $\mathbb{R}$-valued functions is called the open subspace U of X. We can now state our improved definition of manifolds.

1.12 Definition. *Let $k \geq 0$. A C^k-manifold is a topological space with a sheaf of $\mathbb{R}$-valued functions (X, C^k_X), that is locally isomorphic (in the category of topological spaces with sheaves of $\mathbb{R}$-valued functions) to some (U, C^k_U), with U an open subset of some $\mathbb{R}^n$ and C^k_U the sheaf of C^k-functions on U. (Note that n may vary.) A morphism from a C^k-manifold (X, C^k_X) to a C^k-manifold (Y, C^k_Y) is a morphism in the category of topological spaces with a sheaf of $\mathbb{R}$-valued functions.*

Let us now see what it means that this definition is equivalent to the older one. So let, for a moment, **Man** and **Man**$'$ denote the categories of C^k-manifolds in the old and the new sense, respectively. Then we have a functor F from **Man** to **Man**$'$ that sends a set X with a C^k-atlas to the space (X, C^k_X) as explained above, and sends a morphism $f\colon X \to Y$ in **Man** to the morphism (f, f^*) in **Man**$'$. By saying that our two definitions are equivalent, we mean that F is induces an equivalence of categories from **Man** to **Man**$'$. By definition, this means that there is a functor G from **Man**$'$ to **Man** such that GF and FG are isomorphic to $\mathrm{id}_{\mathbf{Man}'}$ and $\mathrm{id}_{\mathbf{Man}}$, respectively. General category nonsense says that such a G exists if and only if F is full and faithful (i.e, for all X and Y in **Man** F gives a bijection from $\mathrm{Hom}_{\mathbf{Man}}(X, Y)$ to $\mathrm{Hom}_{\mathbf{Man}'}(F(X), F(Y)))$ and essentially surjective (i.e., every $(X, \mathcal{O}_X)$ in **Man**$'$ is isomorphic to the image under F of some object of **Man**). Proposition 1.11 means that F is full and faithful. It remains to show that F is essentially surjective. So let $(X, \mathcal{O}_X)$ be an object of **Man**$'$. Then we can cover X by open sets X_i such that the $(X_i, \mathcal{O}_X|_{X_i})$ are isomorphic to some (U, C^k_U) with U open in some $\mathbb{R}^n$. These isomorphisms form an atlas on X for which one easily verifies that it induces the sheaf $\mathcal{O}_X$.

We can give another interpretation of the manifold structure on sets of zeroes of equations that we defined just after Theorem 1.4. The situation is the following: we have an open subset U of some $\mathbb{R}^n$, a C^k-map (some $k \geq 1$) $f\colon U \to \mathbb{R}^m$ such that $(Df)x$ is surjective for all x in $X := f^{-1}0$. Note that X is a closed subset of U. Let Z be any closed subset of U. Then we get a sheaf C^k_Z of $\mathbb{R}$-algebras on Z by letting $C^k_Z(V)$ be the set of $g\colon V \to \mathbb{R}$ such that for all $v \in V$ there exists an open neighborhood V' of v in U and an $h \in C^k(V')$ such that $g = h$ on $V \cap V'$. It is left to the reader to show that for X as above, this space (X, C^k_X) is in fact the manifold obtained by the construction mentioned above. (I realize that in this last part I really skipped a lot of details.)

After these generalities (general nonsense), let us consider some more examples: projective spaces and Grassmannians.

1.13 Example. Let k be a field, and V a k-vector space. Then we define the projective space $\mathbb{P}(V)$ associated to V to be the set of 1-dimensional subspaces of V (i.e., the set of lines through the origin). Clearly, we have a bijection $(V - \{0\})/k^* \to \mathbb{P}(V)$ induced by the map that sends $v \neq 0$ to the subspace kv. For $n \geq 0$ we define $\mathbb{P}^n(k)$ to be $\mathbb{P}(k^{n+1})$. An element in $\mathbb{P}^n(k)$ will be denoted as $(a_0 : a_1 : \cdots : a_n)$ with $a_i \in k$ not all zero; in this notation we have $(a_0 : \cdots : a_n) = (b_0 : \cdots : b_n)$ if and only if there exists $\lambda \in k^*$ with $b_i = \lambda a_i$ for all i. Clearly, $\mathbb{P}^0(k)$ consists of just the one point (1). We can describe $\mathbb{P}^1(k)$ as follows: it is the disjoint union of the set $\{(a : 1) \mid a \in k\}$, that we can identify with k, and the point $(1 : 0)$ "at infinity". In the same way, $\mathbb{P}^n(k)$ is easily seen to be the disjoint union of k^0, $k^1, \ldots, k^n$. If we want to equip $\mathbb{P}^n(k)$ with an atlas (say for $k = \mathbb{R}$ or $\mathbb{C}$), then these disjoint unions are not so useful: we need "open subsets". So we consider the covering of $\mathbb{P}^n(k)$ by the sets $\mathbb{P}^n(k)_i := \{a_0 : \cdots : a_n \mid a_i \neq 0\}$, for $0 \leq i \leq n$. For each i we have a bijection ϕ_i from k^n to $\mathbb{P}^n(k)_i$, sending $(x_1, \ldots, x_n)$ to the point $(a_0 : \cdots : a_n)$ such that $a_i = 1$, $a_j = x_j$ for $j > i$ and $a_j = x_{j+1}$ for $j < i$. The inverse of ϕ_i sends $(a_0 : \cdots : a_n)$ to the $n$tuple of a_j/a_i with $j \neq i$. So, in the usual notation for charts, we have $U_i = k^n$ for all i. For $j > i$ the subset $U_{i,j}$ of $U_i = k^n$ is the set $\{x \in k^n \mid x_j \neq 0\}$, which is an open subset if $k = \mathbb{R}$ or $k = \mathbb{C}$. The map $\phi_j^{-1}\phi_i$ from $U_{i,j}$ to k^n sends $(x_1, \ldots, x_n)$ to the $n$tuple consisting of the x_l/x_j with $l < i$, $1/x_j$, and the x_l/x_j with $j \neq l \geq i$. Clearly, these maps are differentiable if $k = \mathbb{R}$ or $\mathbb{C}$.

1.14 Exercise. Show that the $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ are connected and compact. Show that $\mathbb{P}^1(\mathbb{R})$ is isomorphic to S^1 and that $\mathbb{P}^1(\mathbb{C})$ is isomorphic to S^2.

1.15 Example. For k a field, V a k-vector space and $d \geq 0$, let $\mathrm{Gr}_d(V)$ be the set of d-dimensional subspaces of V. Define $\mathrm{Gr}_{d,n}(k) := \mathrm{Gr}_d(k^n)$. Note that $\mathrm{Gr}_1(V) = \mathbb{P}(V)$ and $\mathrm{Gr}_{1,n}(k) = \mathbb{P}^{n-1}(k)$. For $k = \mathbb{R}$ or $\mathbb{C}$, we want to make the $\mathrm{Gr}_{d,n}(k)$ into manifolds. We will do this in two ways: first by making charts, then by considering $\mathrm{Gr}_{d,n}(k)$ as a quotient of $\mathrm{GL}_n(k)$ by a certain subgroup (in this case we can define charts, or a topological space with a sheaf of functions).

Let $n \geq 0$, $d \geq 0$ and a field k be given. Let V be a k-vector space of dimension n. Let x be in $\mathrm{Gr}_d(V)$, i.e., x is a d-dimensional subspace of V. Choose a subspace y of V such that $V = x \oplus y$. Let $\mathrm{Gr}_d(V)_x$ be the subset $\{z \mid z \cap y = \{0\}\}$ of $\mathrm{Gr}_d(V)$. For each z in $\mathrm{Gr}_d(V)_x$ the projection from z to x along y is an isomorphism. It follows that we have a bijection ϕ_x from $\mathrm{Hom}_k(x, y)$ to $\mathrm{Gr}_d(V)_x$ sending f to $\mathrm{im}(\mathrm{id}_x + f)$. Note that $x = \phi_x(0)$. We view the set $\mathrm{Hom}_k(x, y)$ as a k-vector space in the usual way. We leave it to the reader to study the maps ϕ_x and to verify that, for $k = \mathbb{R}$ or $\mathbb{C}$, they form an atlas. It might be a good idea to look first at what happens when one takes various y's for one x, and then to take one y for various x's. Anyway, it is not necessary to do this, since the next method we use shows that the charts are compatible.

Let us now study $\mathrm{Gr}_{d,n}(k)$ from a different point of view. The group $\mathrm{GL}_n(k)$ acts on the vector space k^n, hence on $\mathrm{Gr}_{d,n}(k)$: an element g sends x to gx, the image of x under g. It is easy to see that $\mathrm{GL}_n(k)$ acts transitively on $\mathrm{Gr}_{d,n}(k)$ (for a given x, choose a basis for x and extend it to a basis of k^n). Let x_0 be the subspace $ke_1 + \cdots + ke_d$ of k^n, where e denotes the standard basis of k^n. The stabilizer $P := \mathrm{GL}_n(k)_x$ of x is the subgroup of $\mathrm{GL}_n(k)$ consisting of those g such that $g_{i,j} = 0$ for all (i, j) with $j \leq d < i$. Hence we get a bijection from $\mathrm{GL}_n(k)/P$ to $\mathrm{Gr}_{d,n}(k)$. Suppose now that $k = \mathbb{R}$ or $\mathbb{C}$. It suffices to equip $\mathrm{GL}_n(k)/P$ with a differentiable structure in order to do so for $\mathrm{Gr}_{d,n}(k)$. As already said above, we can make an atlas for $\mathrm{GL}_n(k)/P$, but we can also make $\mathrm{GL}_n(k)/P$ into a topological space with a sheaf of functions, directly. Since we have already seen numerous atlases, let us first construct the ringed space. For the sake of notation, let $G := \mathrm{GL}_n(k)$, $X := G/P$ and $\pi\colon G \to X$ be the quotient map. We equip X with the quotient topology: a subset U of X is open if and only if $\pi^{-1}U \subset G$ is open. Then we equip the topological space X with the sheaf of functions that are P-invariant. Note that for $U \subset X$ open, P acts on $C^\infty_G(\pi^{-1}U)$ by the formula $(pf)(g) = f(gp)$. For a set S with a P-action, let S^P be the set of elements fixed by P. The we define a sheaf C^∞_X on X by: $C^\infty_X(U) := C^\infty_G(\pi^{-1}U)^P$. Of course, now we have to verify that the topological space with sheaf of functions (X, C^∞_X) is a manifold. To do this, we have to show that every point has an open neighborhood that is isomorphic to some (U, C^∞_U) with U some open subset of some $\mathbb{R}^n$. This is of course almost the same as to make an atlas. Let g be in G. Then the translate gP of P is the orbit of g under P. Let T be the subspace of $\mathrm{M}_n(k)$ consisting of the m with $m_{i,j} = 0$ if $i \leq d$ or $j > d$. Then $\mathrm{M}_n(k)$ is the direct sum of T and the tangent space of P at 1. Let $U_g := T$ and $\phi_g\colon U_g \to X$ be given by $\phi_g(t) = \pi(g(1+t))$ (note that $1+t$ is in G). One verifies that ϕ_g induces an isomorphism between (T, C^∞_T) and $(X_g, C^\infty_X|_{X_g})$ with $X_g = \pi(g(1+T)P)$ open since $(1+T)P$ is exactly the set of g in G with $\det((g_{i,j})_{1\leq i,j\leq d}) \neq 0$. What makes this method work is the fact that the map $f\colon T \times P \to G$, $(t, p) \to (1+t)p$, is an isomorphism from $T \times P$ to the open subset G_e of G consisting of the g in G with $\det((g_{i,j})_{1\leq i,j\leq d}) \neq 0$. That this is so follows from the simple computation $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ ta & tb+d \end{smallmatrix}\right)$. We make this a bit more explicit. It is clear from the computation that the image of f is contained in G_e. Let now $\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right)$ be in G_e, and let us solve the equation $\left(\begin{smallmatrix} a & b \\ ta & tb+d \end{smallmatrix}\right) = \left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right)$. One finds: $a = u$, $b = v$, $t = wu^{-1}$,

$d = x - wu^{-1}v$. This shows that indeed the image of f is G_e, that f is injective, and that the inverse map $f^{-1}\colon G_e \to T \times P$ is given by:

$$\left(\begin{smallmatrix} u & v \\ w & x \end{smallmatrix}\right) \mapsto \left(\left(\begin{smallmatrix} 0 & 0 \\ wu^{-1} & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} u & v \\ 0 & x-wu^{-1}v \end{smallmatrix}\right)\right).$$

It follows that f makes P-invariant open subsets of G_e correspond to open subsets of T, and P-invariant C^∞ functions on a P-invariant open subset of G_e to C^∞ functions on the corresponding open subset of T. Hence we have an isomorphism between $(X_e, C_X^\infty|_{X_e})$ and (T, C_T^∞). One gets similar isomorphisms for all $(X_g, C_X^\infty|_{X_g})$ by using the action of G on itself by left translation.

Let us finish by noting that the charts obtained here and above are in fact the same, so we have now also shown that the charts above are compatible.

1.16 Remark. We note that our definition of manifold is somewhat more general than the one mostly encountered in textbooks, mainly for two reasons that both concern the underlying topological space. Often, manifolds are required to be Hausdorff (also called *separated*), i.e., distinct points lie in disjoint open neighborhoods. The second difference is a finiteness property: one often demands that a manifold has a countable basis for its topology (or, weaker, that it is paracompact (we will encounter this notion later)). We will state these types of assumptions when they are needed. Another difference is that most texts demand the dimension of a manifold, which is a locally constant function in our terminology, to be actually constant. Even worse, some texts demand manifolds to be connected.

1.17 Exercise. Give an example of a C^∞-manifold of which the dimension is not bounded.

1.18 Definition. *Let $k \geq 0$. A morphism of C^k-manifolds $f\colon X \to Y$ is called an* open immersion *if it induces an isomorphism from X to an open submanifold of Y. More explicitly, this means that fX is open in Y, that f induces a homeomorphism from X to fX (equipped with the topology induced from that of Y), and that for every open subset U of fX the map f^* is an isomorphism from $C_Y^k(U)$ to $C_X^k(f^{-1}U)$.*

1.19 Exercise. Manifolds can be "glued" along open subsets. Let I be a set, and $k \geq 0$. For every i in I, let X_i be a C^k-manifold. For every i and j in I, let $X_{i,j}$ be an open subset of X_i, and let $f_{i,j}$ be an isomorphism from $X_{i,j}$ to $X_{j,i}$. Suppose that for every i we have $X_{i,i} = X_i$ and $f_{i,i} = \mathrm{id}$, that for every i and j we have $f_{j,i} = f_{i,j}^{-1}$, and that for every i, j and l we have $f_{i,j}^{-1}(X_{j,i} \cap X_{j,l}) = X_{i,j} \cap X_{i,l}$ and $f_{i,l} = f_{j,l} f_{i,j}$ on $X_{i,j} \cap X_{i,l}$. Show that there is a C^k-manifold X, and open immersions g_i from X_i in X, such that X is the union of the $g_i X_i$, and such that for all i and j one has $g_j \circ f_{i,j} = g_i$ on $X_{i,j}$.

1.20 Exercise. Give an example of a C^∞-manifold that it not Hausdorff. (See Looijenga's syllabus if necessary.)

1.21 Exercise. Give an example of a C^∞-manifold that does not admit a countable basis for its topology.

2 Vector bundles

Before defining what a vector bundle is, let us study a most important example: the tangent bundle of a manifold. So first we recall what the tangent spaces of a manifold at its points are.

2.1 Tangent spaces

Let $k \geq 1$ and let (X, C_X^k) be a C^k-manifold. For x in X we want to define its tangent space. There are several ways to do this (which are of course equivalent). For a detailed discussion of all of those I know, see Spivak's book, Volume I, Chapter 3. We will discuss some of them. Intuitively, the tangent space of X at a point x in X is the first order approximation of X at x. We need it in order to speak of the derivatives of morphisms of manifolds.

Suppose that we have an atlas (X, I, n, U, ϕ) for X. Let x be in X. A tangent vector v at x will then be a compatible system of pairs (i, v_i), with v_i in $\mathbb{R}^{n_i}$, for the i in I such that x is in X_i. The compatibility is defined as follows. Let i and j be in I with $x \in X_i$ and $x \in X_j$. Then the transition isomorphism $\phi_j^{-1}\phi_i$ from $U_{i,j}$ to $U_{j,i}$ has the property that:

(2.1.1) $$(D(\phi_j^{-1}\phi_i))(\phi_i^{-1}x) \text{ sends } v_i \text{ to } v_j.$$

Since for every such pair (i, j) the map $(D(\phi_j^{-1}\phi_i))(\phi_i x)$ is an isomorphism of $\mathbb{R}$-vector spaces from $\mathbb{R}^{n_i}$ to $\mathbb{R}^{n_j}$, it is clear that a compatible system of (i, v_i) is determined by any of its elements, and that such an element can be arbitrary in $\mathbb{R}^{n_i}$. So to give such a compatible system, it is equivalent to give, for one i in I with $X_i \ni x$, an element v_i of $\mathbb{R}^{n_i}$. In particular, the set of such compatible systems, that we call the tangent space of X at x and that we denote $\mathrm{T}_X(x)$, has a natural structure of $\mathbb{R}$-vector space and is, via this construction, isomorphic to the $\mathbb{R}$-vector space $\mathbb{R}^{n_i}$.

Our second description of $\mathrm{T}_X(x)$ uses parametrized curves, and does not require charts. Let x be in X. A parametrized curve at x is a differentiable map $c\colon U \to X$ with $U \subset \mathbb{R}$ an open interval containing zero and with $c(0) = x$. We want to define the tangent space at x as the set of equivalence classes of such curves, where c_1 and c_2 are to be equivalent if and only if they give the same tangent vector. Of course, we do not want to use the previous definition in terms of charts, so we want another way to say that c_1 and c_2 define the same tangent vector. One way to do this is the following. Let $c\colon U \to X$ be a parametrized curve at x, and f in $C_X(V)$ with V an open neighborhood of x. Then, after shrinking U if necessary, $c^*f := fc$ is a differentiable function on U; let $(fc)'(0)$ be its derivative at 0. Then we say that c_1 and c_2 are equivalent if for all open neighborhoods V of x in X and all f in $C_X(V)$ we have $(fc_1)'(0) = (fc_2)'(0)$. Now we have some work to do: we have to show that this relation is an equivalence relation, and that the set of equivalence classes is in some natural way (this will be made precise below) an $\mathbb{R}$-vector space and as such isomorphic to the one defined above. The relation is clearly an equivalence relation. Let c be a parametrized curve at x, and i in I with $X_i \ni x$. Then $\phi_i^{-1}c$ is defined at 0; we get an element $v_i := (D(\phi_i^{-1}c))0$ of $\mathbb{R}^{n_i}$. One checks immediately that this is a compatible system of (i, v_i) in the sense explained above. If c_1 and c_2 are equivalent, then they give the same v_i (view ϕ_i^{-1} as an $n_i$tuple of functions). Suppose now that c_1 and c_2 give the same v_i. We want to show that c_1 and c_2 are equivalent. For doing this, we may suppose that $X = U_i$ and that $\phi_i = \mathrm{id}_X$. Then we know that $(Dc_1)0 = (Dc_2)0$ (consider partial derivatives). But then we have, for f in

$C_X(V)$, that

(2.1.2) $$(fc_1)'0 = ((Df)x)((Dc_1)0) = ((Df)x)((Dc_2)0) = (fc_2)'0.$$

This shows that, as a set, the set of equivalence classes of c is the same as $\mathrm{T}_X(x)$ as above, so we will use the same notation for both. The $\mathbb{R}$-vector space structure on $\mathrm{T}_X(x)$ has the following interpretation (we denote the class of a curve c by $[c]$): $[c_1]+[c_2]=[c_3]$ if and only if for all V and f we have $(fc_1)'(0)+(fc_2)'(0)=(fc_3)'(0)$. Likewise: $a[c_1]=[c_2]$, for a in $\mathbb{R}$, if and only if $a(fc_1)'(0)=(fc_2)'(0)$ for all f. So indeed we have a description of $\mathrm{T}_X(x)$ that does not use charts.

From the previous description it is just a small step to the third and last one. But in order to define it, it is really convenient to use the notions of germ and stalk. So here follows a short intermezzo.

Suppose that X is a topological space, F a sheaf (of sets) on X and x an element of X. The stalk F_x of F at x is then defined as the direct limit $\varinjlim F(U)$ taken over all open neighborhoods U of x (see for example in Hartshorne's book for the notion of direct limit). Concretely, this means that F_x is the set of equivalence classes of pairs (U,s), with U an open neighborhood of x and s in $F(U)$, for the following equivalence relation. Two such pairs (U_1,s_1) and (U_2,s_2) are equivalent if and only if there is an open neighborhood V of x contained in $U_1\cap U_2$, such that the elements $s_1|_V$ and $s_2|_V$ are equal. The elements of F_x are called germs of sections of F, and the class s_x of (U,s) is called the germ of s at x. If F is a sheaf of $\mathbb{R}$-algebras, then F_x is an $\mathbb{R}$-algebra.

Let us now go back to our tangent spaces. So X is again a C^k-manifold with $k\geq 1$, and x is in X. For c a parametrized curve at x, we get a map $\partial_c\colon C^k_{X,x}\to\mathbb{R}$ defined by: $\partial_c(f)=(fc)'0$ (note that this makes sense). This map satisfies the following properties:

1. it is $\mathbb{R}$-linear;
2. $\partial_c(fg)=f(x)\partial_c(g)+g(x)\partial_c(f)$ (Leibniz's or product rule);
3. $\partial_c(f)=0$ if $(\phi^*f)y=o(||y||)$, for ϕ a chart at x such that $\phi^{-1}(x)=0$.

For A a commutative ring, B a commutative A-algebra and M a B-module, an A-linear derivation from B to M is an A-linear map $D\colon B\to M$ satisfying $D(bb')=D(b)b'+b'D(b)$ for all b and b' in B. The set $\mathrm{Der}_A(B,M)$ of such maps is clearly a B-module: $(bD+D')b'=bD(b')+D'(b')$. Hence the first two of the properties of ∂_c above say that ∂_c is in $\mathrm{Der}_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$.

We let $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$ be the set of maps $\partial\colon C^k_{X,x}\to\mathbb{R}$ that satisfy:

1. ∂ is $\mathbb{R}$-linear;
2. $\partial(fg)=f(x)\partial(g)+g(x)\partial(f)$;
3. $\partial_c(f)=0$ if $(\phi^*f)y=o(||y||)$, for ϕ a chart at x such that $\phi^{-1}(x)=0$.

To emphasize the third condition, we use the notation Der' in stead of Der. Since $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$ is closed under $\mathbb{R}$-linear combinations, it is an $\mathbb{R}$-vector space.

By definition, we have an injective $\mathbb{R}$-linear map from $\mathrm{T}_X(x)$ to $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$, sending $[c]$ to ∂_c. We claim that this map is an isomorphism. To prove that, it suffices to show that every element of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x},\mathbb{R})$ is of the form ∂_c. Since everything is defined intrinsically, and the question is local, we may suppose that X is an open subset of $\mathbb{R}^n$ and that $x=0$. Let ∂ be in

$\mathrm{Der}'_{\mathbb{R}}(C^k_{X,0},\mathbb{R})$. We claim that $\partial = \sum_{i=1}^n \partial(x_i)\partial_i$, with ∂_i be the element of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,0},\mathbb{R})$ that sends f to its ith partial derivative at 0. The elements ∂_i are linearly independent because of the relations $\partial_i x_j = \delta_{i,j}$ (where x_j is the jth coordinate function, and $\delta_{i,j}$ the Kronecker symbol). To prove the identity, let f be in $C^k_{X,0}$. Rewriting the equality:

$$\int_0^1 \left(\frac{d}{dt}f(tx)\right) dt = f(x) - f(0),$$

with x in some neighborhood of 0, gives:

$$f(x) = f(0) + \sum_{i=1}^n x_i \int_0^1 (D_i f)(tx)dt = f(0) + \sum_{i=1}^n \partial_i(f)x_i + \sum_{i=1}^n x_i g_i(x) = f(0) + \sum_{i=1}^n \partial_i(f)x_i + g(x),$$

with D_i the ith partial derivative, and $g_i(x) = \int_0^1((D_i f)(tx) - (D_i f)(0))dt$. As g is the difference of two C^k-functions, it is C^k. Since the $D_i f$ are C^{k-1}-functions, the g_i are C^{k-1}. As $g_i(0) = 0$, we see that $g(x) = o(||x||)$. Applying ∂ to the last identity gives the desired result:

$$\partial = \sum_{i=1}^n \partial(x_i)\partial_i.$$

Now we know that the ∂_i form a basis of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,0},\mathbb{R})$. It follows that our map from $\mathrm{T}_X(x)$ to $\mathrm{Der}'_{\mathbb{R}}(C_{X,x},\mathbb{R})$ is an isomorphism.

In the case $k = \infty$, the g_i above are C^∞-functions, hence, without assuming that ∂ satisfies property 3, one can apply ∂ to the g_i, and get that $\partial(x_i g_i) = 0$ for all i, and hence the same conclusion: $\partial = \sum_{i=1}^n \partial(x_i)\partial_i$. Hence in the C^∞-case we see that $\mathrm{T}_X(x) = \mathrm{Der}_{\mathbb{R}}(C^\infty_{X,x},\mathbb{R})$.

Let now $f\colon X \to Y$ be a morphism of C^k-manifolds, with $k \geq 1$, and x in X. Then we get an $\mathbb{R}$-linear map $\mathrm{T}_f(x)\colon \mathrm{T}_X(x) \to \mathrm{T}_Y(fx)$ as follows. In the first description of tangent spaces, the map is $(D(\psi_j^{-1} f \phi_i))(\phi_i^{-1}x)$ (in the notation of Definition 1.3). In the second description it is the map that sends $[c]$ to $[fc]$. In the last description it is the map that sends ∂ to $\partial \circ f^*$, with $f^*\colon C_{Y,fx} \to C_{X,x}$ given by $g \mapsto gf$. It is left to the reader to show that these three maps are compatible with respect to the identifications between the various kinds of tangent spaces. The map $\mathrm{T}_f(x)$ is called the tangent map of f at x, or also the derivative of f at x. One can say that the main purpose of defining tangent spaces is just to have these tangent maps. In the same way, the main purpose of defining the tangent bundle is to have a tangent map T_f for all x at once.

2.1.3 Exercise. Let n and m be positive integers, and $k \geq 1$. Let $U \subset \mathbb{R}^n$ be open, and $f\colon U \to \mathbb{R}^m$ be a C^k-map. As in Section 1, let $X := f^{-1}0$, suppose that f is a submersion at all x in X, and consider X as a C^k-manifold. Let $i\colon X \to U$ be the inclusion map. Show that for x in X the map $\mathrm{T}_i(x)$ identifies $\mathrm{T}_X(x)$ with $\ker(\mathrm{T}_f(x))$.

2.2 Vector bundles, the tangent bundle

Let X be a C^k-manifold with $k \geq 1$. We want to make an object T_X, called the tangent bundle of X, that combines all the $\mathrm{T}_X(x)$. As a set, T_X is just the disjoint union of all the $\mathrm{T}_X(x)$, $x \in X$. But in order to have something useful, for example second derivatives of morphisms of manifolds, we need to equip T_X with the structure of a manifold, reflecting the fact that it is a disjoint union of vector spaces. The notion of vector bundle is made exactly for doing this. Note that we have a canonical map $p\colon \mathrm{T}_X \to X$, such that the fibre $p^{-1}\{x\}$ over x is $\mathrm{T}_X(x)$. The following definition is meant to be a warming up for what comes after it.

2.2.1 Definition. *Let $k \geq 0$ and $p\colon E \to X$ be a morphism of C^k-manifolds. Then p is called a C^k-fibration if for every x in X there exists an open neighborhood U, a C^k-manifold F and an isomorphism $\phi\colon F \times U \to p^{-1}U$ such that $p \circ \phi$ is the projection $F \times U \to U$. The triple (U, F, ϕ) is called a trivialization over U. A fibration is called trivial if there exists such an isomorphism with $U = X$.*

2.2.2 Remark. The reader is supposed to understand what a product such as $F \times U$ is. See Section 1.

2.2.3 Example. The Möbius strip with its map to the circle is a non-trivial fibration with fibre the open interval $]-1, 1[$. Another example is the Hopf fibration $S^3 \to S^2$ with fibre S^1, obtained from the identifications of $\mathbb{P}^1(\mathbb{C})$ with S^2, of $(\mathbb{C}^2 - \{0\})/\mathbb{R}^*_{>0}$ with S^3, and $\mathbb{C}^*/\mathbb{R}^*_{>0}$ with S^1.

Roughly speaking, a vector bundle is a fibration in which all the fibres are vector spaces (say over $\mathbb{R}$), such that there are local trivializations compatible with the vector space structures.

2.2.4 Definition. *Let $k \geq 0$ and X a C^k-manifold. A real vector bundle over X is a five tuple $(E, p, 0, +, \cdot)$ with $p\colon E \to X$ a C^k-fibration, $(0, +, \cdot)$ the structure of $\mathbb{R}$-vector space on all fibres of p, such that for all x in X there exists a local C^k-trivialization $\phi\colon F \times U \to p^{-1}U$, with $U \ni x$, respecting the vector space structures. A complex vector bundle is defined analogously.*

2.2.5 Remark. If one wants to give a set theoretic meaning to the triple $(0, +, \cdot)$ above, it is the following. The element 0 is a section of p, i.e., it is a map from X to E such that $p \circ 0 = \mathrm{id}_X$. The element $+$ is a map from the fibred product $E \times_X E$ to E. The fibred product $E \times_X E$ is the subset of elements (e_1, e_2) of $E \times E$ with $p(e_1) = p(e_2)$, i.e., it is the set of pairs of elements of E that lie in the same fibre over X. The map $+$ is then of course the sum map of the vector space structure on the fibres. The element $\cdot$ is a map from $\mathbb{R} \times E \to E$ that gives the multiplication in the fibres.

It is certainly possible to equip the sets above with the structure of manifolds, in a natural way. The maps 0, $+$ and $\cdot$ are then morphisms of manifolds.

Now that we know what a vector bundle is, let us construct tangent bundles. So let X be a C^k-manifold with $k \geq 1$. As a set, T_X is the disjoint union of the $\mathrm{T}_X(x)$, for x in X. The map p from T_X to X is the unique map such that $p^{-1}x = \mathrm{T}_X(x)$ for all x. Suppose that we have an atlas (X, I, n, U, ϕ). Then we get an atlas for T_X as follows. For i in I, let $\mathrm{T}_{X,i} := p^{-1}X_i$. Put $V_i := \mathbb{R}^{n_i} \times U_i$ and define $\psi_i\colon V_i \to \mathrm{T}_{X,i}$ by: $\psi_i(v, x) = [(i, v)] \in \mathrm{T}_X(x)$, where $[(i, v)]$ denotes the compatible system corresponding to (i, v) as in our first description of $\mathrm{T}_X(\phi_i x)$. These ψ_i are easily seen to form a C^{k-1}-atlas, since

$$\psi_j^{-1}\psi_i\colon \mathbb{R}^{n_i} \times U_{i,j} \to \mathbb{R}^{n_j} \times U_{j,i}, \quad (v, x) \mapsto (((D(\phi_j^{-1}\phi_i))(x))v, (\phi_j^{-1}\phi_i)x). \tag{2.2.6}$$

The C^{k-1}-manifold thus obtained does not depend on the choice of the atlas (verification left to the reader). It remains now to show that the five tuple $(\mathrm{T}_X, p, 0, +, \cdot)$ is a vector bundle, i.e, that it has local trivializations as in Definition 2.2.4. But such trivializations are given by our maps ψ_i.

Suppose that $f\colon X \to Y$ is a morphism of C^k-manifolds, with $k \geq 1$. Let $\mathrm{T}_f\colon \mathrm{T}_X \to \mathrm{T}_Y$ be the map that is $\mathrm{T}_f(x)$ on $\mathrm{T}_X(x)$. Then T_f is a morphism of C^{k-1}-manifolds, it induces a commutative diagram:

$$
\begin{array}{ccc}
\mathrm{T}_X & \xrightarrow{\mathrm{T}_f} & \mathrm{T}_Y \\
\downarrow^{p_X} & & \downarrow^{p_Y} \\
X & \xrightarrow{f} & Y
\end{array}
\tag{2.2.7}
$$

and it is $\mathbb{R}$-linear on the fibres. This motivates the following definition.

2.2.8 Definition. *Let $f\colon X \to Y$ be a morphism of C^k-manifolds, and let $p_E\colon E \to X$ and $p_F\colon F \to Y$ be C^l-vector bundles (with $l \leq k$ of course). A C^l-morphism from E to F over f is then a morphism of C^l-manifolds $g\colon E \to F$ with $p_F g = f p_E$ that is a morphism of vector spaces on the fibres.*

It is clear that morphisms of vector bundles can be composed, and that we get a category of vector bundles. So finally we can say that associating to a manifold its tangent bundle, and to morphisms their derivative, is a functor T from the category of manifolds to the category of vector bundles.

2.3 Vector bundles as sheaves of modules

In this section we are interested in vector bundles over a fixed manifold X. A morphism of vector bundles over X is a morphism as in (2.2.7) with $f = \mathrm{id}_X$.

Let $p\colon E \to X$ be a real C^l-vector bundle over a C^k-manifold X (with $k \geq l$, of course). In practice one is very often more interested in the sections of E then in E itself. A C^l-section of E over an open subset U of X is a morphism of C^l-manifolds $s\colon U \to E$ such that $ps = \mathrm{id}_U$. The set of C^l-sections of E over U will be denoted $E(U)$. (For example, $\mathrm{T}_X(X)$ is the set of C^{k-1}-vector fields on X.) If $V \subset U$ is an inclusion of open subsets of X, then we have a restriction map from $E(U)$ to $E(V)$. These restriction maps clearly make the system of $E(U)$'s into a sheaf that we will still denote by E. Consider a set $E(U)$. It has the structure of $\mathbb{R}$-vector space, and as such it is usually infinite dimensional. But it also has the structure of a module over the $\mathbb{R}$-algebra $C^l_X(U)$. This structure is compatible with the restriction maps, hence the following definition says that the sheaf E is a sheaf of C^l_X-modules.

2.3.1 Definition. *Let X be a C^k-manifold, and $l \leq k$. A sheaf of C^l_X-modules (or just a C^l_X-module) is then a sheaf $\mathcal{M}$ on X together with the structure, for all open U in X, of $C^l_X(U)$-module on $\mathcal{M}(U)$, compatible with the restriction maps. A morphism of C^l_X-modules is a morphism of sheaves such that on each open subset U of X it gives a morphism of $C^l_X(U)$-modules.*

Associating to a vector bundle its sheaf of sections is a functor from the category of C^l-vector bundles to the category of C^l_X-modules. We will show that this functor induces an equivalence of categories from the category of C^l-vector bundles to the full subcategory of C^l_X-modules that are "locally free of finite rank".

If $\mathcal{M}$ is a C^l_X-module and $U \subset X$ an open subset, then the restriction $\mathcal{M}|_U$ is a C^l_U-module. For $\mathcal{M}_1$ and $\mathcal{M}_2$ two C^l_X-modules we define a presheaf $\mathcal{M}_1 \oplus \mathcal{M}_2$ by: $(\mathcal{M}_1 \oplus \mathcal{M}_2)U = \mathcal{M}_1(U) \oplus \mathcal{M}_2(U)$. This presheaf is in fact a sheaf (exercise), of C^l_X-modules (trivial), with canonical morphisms from $\mathcal{M}_1$ and $\mathcal{M}_2$ to it, such that it is the direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ in the category of C^l_X-modules (exercise). The same then works of course for arbitrary finite direct sums. A finite direct sum of C^l_X-modules has projection morphisms to its components, making it into the direct product of those (exercise).

Suppose now that $\mathcal{M}$ is a C^l_X-module, and that we have global sections $m_1, \ldots, m_r$ of it. Then we get a morphism from $(C^l_X)^r := \oplus_{i=1}^r C^l_X$ to $\mathcal{M}$ such that for all open $U \subset X$ and all $f_1, \ldots, f_r$ in $C^l_X(U)$ the element $(f_1, \ldots, f_r)$ of $(C^l_X)^r(U)$ is sent to $f_1 m_1 + \cdots + f_r m_r$ in $\mathcal{M}(U)$ (we omit the restriction maps). The sequence $(m_1, \ldots, m_r)$ is said to be a basis of $\mathcal{M}$ if this morphism is an isomorphism. A C^l_X-module $\mathcal{M}$ is called free of rank r if it is isomorphic to $(C^l_X)^r$, or, equivalently, if it has a basis with r elements. A C^l_X-module $\mathcal{M}$ is called locally free of rank r (with r a locally constant function on X) if every x in X has an open neighborhood U such that $\mathcal{M}|_U$ is a free C^l_U-module of rank $r(x)$.

2.3.2 Exercise. Let E be a real C^l-vector bundle on X. Show that its sheaf of sections E is locally free of finite rank as C^l_X-module. Show that for $U \subset X$ open and $e_1, \ldots, e_r$ in $E(U)$ the sequence $(e_1, \ldots, e_r)$ is a basis for $E|_U$ if and only if for all x in U the sequence $(e_1(x), \ldots, e_r(x))$ is a basis for the fibre $E(x)$ of E at x.

2.3.3 Theorem. *Let X be a C^k-manifold, and let $l \leq k$. The functor that sends a C^l-vector bundle to its sheaf of sections, viewed as a C^l_X-module, is an equivalence from the category of C^l-vector bundles to the full subcategory of the category of C^l_X-modules consisting of the C^l_X-modules that are locally free of finite rank. A quasi-inverse of this functor is described in the proof below.*

Proof. We will first describe a functor G from the category of locally free C^l_X-modules of finite rank to the category of C^l-vector bundles on X and then show that it is a quasi-inverse of the functor F mentioned in the Theorem (i.e., FG and GF are isomorphic to the identity funtors of the two categories in question).

So let $\mathcal{M}$ be a locally free C^l_X-module of finite rank r. Let x be in X. We consider the stalk $C^l_{X,x}$ of C^l_X at x and the stalk $\mathcal{M}_x$ of $\mathcal{M}$ at x. It follows from the definition of stalk that $\mathcal{M}_x$ is a $C^l_{X,x}$-module. Since $\mathcal{M}$ is locally free of rank r, its stalk $\mathcal{M}_x$ is a free $C^l_{X,x}$-module of rank $r(x)$. Let $C^l_{X,x} \to \mathbb{R}$ be the map that sends f to $f(x)$. It is a morphism of $\mathbb{R}$-algebras; let $m_x \subset C^l_{X,x}$ be its kernel. Then we define:

$$E(x) := \mathcal{M}_x / m_x \mathcal{M}_x = \mathbb{R} \otimes_{C^l_{X,x}} \mathcal{M}_x, \quad E := \coprod_{x \in X} E(x).$$

By construction, we get a map $p \colon E \to X$, such that $p^{-1}x = E(x)$. The $E(x)$ are clearly $\mathbb{R}$-vector spaces. Let U be an open subset of X on which $\mathcal{M}$ is trivial; let $m := (m_1, \ldots, m_d)$ be a basis of $\mathcal{M}|_U$. For x in U and s in $\mathcal{M}(U)$, let $s(x)$ be the image of s in $E(x)$. Then for all x in U, the $m_i(x)$ form an $\mathbb{R}$-basis of $E(x)$. Hence we get a bijection:

$$\phi_{U,m} \colon \mathbb{R}^d \times U \to p^{-1}U, \quad (\lambda, x) \mapsto (\sum_i \lambda_i m_i, x).$$

This bijection gives $p^{-1}U$ the structure of a C^l-manifold. One checks that this structure does not depend on the choice of the basis m, since if m' is another basis, one has a (unique) element g in $\mathrm{GL}_d(C^l_X(U))$ such that $m'_i = g m_i$ for all i. It is now clear that $p \colon E \to X$, with the $\mathbb{R}$-vector space structures on the $E(x)$, is a C^l-vector bundle. From the construction it is also clear that a morphism between C^l_X-modules that are locally free of finite rank induces a morphism of C^l-vector bundles on X. We have thus defined our functor G.

To finish the proof, we have to show that GF (resp., FG) is isomorphic to the identity functor of the category of C^l-vector bundles (resp., the category of locally free C^l_X-modules of finite rank).

Let E be a C^l-vector bundle on X. Then $F(E)$ is the sheaf of sections of E. From the definition of $F(E)$ and the definition of stalk, it follows that we have a map $F(E)_x \to E(x)$ that sends s to $s(x)$. This map of $C^l_{X,x}$-modules is surjective and its kernel is $m_xF(E)_x$ (use a local trivialization of E at x). Hence $E(x)$ is canonically isomorphic to $F(E)_x/m_xF(E)_x$, i.e., to $(GF(E))(x)$. It is left to the reader to verify that this fibre wise isomorphism between E and $(GF)E$ is an isomorphism of C^l-vector bundles, and that it is functorial.

Let $\mathcal{M}$ be a locally free C^l_X-module of finite rank. Let U be an open subset of X, and s in $\mathcal{M}(U)$. For x in U, let $s(x)$ be the image of s in $(G\mathcal{M})(x)$. Then $x \mapsto s(x)$ is a C^l-section of $G\mathcal{M}$ over U (the verification of this, which can be done locally, is left to the reader). Hence we have a map from $\mathcal{M}(U)$ to $(FG\mathcal{M})U$. It is again left to the reader to check that these maps define an isomorphism of C^l_X-modules, and that this isomorphism is functorial. $\square$

3 Tensor constructions

In multi-linear algebra, there are constructions that associate, to a given collection of vector spaces, a vector space. For example, to a k-vector space V (k a field) one can associate its dual $V^\vee := \mathrm{Hom}_k(V,k)$. For k-vector spaces V and W, one has $\mathrm{Hom}_k(V,W)$ and $V \otimes_k W$. We will show that the constructions in the examples carry over to vector bundles, and to their sheaves of sections. Before we do that, we recall some facts about tensor products, the symmetric algebra and the exterior algebra, mainly for free modules over a ring (that is assumed to be commutative, as usual). As a reference for multi-linear algebra one can consult any algebra book, for example Lang's "Algebra", Bourbaki, or Jacobson's "Basic algebra I and II".

3.1 Multi-linear algebra

Let A be a commutative ring. For A-modules M and N we have the A-module $M \otimes_A N$, called the tensor product of M and N over A. This A-module $M \otimes_A N$ is defined as follows: we have a universal A-bilinear map $M \times N \to M \otimes_A N$, denoted by $(m,n) \mapsto m \otimes n$. (This means that for all A-bilinear maps $b\colon M \times N \to P$, there exists a unique A-linear map $\overline{b}\colon M \otimes_A N \to P$ such that $b(m,n) = \overline{b}(m \otimes n)$ for all (m,n).) If M and N are free, with bases m_i, $i \in I$, and n_j, $j \in J$, then $M \otimes_A N$ is free and $m_i \otimes n_j$, $(i,j) \in I \times J$, is a basis. For an A-module M we define $M^\vee$ to be the A-module $\mathrm{Hom}_A(M,A)$.

3.1.1 Proposition. *Let A be a commutative ring, and M and N A-modules. Then we have an A-linear map $M^\vee \otimes_A N \to \mathrm{Hom}_A(M,N)$ that sends $l \otimes n$ to $m \mapsto l(m)n$. If M is free of finite rank, then this map is an isomorphism of A-modules.*

Proof. Since $m \mapsto l(m)n$ is bilinear in l and n, the required map exists and is unique. Assume now that M is free of some rank r. To prove that the map is an isomorphism, we may suppose that $M = A^r$, because the map is functorial in M. But then we may identify $M^\vee$ with A^r, via the dual basis of the standard basis. Hence $M^\vee \otimes_A N = A^r \otimes_A N = N^r$. On the other hand, $\mathrm{Hom}_A(A^r, N) = N^r$. We leave it to the reader to see that our map is this identification. $\square$

3.1.2 The tensor algebra

Let A be a commutative ring, and M an A-module. For $i \geq 0$ let $\mathrm{T}^i(M) := M^{\otimes i}$ be the ith tensor power of M. One way to define $\mathrm{T}^i(M)$ is to say that we have a universal i-linear map $M^i \to \mathrm{T}^i(M)$, sending $(m_1, \ldots, m_i)$ to $m_1 \otimes \cdots \otimes m_i$. We define the tensor algebra of M to be the A-module $\mathrm{T}(M) := \oplus_{i \geq 0} \mathrm{T}^i(M)$, with the A-algebra structure defined as follows. Let i and j be ≥ 0. Consider the map $M^i \times M^j = M^{i+j} \to \mathrm{T}^{i+j}(M)$. Since this map is i-linear in the first variable, it induces a map $\mathrm{T}^i(M) \times M^j \to \mathrm{T}^{i+j}(M)$ that is linear in the first variable. This last map is j-linear in the second variable, hence induces a map $\mathrm{T}^i(M) \times \mathrm{T}^j(M) \to \mathrm{T}^{i+j}(M)$ that is bilinear and defines our multiplication map. The reader will check that $\mathrm{T}(M)$ becomes an associative graded A-algebra. We have $\mathrm{T}^0(M) = A$ and $\mathrm{T}^1(M) = M$. Let B be an associative A-algebra, and $\phi\colon M \to B$ a morphism of A-modules. Then there exists a unique morphism of A-algebras $\tilde{\phi}\colon \mathrm{T}(M) \to B$ such that the restriction of $\tilde{\phi}$ to $\mathrm{T}^1(M)$ is ϕ. This situation gives us an example of adjoint functors: we have a functor F from the category of associative A-algebras to the category of A-modules, that sends B to B viewed as an A-module, and the functor T in the other direction, such that

$$\mathrm{Hom}_{A-\mathrm{mod}}(M, F(B)) = \mathrm{Hom}_{\mathrm{ass}-A-\mathrm{alg}}(\mathrm{T}(M), B),$$

functorially in M and B. If M is a free A-module of rank n, then $\mathrm{T}^i(M)$ is free of rank n^i (the reader will provide a basis).

3.1.3 The symmetric algebra

Let again A be a commutative ring, and M an A-module. We define the symmetric algebra $\mathrm{S}(M)$ to be the quotient of $\mathrm{T}(M)$ by the ideal generated by all $x \otimes y - y \otimes x$ with x and y in M. Since $\mathrm{T}(M)$ is generated, as A-algebra, by M, $\mathrm{S}(M)$ is a commutative A-algebra. Since the kernel of $\mathrm{T}(M) \to \mathrm{S}(M)$ is generated by homogeneous elements, the grading on $\mathrm{T}(M)$ induces a grading on $\mathrm{S}(M)$. For $i \geq 0$, $\mathrm{S}^i(M)$ is called the ith symmetric product of M, and denoted $\mathrm{Sym}^i_A(M)$. As in the case of the tensor algebra, the functor S, from the category of A-modules to the category A-alg of commutative A-algebras, is the left-adjoint of the forget functor in the opposite direction:

$$\mathrm{Hom}_{A-\mathrm{mod}}(M, B) = \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{S}(M), B).$$

If M is free of rank n, say with basis $m_1, \ldots, m_n$, then the morphism of A-algebras from the polynomial ring $A[x_1, \ldots, x_n]$ to $\mathrm{S}(M)$, sending x_i to m_i, is an isomorphism (use the universal property of $\mathrm{S}(M)$ to define the inverse). In particular, $\mathrm{S}^i(M)$ has basis $m_1^{i_1} \cdots m_n^{i_n}$, $i_1 + \cdots + i_n = i$, hence $\mathrm{S}^i(M)$ is free of rank $\binom{i+n-1}{n-1}$. It is left as an exercise to see that, for any A-module M, the map $M^i \to \mathrm{S}^i(M)$ that sends $(m_1, \ldots, m_i)$ to $m_1 \cdots m_i$ is a universal symmetric i-linear map.

3.1.4 The exterior algebra

This section is somewhat less trivial than the previous two, since as a special case we construct the determinant of a square matrix. Let again A be a commutative ring, and M an A-module. The exterior algebra $\Lambda(M)$ of M is then defined to be the quotient of $\mathrm{T}(M)$ by the ideal generated by the $x \otimes x$, with x in M. Then $\Lambda(M)$ is a graded associative A-algebra. We have $\Lambda^0(M) = A$ and $\Lambda^1(M) = M$. We claim that $\Lambda(M)$ is what is called "graded-commutative":

$$yx = (-1)^{ij} xy, \quad \text{for all } i, j \geq 0,\ x \text{ in } \Lambda^i(M),\ y \text{ in } \Lambda^j(M).$$

To see this, we first note that this is true when i and j are both equal to one, since then $0 = (x+y)(x+y) - xx - yy = xy + yx$. Then it follows for all i and j, since M generates $\Lambda(M)$. The product in $\Lambda(M)$ is called the wedge product, and is sometimes denoted $(x, y) \mapsto x \wedge y$. The exterior algebra $\Lambda(M)$ has the following universal property: if $\phi\colon M \to B$ is an A-linear map from M to an A-algebra B, such that $\phi(m)\phi(m) = 0$ for all m in M, then there exists a unique morphism of A-algebras $\tilde{\phi}$ from $\Lambda(M)$ to B whose restriction to M is ϕ. We leave it as an exercise to show that the map $M^i \to \Lambda^i(M)$ that sends $(m_1, \ldots, m_i)$ to $m_1 \cdots m_i$ is a universal alternating i-linear map. If M is free of rank n, say with basis $m_1, \ldots, m_n$, then $\Lambda^i(M)$ is free of rank $\binom{n}{i}$, and $m_{j_1} \cdots m_{j_i}$, $j_1 < \cdots < j_i$, is a basis. Since this is not so obvious, we will give a proof.

So assume that $m_1, \ldots, m_n$ is a basis of M. Let $i \geq 0$. It is clear that the $m_{j_1} \cdots m_{j_i}$, $j_1 < \cdots < j_i$, generate $\Lambda^i(M)$, so it remains to show that they are linearly independent. This follows if we can construct, for all $j = (j_1, \ldots, j_i)$, with $j_1 < \cdots < j_i$, an A-linear map ϕ_j from $\Lambda^i(M)$ to A, such that for all $k = (k_1, \ldots, k_i)$ with $k_1 < \cdots < k_i$ we have $\phi_j(m_{k_1} \cdots m_{k_i}) = \delta_{j,k}$.

Note that $\mathrm{T}^i(M)$ has basis $m_{l_1}\cdots m_{l_i}$, $l_k \in \{1,\ldots,n\}$. Define an A-linear map ϕ_j from $\mathrm{T}^i(M)$ to A by:

$$\begin{aligned}\phi_j(m_{l_1}\cdots m_{l_i}) &= 0 \quad \text{if } \{l_1,\ldots,l_i\} \neq \{j_1,\ldots,j_i\},\\ \phi_j(m_{l_1}\cdots m_{l_i}) &= \varepsilon(\sigma) \quad \text{if } \{l_1,\ldots,l_i\} = \{j_1,\ldots,j_i\},\end{aligned}$$

where σ is the permutation sending j_k to l_k (ε is the sign of a permutation). Then ϕ_j induces the desired alternating i-linear form on M.

3.1.5 Exercise. Let k be a field, V a k-vector space and $d \geq 0$ an integer. Recall that we have defined the Grassmannian $\mathrm{Gr}_d(V)$ of d-dimensional subspaces of V, with the structure of C^∞-manifold if $k = \mathbb{R}$ and V is finite dimensional.

Let W be a d-dimensional subspace of V, and let $w_1,\ldots,w_d$ be a basis of W. Then we get an element $w_1\cdots w_d$ of $\Lambda^d(V)$. Show that this element is non-zero, and that its image in $\mathbb{P}(\Lambda^d(V))$ only depends on W. Hence we have a map ϕ from $\mathrm{Gr}_d(V)$ to $\mathbb{P}(\Lambda^d(V))$. Show that this map is injective. It is called the Plücker embedding.

3.2 Tensor products of vector bundles and locally free sheaves

Let X be a C^k-manifold ($k \geq 0$), and $p\colon E \to X$ a real (or complex) C^l-vector bundle on it (with $l \leq k$). Then we define the dual $E^\vee$ of E, which is also a real (or complex) C^l-vector bundle, as follows. As a set, $E^\vee$ is the disjoint union of the $E(x)^\vee$, x in X. Let $q\colon E^\vee \to X$ be the map with $q^{-1}x = E(x)^\vee$. To give $E^\vee$ the structure of a C^l-vector bundle, choose a covering of X by open subsets X_i, and trivializations $\phi_i\colon F_i \times X_i \to p^{-1}X_i$ of E. Then, for x in X_i, we have an isomorphism of vector spaces $\phi_i(x)\colon F_i \to E(x_i)$. Consequently, we have an isomorphism $\psi_i(x) := (\phi_i(x)^\vee)^{-1}$ from $F_i^\vee$ to $E^\vee(x_i)$. For all i, we have a bijection $\psi_i\colon F_i^\vee \times X_i \to q^{-1}X_i$. These bijections give $E^\vee$ the structure of a C^l-vector bundle. One checks that this structure does not depend on the choice of the X_i and the ϕ_i. Defined like this, we have, for each x in X, a bilinear map $\langle\cdot,\cdot\rangle_x$ from $E^\vee(x) \times E(x)$ to $\mathbb{R}$ (or $\mathbb{C}$), given by evaluation. For $U \subset X$ open, s in $E(U)$ and t in $E^\vee(U)$, the function $\langle t,s\rangle_U$ on U that sends x to $\langle t(x), s(x)\rangle_x$ is in $C^l_X(U)$. We want to show that these maps define an isomorphism from the C^l_X-module $E^\vee$ to the dual of E, as C^l_X-module. But in order to do this, we first have to define what such a dual is supposed to be.

3.2.1 Definition. *Let $(Y, \mathcal{O}_Y)$ be a ringed space, $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_Y$-modules. Then we define the presheaf $\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M},\mathcal{N})$ as follows:*

$$\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M},\mathcal{N})(U) = \mathrm{Hom}_{\mathcal{O}_Y|_U}(\mathcal{M}|_U, \mathcal{N}|_U), \quad U \subset Y \text{ open},$$

with the obvious restriction maps. This presheaf is actually a sheaf (the rather long verification is left to the reader), and moreover a $\mathcal{O}_Y$-module. The $\mathcal{O}_Y$-module $\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M},\mathcal{O}_Y)$ is called the dual of $\mathcal{M}$ and will be denoted $\mathcal{M}^\vee$.

We will now first define a morphism of C^l_X-modules from the sheaf of sections of $E^\vee$ to the sheaf $\mathbf{Hom}_{C^l_X}(E, C^l_X)$, and then show that it is an isomorphism. So let $U \subset X$ be open, and t in $E^\vee(U)$. Let $V \subset U$ be open, and s in $E(V)$. Then we have $\langle t|_V, s\rangle_V$ in $C^l_X(V)$. This defines a morphism of $C^l_X(U)$-modules from $E^\vee(U)$ to $\mathrm{Hom}_{C^l_X|_U}(E^\vee|_U, C^l_X|_U)$. One verifies that we get indeed a morphism from $E^\vee$ to $\mathbf{Hom}_{C^l_X}(E, C^l_X)$ as desired. Let us now show that it is an isomorphism. So let $U \subset X$ be open and let ϕ be in $\mathbf{Hom}_{C^l_X}(E, C^l_X)(U)$. Then, for each x in U, ϕ induces, via the equivalence between vector bundles and sheaves of modules (that is, Theorem 2.3.3), an element $\phi(x)$ of $E^\vee(x)$. It is easy to check that these $\phi(x)$ form an element of $E^\vee(U)$.

So we have now seen that the operation $E \mapsto E^\vee$ on C^l-vector bundles corresponds to $E \mapsto \mathbf{Hom}_{C^l_X}(E, C^l_X)$ on C^l_X-modules. Our next goal is to define tensor products, on both sides. Let us first treat the case of vector bundles. So let E and F be C^l-vector bundles on X. We define a vector bundle $E \otimes F$ as follows. As a set, it is the disjoint union of the vector spaces $E(x) \otimes F(x)$, for x in X. The map from it to X is clear. Then $E \otimes F$ is given the structure of C^l-manifold via local trivializations of E and F. Let $U \subset X$ be open, s in $E(U)$ and t in $F(U)$. For each x in U we get $s(x) \otimes t(x)$ in $(E \otimes F)(x)$. These define an element $s \otimes t$ of $(E \otimes F)(U)$. The map from $E(U) \times F(U)$ to $(E \otimes F)(U)$ sending (s, t) to $s \otimes t$ is $C^l_X(U)$-bilinear. Varying U, we get a C^l_X-bilinear map from $E \times F$ to $E \otimes F$. We will prove a bit further that this map is the universal C^l_X-bilinear map from $E \times F$ to C^l_X-modules. Before that, we define the tensor product on the side of sheaves.

3.2.2 Definition. *Let $(Y, \mathcal{O}_Y)$ be a ringed space, $\mathcal{M}$ and $\mathcal{N}$ locally free $\mathcal{O}_Y$-modules of finite rank. By definition, every y in Y has an open neighborhood U on which $\mathcal{M}$ and $\mathcal{N}$ are free, say of ranks m and n. For such open subsets U we put: $\mathcal{T}(U) := \mathcal{M}(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{N}(U)$. For V an open subset of such a U we have a restriction map $\mathcal{T}(U) \to \mathcal{T}(V)$ (note that indeed $\mathcal{M}$ and $\mathcal{N}$ are free on V). Choosing, for such a U, isomorphisms $(\mathcal{O}_Y|_U)^m \to \mathcal{M}|_U$ and $(\mathcal{O}_Y|_U)^n \to \mathcal{N}|_U$, we see that $V \mapsto \mathcal{T}(V)$ is a $\mathcal{O}_Y|_U$-module. The next lemma implies, among other things, that there is a unique $\mathcal{O}_Y$-module $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ on Y such that for all U as above we have $(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U) = \mathcal{T}(U)$. This $\mathcal{O}_Y$-module is locally free of finite rank.*

3.2.3 Lemma. *Let X be a topological space, $\mathcal{U}$ a collection of open subsets of X that covers X and that is a sieve on X (i.e., $U \in \mathcal{U}$ and $V \subset U$ open imply $V \in \mathcal{U}$). A presheaf on $\mathcal{U}$ is defined to be a contravariant functor from $\mathcal{U}$ (morphisms are just the inclusions) to the category of sets. A presheaf F on $\mathcal{U}$ is called a sheaf if, for all U in $\mathcal{U}$, F defines a sheaf on U. Let $\mathrm{Sh}(X)$ and $\mathrm{Sh}(\mathcal{U})$ denote the categories of sheaves on X and $\mathcal{U}$, respectively. Then the functor $F \mapsto F|_{\mathcal{U}}$ from $\mathrm{Sh}(X)$ to $\mathrm{Sh}(\mathcal{U})$ is an equivalence of categories. A quasi-inverse is described in the proof.*

Proof. Let F be a sheaf on $\mathcal{U}$. Let V be an open subset of X. Let $\mathcal{U}|_V$ be the set of U in $\mathcal{U}$ with $U \subset V$. We define $F^+(V)$ to be the projective limit $\varprojlim F(U)$, taken over the U in $\mathcal{U}|_V$. Concretely, this means that an element of $F^+(V)$ is a compatible system of s_U in $F(U)$, indexed by $\mathcal{U}|_V$. Since the restriction of F to any U in $\mathcal{U}$ is a sheaf, we see that $F^+(U) = F(U)$ for such U. It is left to the reader to define the restriction maps for F^+, and to verify that F^+ is a sheaf. The verifications that $F \mapsto F^+$ is a functor, and that it is a quasi-inverse of $G \mapsto G|_{\mathcal{U}}$, are left to the reader. $\square$

3.2.4 Proposition. *Let $(Y, \mathcal{O}_Y)$ be a ringed space and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_Y$-modules. Let $\mathcal{U}$ be the sieve on X consisting of the U on which both $\mathcal{M}$ and $\mathcal{N}$ are trivial. Lemma 3.2.3 implies that the maps:*

$$\mathcal{M}(U) \times \mathcal{N}(U) \to \mathcal{M}(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{N}(U) = (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U)$$

define a morphism of sheaves $\mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ on X. This morphism is a universal $\mathcal{O}_Y$-bilinear map from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{O}_Y$-modules.

Proof. Everything but the universality is clear. So let $\mathcal{P}$ be a $\mathcal{O}_Y$-module and b a bilinear map from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{P}$. Then, for each U in $\mathcal{U}$, we get a unique morphism of $\mathcal{O}_Y(U)$-modules from $(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U)$ to $\mathcal{P}(U)$ that, composed with the universal bilinear map from $\mathcal{M}(U) \times \mathcal{N}(U)$, is $b(U)$. Lemma 3.2.3 shows that these maps give the desired unique morphism of $\mathcal{O}_Y$-modules from $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ to $\mathcal{P}$ that, composed with the bilinear map from $\mathcal{M} \times \mathcal{N}$, is b. $\square$

Let us now go back to our manifolds and tensor products of vector bundles: we had a C^k-manifold X and C^l-vector bundles E and F. We have already defined the C^l-vector bundle $E \otimes F$. It is easy to check that, for an open subset U of X on which E and F are trivial, we have $(E \otimes F)U = E(U) \otimes_{C^l_X(U)} F(U)$ (or, more precisely, the natural map between them is an isomorphism). Lemma 3.2.3 tells us that (the sheaf of sections of) $E \otimes F$ is the same as (more precisely, uniquely isomorphic to) $E \otimes_{C^l_X} F$.

From what we have done up to now, it is clear how to define, for a C^l-vector bundle E on a manifold X, the bundle analogs $\mathrm{T}(E)$, $\mathrm{S}(E)$ and $\Lambda(E)$ of the tensor algebra, the symmetric algebra and the exterior algebra, and that these constructions coincide with their analogs for C^l_X-modules that are locally free of finite rank. (Here we forget for a moment that vector bundles, as we have defined them, have finite dimension.)

In order to consider complex C^l-vector bundles on a C^k-manifold X it suffices to consider the sheaf $C^l_{X,\mathbb{C}}$ of complex valued C^l-functions on X, where, as explained in §1, a $\mathbb{C}$-valued function on X is called C^l if both its real and imaginary part are. Before indulging in differential forms and de Rham cohomology we take a brief look at metrics on vector bundles.

3.3 Metrics on vector bundles

Let X be a C^k-manifold, with $k \geq 0$, and E a C^l-vector bundle on X. There are various ways to describe what a metric on E is. Viewing E as a bundle, a metric on E is a collection of non-degenerate symmetric bilinear forms $\langle\cdot,\cdot\rangle_x$ on the $E(x)$, "varying C^l with x". This last condition means that after local trivialization, the coefficients of the matrix describing the $\langle\cdot,\cdot\rangle_x$ are C^l-functions. Equivalently, for U open in X and s and t in $E(U)$, $\langle s,t\rangle_U : x \mapsto \langle s(x), t(x)\rangle_x$ is in $C^l_X(U)$. Viewing E as a locally free C^l_X-module, a metric is a symmetric bilinear map $b\colon E \times E \to C^l_X$ such that, for all x in X, the symmetric bilinear form $b(x)$ on $E(x)$ is non-degenerate. Equivalently, b induces an isomorphism of C^l_X-modules from E to $E^\vee$. Or also: a metric on E is a symmetric isomorphism of C^l_X-modules from E to $E^\vee$. Considering the universal symmetric bilinear form on E, one sees that a metric on E is an element b of $\mathrm{S}^2(E)^\vee(X)$ such that all $b(x)$ are non-degenerate. To conclude: all various equivalent descriptions of symmetric bilinear forms that one sees in a linear algebra course work in the contexts of vector bundles and C^l_X-modules. For example, a metric b on E has a signature, which is a locally constant function $s\colon X \to \mathbb{Z}^2$ such that $s(x)_1$ (resp., $s(x)_2$) is the number of positive (resp., negative) coefficients of $b(x)$ in any diagonal form.

Usually when working with vector bundles with a metric, the metric comes naturally with the vector bundle. But sometimes it is useful to just choose a metric on a given vector bundle, if one exists (for example, if one wants to split short exact sequences of vector bundles). So a natural question to ask is: under what conditions does a vector bundle E admit a metric b (say with a fixed signature)? We will see, when discussing partitions of unity, that, for X paracompact (i.e., X is separated and every open cover has a locally finite refinement), every vector bundle has a positive definite metric. This has to do with the fact that the set of positive definite symmetric bilinear forms on $\mathbb{R}^n$ is convex. On the other hand, there are topological obstructions against the existence of metrics of signature $(1,1)$, because the set of symmetric bilinear forms on $\mathbb{R}^2$ of that signature is homotopically equivalent to the circle. For example, it can be seen that the tangent bundle of the two-sphere S^2 does not admit a metric of signature $(1,1)$ (namely, from such a metric one can construct a nowhere zero vector field on S^2, and everybody knows that the sphere can't be combed).

Before going on, let us look a bit at what is happening here. So let E be a vector bundle of constant rank r on a manifold X, and let s be a fixed signature. For x in X, the set of metrics of signature s on $E(x)$ is an open subset $Y(x)$ of $\mathrm{S}^2(E(x))^\vee$. The $Y(x)$ are all (non-canonically) isomorphic, as manifolds, to the open subset F of $\mathrm{S}^2(\mathbb{R}^r)^\vee$ consisting of metrics of signature s. Using that E is locally trivial, it is easy to equip the disjoint union Y of the $Y(x)$ with the structure of a fibration over X with fibre F. The question of whether or not E has a metric of signature s is then the same as the question of whether or not this fibration has a section. Let us now give two fibrations, with non-empty fibre, that have no section. The first example is the complement of the zero section of the Möbius strip, viewed as a fibration over S^1 with fibre $[-1,1]-\{0\}$. The other example is the Hopf fibration of S^3 over S^2; it is obtained as follows. View $\mathbb{P}^1(\mathbb{C})$ as the two-sphere S^2. Then S^2 is the quotient of $\mathbb{C}^2-\{0\}$ by the action of $\mathbb{C}^*$. View S^3 as the quotient of $\mathbb{C}^2-\{0\}$ by the subgroup $\mathbb{R}^*_{>0}$ of $\mathbb{C}^*$. Then we see that S^2 is the quotient of S^3 by the group $\mathbb{C}^*/\mathbb{R}^*_{>0}$, which is isomorphic to the subgroup S^1 of $\mathbb{C}^*$. Since the action of $\mathbb{C}^*$ on $\mathbb{C}^2-\{0\}$ is free, the action of S^1 on S^3 is so too. Hence we have our fibration. To see that there is no section, note that if there is a section, we get an isomorphism from $S^1\times S^2$ to S^3, but the first of these two is not simply connected whereas the second is.

3.3.1 Definition. *Let X be a C^k-manifold, with $k\geq 1$. A Riemannian metric on X is then a positive definite metric on T_X.*

As we have said, we will show later that every paracompact manifold has a Riemannian metric. Suppose now that X is a C^k-manifold and that $\langle\cdot,\cdot\rangle$ is a Riemannian metric on it. Suppose that $c\colon I\to X$ is C^1, with $I=[a,b]$ some non-empty closed interval in $\mathbb{R}$. Then we can define the length of c as follows:

$$\text{length}(c) := \int_a^b \|c'(t)\|_{c(t)}\,dt, \tag{3.3.2}$$

with $\|\cdot\|$ the norm associated to $\langle\cdot,\cdot\rangle$. The reader should note that when X is $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the standard Riemannian metric, this definition of length coincides with the standard one. An important fact is that if $c_1 := c\circ\phi$ with $\phi\colon J\to I$ a diffeomorphism of closed intervals, say with $J=[a_1,b_1]$, then the length of c_1 equals that of c:

$$\int_{a_1}^{b_1} \|c_1'(s)\|_{c_1(s)}\,ds = \int_{a_1}^{b_1} |\phi'(s)|\,\|c'(\phi(s))\|_{c(\phi(s))}\,ds = \int_a^b \|c'(t)\|_{c(t)}\,dt. \tag{3.3.3}$$

This implies that the length of a curve c is independent of the choice of the parametrization. Suppose now moreover that X is connected. Then X is arcwise connected, hence we can define a real valued function d on $X\times X$ by:

$$d(x,y) := \inf\{\text{length}(c)\mid c \text{ a smooth curve from } x \text{ to } y\}. \tag{3.3.4}$$

It is quite clear that d is symmetric and that it satisfies the triangle inequality. One can show without too much pain that if X is separated, then one has $d(x,y)=0$ if and only if $x=y$. See for example Chapter 9 of Spivak, Volume 1. The problem of finding the shortest path between two given points leads to variational calculus and to the definition of a geodesic.

Not all metrics that arise naturally are positive definite. For example, in the theory of general relativity one studies four-dimensional manifolds with a metric of signature $(1, 3)$; so-called Lorentzian manifolds. The path from x to y that corresponds to a free fall is then a path of maximal length from x to y (of course one only considers paths that do respect the speed limit imposed by the speed of light, because otherwise the square root in the definition of $\|\cdot\|$ becomes imaginary). There is an excellent book on this matter, by Sachs and Wu, with the title "General relativity for mathematicians".

To finish this section: not even all bilinear forms that occur naturally on vector bundles are symmetric. For example, anti-symmetric bilinear forms, also called symplectic forms, play an important role in classical mechanics (Hamilton systems). Here the reader should think of formulas such as $\sum_i dp_i \wedge dq_i$.

3.4 Differential forms

Let X be a C^k-manifold, with $k \geq 1$. Then we have the C^{k-1}-vector bundle T_X on X. The dual $\mathrm{T}_X^\vee$ of T_X is called the bundle of 1-forms on X, and is denoted Ω_X^1. Note that with our conventions, Ω_X^1 also denotes the sheaf of sections of Ω_X^1. We define Ω_X to be $\Lambda(\Omega_X^1)$. This Ω_X is a graded-commutative C_X^{k-1}-algebra, and its degree i component Ω_X^i is called the sheaf or bundle of i-forms. We will do two important things in the next two sections: we define the usual morphisms of sheaves $d\colon \Omega_X^i \to \Omega_X^{i+1}$ that give us the de Rham complex of X, and we define what integration of forms of top degree is.

Our first step is to define a map $d\colon C_X^k \to \Omega_X^1$. But even to do this, we have to go back to T_X itself: namely, we have to give an interpretation of $\mathrm{T}_X(U)$ for $U \subset X$ open. So let $U \subset X$ open, and let ∂ in $\mathrm{T}_X(U)$. We will show that ∂ defines a derivation from $C_X^k|_U$ to $C_X^{k-1}|_U$. So let $V \subset U$ be open, and let f be in $C_X^k(V)$. Let x be in V. Then we have $\partial(x)$ a tangent vector at x, and f_x in the stalk $C_{X,x}^k$. We define: $(\partial f)x := \partial(x) f_x$, which is in $\mathbb{R}$. No matter how we view tangent vectors, this number is simply the derivative at x of f in the direction $\partial(x)$. Looking in a chart, it is clear that the function ∂f from V to $\mathbb{R}$ is in $C_X^{k-1}(V)$, that ∂ is indeed a morphism of sheaves from $C_X^k|_U$ to $C_X^{k-1}|_U$ and that it is a derivation: $\partial(fg) = f\partial(g) + \partial(f)g$. Moreover, for every x in U, the map $\partial(x)\colon C_{X,x}^k \to \mathbb{R}$ that sends f to $(\partial f)x$ is in $\mathrm{Der}'_{\mathbb{R}}(C_{X,x}^k, \mathbb{R})$ (notation as in Section 2). We define $\mathbf{Der}'_{\mathbb{R}}(C_X^k, C_X^{k-1})$ to be the sheaf on X whose sections over U are the morphisms of sheaves $\partial\colon C_X^k|_U \to C_X^{k-1}|_U$ that are $\mathbb{R}$-derivations such that for every x in X the map $C_{X,x}^k \to \mathbb{R}$, $f \mapsto (\partial f)(x)$, is in $\mathrm{Der}'_{\mathbb{R}}(C_{X,x}^k, \mathbb{R})$. Then the construction that we have just given is an isomorphism of C_X^{k-1}-modules:

$$\mathrm{T}_X \longrightarrow \mathbf{Der}'_{\mathbb{R}}(C_X^k, C_X^{k-1}).$$

We can now define our map $d\colon C_X^k \to \Omega_X^1$. Let $U \subset X$ be open and let f be in $C_X^k(U)$. By construction, $\Omega_X^1(U)$ is equal to $\mathbf{Hom}_{C_X^{k-1}}(\mathrm{T}_X, C_X^{k-1})(U)$. We define df to be the element in $\Omega_X^1(U)$ that sends ∂ in $\mathrm{T}_X(V)$, with $V \subset U$ open, to ∂f. The reader will verify that d is a morphism of sheaves, that it is $\mathbb{R}$-linear and that it satisfies:

$$d(fg) = f\,dg + g\,df, \tag{3.4.1}$$

for U open in X and f and g in $C_X^k(U)$. Intuitively, the expression $(df)x$ can be thought of as a measure for the infinitesimal rate of change of f at x in an unspecified direction, and that, when evaluated on a tangent vector at x, it gives the derivative at x of f in that direction. The map d itself can be thought of as a universal derivation from C^k_X to C^{k-1}_X-modules satisfying the third property in the definition of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x}, \mathbb{R})$ at all x. One should note that the morphism of sheaves $d\colon C^k_X \to \Omega^1_X$ is not a morphism of vector bundles (except in the case where Ω^1_X is zero, of course), because it is not C^k_X-linear (it is a derivation, after all).

3.4.2 Proposition. *Let X be a C^k-manifold with $k \geq 1$. Let $U \subset X$ be an open set and $x_1, \ldots, x_n$ local coordinates on U, i.e., the x_i are in $C^k_X(U)$ and the map $x\colon U \to \mathbb{R}^n$ sending u to $(x_1(u), \ldots, x_n(u))$ is an isomorphism of C^k-manifolds from U to an open subset V of $\mathbb{R}^n$. Then $\Omega^1_X|_U$ is a free C^{k-1}_U-module and $(dx_1, \ldots, dx_n)$ is a basis. This basis is the dual basis of the basis $D_1, \ldots, D_n$ of $\mathrm{T}_X|_U$ given by the partial derivatives. For f in $C^k_X(U)$ we have the formula:*

$$df = \sum_{i=1}^{n} (D_i f) dx_i.$$

Proof. The fact that the D_i form a basis of $\mathrm{T}_X|_U$ was proved, point-wise, in §2.1. By construction, the dx_i form the dual basis. The formula above follows from evaluating both sides on the D_j. □

3.4.3 Corollary. *Let X be a C^k-manifold with $k \geq 1$. Let $U \subset X$ be open, $x_1, \ldots, x_n$ local coordinates on U, and $r \geq 0$ an integer. Then $\Omega^r_X|_U$ is a free C^{k-1}_U-module and the $dx_{i_1} \cdots dx_{i_r}$ with $i_1 < \cdots < i_r$ form a basis.*

3.4.4 Proposition. *Let X be a C^∞-manifold. There exists a unique morphism of sheaves $d\colon \Omega_X \to \Omega_X$ such that:*

1. *d is $\mathbb{R}$-linear and maps Ω^i_X to Ω^{i+1}_X;*
2. *the restriction of d to Ω^0_X is $d\colon C^\infty_X \to \Omega^1_X$;*
3. *for $U \subset X$ open, x in $\Omega^r_X(U)$ and y in $\Omega^s_X(U)$, we have $d(xy) = (dx)y + (-1)^r x dy$;*
4. *$d^2 = 0$.*

Proof. Since Ω^1_X generates Ω_X as a C^∞_X-algebra, there exists at most one such morphism. Because of this uniqueness, it suffices to prove the existence locally. So we may and do assume that X is an open subset of $\mathbb{R}^n$. Let $r \geq 0$ be an integer. The $dx_{i_1} \cdots dx_{i_r}$ with $i_1 < \cdots < i_r$ form a C^∞_X-basis of Ω^r_X. Conditions 2 and 4 of the proposition we are proving imply that we must define

$$d(f dx_{i_1} \cdots dx_{i_r}) = df\ dx_{i_1} \cdots dx_{i_r} = \sum_{i=1}^{n} (D_i f) dx_i dx_{i_1} \cdots dx_{i_r}. \tag{3.4.5}$$

Let us now show that the morphism d defined by this formula satisfies all the conditions of the proposition. Conditions 1 and 2 are clearly satisfied. Let us now do 4. One computes:

$$\begin{aligned} d(d(f dx_{i_1} \cdots dx_{i_r})) &= d(\sum_i (D_i f) dx_i dx_{i_1} \cdots dx_{i_r}) = \sum_i d(D_i f) dx_i\ dx_{i_1} \cdots dx_{i_r} = \\ &= \sum_i \sum_j D_j(D_i f) dx_j\ dx_i\ dx_{i_1} \cdots dx_{i_r} = \\ &= \left(\sum_{i,j} D_j(D_i f) dx_i\ dx_j \right) dx_{i_1} \cdots dx_{i_r} = 0. \end{aligned}$$

To prove 3, we may write $x = f dx_{i_1} \cdots dx_{i_r}$ and $y = g dx_{j_1} \cdots dx_{j_s}$. One computes:

$$\begin{aligned} d(xy) &= d(fg dx_{i_1} \cdots dx_{i_r}\, dx_{j_1} \cdots dx_{j_s}) = (f dg + g df) dx_{i_1} \cdots dx_{i_r}\, dx_{j_1} \cdots dx_{j_s} = \\ &= df\, dx_{i_1} \cdots dx_{i_r}\, g dx_{j_1} \cdots dx_{j_s} + (-1)^r f dx_{i_1} \cdots dx_{i_r}\, dg\, dx_{j_1} \cdots dx_{j_s} = \\ &= dx\, y + (-1)^r x dy. \end{aligned}$$

□

3.4.6 Remark. Note that the previous Proposition only talks about the C^∞-case. It is certainly possible to formulate an analogous result for $k \geq 2$, but I find that too much of a hassle.

Let us consider the case where X is open in $\mathbb{R}^3$. For f in $C_X^k(X)$ with $k \geq 1$ one has $df = (D_1 f)dx_1 + (D_2 f)dx_2 + (D_3 f)dx_3$, which is just an expression for the gradient of f. The reader should verify that $d(f_1 dx_1 + f_2 dx_2 + f_3 dx_3)$ gives the curl and $d(f_1 dx_2 dx_3 + f_2 dx_3 dx_1 + f_3 dx_1 dx_2)$ the divergence. Usually, in calculus, the gradient of a function is a vector field, and not a one-form; this comes from the identification between T_X and Ω_X^1 given by the standard Riemannian metric. Likewise, in calculus one applies divergence to vector fields, not to two-forms; here one uses that the multiplication $\Omega_X^2 \times \Omega_X^1 \to \Omega_X^3$ is a perfect pairing (i.e., it identifies both sides with the dual of the other). A similar remark holds for the curl.

3.4.7 Exercise. Let X be a C^k-manifold with $k \geq 0$. Show that X is the disjoint union (as a topological space) of its arcwise connected components; let $\pi_0(X)$ denote the set of connected components of X. Assume now that $k \geq 1$. Show that the kernel of $d\colon C_X^k \to \Omega_X^1$ is the constant subsheaf $\mathbb{R}_X$ of C_X^k (i.e., the subsheaf of locally constant functions). Conclude that:

$$\ker(d\colon C_X(X) \to \Omega_X^1(X)) = \mathbb{R}^{\pi_0(X)}.$$

3.4.8 Remark. We will see later that the complex of sheaves (Ω_X, d) is exact in all degrees $i > 0$. This means that (Ω_X, d) is a resolution of the sheaf $\mathbb{R}_X$. We will show that the sheaves Ω_X^i are acyclic for the functor $\Gamma(X, \cdot)$ when X is paracompact. It follows that under that condition, the complex $(\Omega_X(X), d)$ computes the cohomology of $\mathbb{R}_X$. Since the cohomology of this complex is by definition the de Rham cohomology, we see that the de Rham cohomology of X is the cohomology of $\mathbb{R}_X$.

3.5 Volume forms, integration and orientation

Thinking about what kind of objects one can expect to be able to integrate over manifolds, one comes to the following definition.

3.5.1 Definition. *Let V be a finite dimensional $\mathbb{R}$-vector space, say of dimension n. Let W be an $\mathbb{R}$-vector space. A volume form on V with values in W is then a map $v\colon V^n \to W$ such that:*

1. $v(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = v(v_1, \ldots, v_n)$, *for all* $v_1, \ldots, v_n$ *in* V *and* σ *in* S_n;
2. $v(\lambda v_1, v_2 \ldots, v_n) = |\lambda| v(v_1, \ldots, v_n)$, *for all* $v_1, \ldots, v_n$ *in* V *and* λ *in* $\mathbb{R}$;
3. $v(v_1 + v_2, v_2, \ldots, v_n) = v(v_1, \ldots, v_n)$, *for all* $v_1, \ldots, v_n$ *in* V.

All volume forms as in the definition are obtained as follows (proof left to the reader). Let w be in W and l in $\Lambda^n(V)^\vee$. Then the map v defined by $v(v_1,\ldots,v_n) := |l(v_1,\ldots,v_n)|\,w$ is a volume form. It follows that for v a volume form, $v_1,\ldots,v_n$ in V and g in $\mathrm{GL}(V)$, we have $v(g(v_1),\ldots,g(v_n)) = |\det(g)|\,v(v_1,\ldots,v_n)$. The set $\mathrm{Vol}(V,W)$ of W-valued volume forms on V his itself an $\mathbb{R}$-vector space (sum and scalar multiplication are defined as usual). For $f\colon W \to W'$ an $\mathbb{R}$-linear map of $\mathbb{R}$-vector spaces, we get an $\mathbb{R}$-linear map f_* from $\mathrm{Vol}(V,W)$ to $\mathrm{Vol}(V,W')$, that sends v to $f \circ v$. In fact, $\mathrm{Vol}(V,\cdot)$ is a covariant functor.

3.5.2 Lemma. *Let V be a finite dimensional $\mathbb{R}$-vector space, and let n be its dimension. Then the functor $\mathrm{Vol}(V,\cdot)$ is representable, by a one-dimensional $\mathbb{R}$-vector space that we denote $|\Lambda^n(V)|$. Equivalently, we have a universal volume form $V^n \to |\Lambda^n(V)|$.*

Proof. Take l to be a non-zero element of $\Lambda^n(V)$, and consider the $\mathbb{R}$-valued volume form $|l|$. The discussion above shows that this volume form is universal. □

The set of $\mathbb{R}$-valued volume forms on V is $|\Lambda^n(V)|^\vee$. An $\mathbb{R}$-valued volume form is called positive if all its values are ≥ 0. We have a map $\Lambda^n(V)^\vee \to |\Lambda^n(V)|^\vee$ that sends l to $|l| := |\cdot| \circ l$. The image of this map is the set of positive volume forms.

3.5.3 Definition. *An orientation on a one-dimensional $\mathbb{R}$-vector space L is a connected component of $L - \{0\}$. The union of this component with $\{0\}$ will be denoted L^+ and it will be called the positive component.*

It is clear from the definition that $|\Lambda^n(V)|^\vee$ has a given orientation, for which the positive component consists of the positive volume forms. We are now ready to apply the notion of a volume form to manifolds.

3.5.4 Definition. *Let X be a C^k-manifold, for some $k \geq 1$. We define the vector bundle Vol_X of volume forms on X to be the C^{k-1}-vector bundle with $\mathrm{Vol}_X(x) = |\Lambda^{\dim_X(x)}\mathrm{T}_X(x))|^\vee$ for all x in X, with local trivializations induced by those of T_X. For W a finite dimensional $\mathbb{R}$-vector space $W \otimes_{\mathbb{R}} \mathrm{Vol}_X$ is defined to be the C^{k-1}-vector bundle with fibres $(W \otimes_{\mathbb{R}} \mathrm{Vol}_X)(x) = W \otimes_{\mathbb{R}} \mathrm{Vol}_X(x)$ for all x in X, with local trivializations induced by those of T_X. For $U \subset X$ open and $l \leq k-1$, a C^l-section of $W \otimes_{\mathbb{R}} \mathrm{Vol}_X$ over U is called a W-valued volume form on U.*

3.5.5 Remark. The sheaf of C^{k-1}-sections of $W \otimes_{\mathbb{R}} \mathrm{Vol}_X$ is $W_X \otimes_{\mathbb{R}_X} \mathrm{Vol}_X$, where W_X and $\mathbb{R}_X$ denote the constant sheaves on X associated to W and $\mathbb{R}$.

It follows immediately from the definitions that, for $U \subset X$ open with local coordinates $x_1,\ldots,x_n$, every element of $(W \otimes_{\mathbb{R}} \mathrm{Vol}_X)(U)$ can be uniquely written in the form $w \cdot |dx_1 \cdots dx_n|$, with $w\colon U \to W$ a C^{k-1}-function.

The finite dimensionality of W is there just because we have decided that vector bundles should have finite rank (they have to be manifolds themselves). Working with sheaves, there is no problem whatsoever to allow W to have infinite dimension.

3.5.6 Definition. *Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X and f an element of $\mathcal{F}(X)$. The support of f, denoted $\mathrm{Supp}(f)$, is defined to be the set $\{x \in X \mid f_x \neq 0\}$; it is a closed subset of X.*

We will now be concerned with defining the integral of continuous W-valued volume forms with quasi-compact support (recall that a topological space is called quasi-compact if every open cover of it has a finite subcover). Intuitively, the integral of such a form is the sum of all its values (which are elements of W). More precisely, one should think of Riemann sums; the volume form itself tells us how to measure the size of "small cubes". But the terms "cube" and "block" do not make much sense in X. So first we explain what we want to do locally; after that we set up an administration system to make sure that everything gets counted exactly once. I should also admit that the usual definition of integration of volume forms is in terms of paritions of unity. In this course I want to show that there is another definition that is closer to the way one actually computes integrals, and which does not need that the manifold is separated. Of course, both definitions give the same result for separated manifolds. We will probably need the usual definition do prove some general results on integration (such as Stokes's theorem).

So let X be a C^k-manifold, with $k \geq 1$. Let W be a finite dimensional $\mathbb{R}$-vector space and v a W-valued volume form on X. Let $\phi\colon U \to X$ be a chart, with $U \subset \mathbb{R}^n$ open. For all u in U, $\mathrm{T}_\phi(u)$ is an isomorphism from $\mathrm{T}_U(u)$ to $\mathrm{T}_X(\phi(u))$. This gives us isomorphisms from $(W \otimes_\mathbb{R} \mathrm{Vol}_U)(u)$ to $(W \otimes_\mathbb{R} \mathrm{Vol}_X)(\phi(u))$, and hence a W-valued volume form ϕ^*v on U. We have, uniquely, $\phi^*v = w|dx_1 \cdots dx_n|$, with $w\colon U \to W$ a C^{k-1}-map. Let $V \subset U$ be a measurable subset (in the sense of Lebesgue) of U whose closure $\overline{U}$ in $\mathbb{R}^n$ is compact and contained in U (for example, a bounded closed subset of $\mathbb{R}^n$ that is contained in U, or an open subset of such a closed subset). Then we define:

$$\int_V \phi^*v := \int_V w,$$

where the last integral is in the sense of Lebesgue. To make things a bit more concrete: if $(w_1, \ldots, w_d)$ is a basis for W, and $w = \sum_i f_i w_i$, then $\int_V w = \sum_i (\int_V f_i) w_i$. We are now ready to define the integral on X itself.

3.5.7 Construction. Let X be a C^k-manifold with $k \geq 1$. Let W be a finite dimensional $\mathbb{R}$-vector space. Let v be a W-valued volume form on X with quasi-compact support. Suppose that we have an integer $m \geq 0$, charts $\phi_i\colon U_i \to X$, $1 \leq i \leq m$, $V_i \subset U_i$ compact such that $\mathrm{Supp}(v) \subset \cup_i \phi_i V_i$. Then we define the integral of v over X, with respect to these data, to be:

$$\int_X v := \sum_{r=1}^{m} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} \int_{V_{i_1,\ldots,i_r}} \phi_{i_1}^* v,$$

where $V_{i_1,\ldots,i_r} := \phi_{i_1}^{-1} \cap_{j=1}^r \phi_{i_j} V_{i_j}$.

The sum over r, and the signs in it, are there to make that we do not count the intersections twice, etc.; it is called the inclusion exclusion principle, easily understood in terms of the characteristic functions of the V_i. Note that the $V_{i_1,\ldots,i_r}$ are indeed measurable subsets of U_{i_1}: $\phi_{i_1} V_{i_1} \cap \phi_{i_2} U_{i_2}$ is open in $\phi_{i_1} V_{i_1}$, and $\phi_{i_1} V_{i_1} \cap \phi_{i_2} V_{i_2}$ is closed in $\phi_{i_1} V_{i_1} \cap \phi_{i_2} U_{i_2}$, etc. Let us show how charts ϕ_i and compact subsets V_i of U_i with $\mathrm{Supp}(v) \subset \cup_i \phi_i V_i$ can be obtained. For each x in $\mathrm{Supp}(v)$, choose a chart $\phi_x\colon U_x \to X$ and a compact neighborhood V_x of $\phi_x^{-1} x$ in U_x. Since $\mathrm{Supp}(v)$ is quasi-compact, it is covered by a finite number of the V_x. Numbering those x gives the desired charts and measurable subsets. Of course, in practice one usually tries to take the V_i disjoint, or at least such that the $V_i \cap V_j$ have measure zero for $i \neq j$.

3.5.8 Proposition. *The integral of v as defined in Construction 3.5.7 does not depend on the choice of the charts ϕ_i and the sets V_i.*

Proof. Suppose we have two sets of data: m, m', ϕ_i, ϕ'_j, etc. Then we construct two new sets of data as follows: $m'' := mm'$, $U_{i,j} := \phi_i^{-1}(\phi_i U_i \cap \phi'_j U'_j)$, $\phi_{i,j} := \phi_i|_{U_{i,j}}$, $U'_{i,j} := (\phi'_i)^{-1}(\phi_i U_i \cap \phi'_j U'_j)$, $\phi'_{i,j} := \phi'_j|_{U'_{i,j}}$, $V_{i,j} := \phi_i^{-1}(\phi_i V_i \cap \phi'_j V'_j)$, $V'_{i,j} := (\phi'_i)^{-1}(\phi_i V_i \cap \phi'_j V'_j)$. Let us first argue that these new two sets of data give the same integral. For that purpose, consider a pair (i,j). Let $U := U_{i,j}$ and $U' := U'_{i,j}$. Then $f := {\phi'_j}^{-1}\phi_i$ defines an isomorphism from U to U', such that $V := V_{i,j}$ has image $V' := V'_{i,j}$. Let us write $\phi_i^* v = v|dx_1 \cdots dx_n|$ and ${\phi'_j}^* v = v'|dx'_1 \cdots dx'_n|$. The partial derivatives $\partial/\partial x_i$, $1 \le i \le n$ form a basis of T_U, and likewise for $\partial/\partial x'_j$ for $\mathrm{T}_{U'}$. Written in this basis, the tangent map T_f is given by the matrix whose (i,j)th coefficient is $\partial f_j/\partial x_i$, where f is written $(f_1,\ldots,f_n)$. Note that $|dx_1 \cdots dx_n|$ is a basis for Vol_U, and that $|dx'_1 \cdots dx'_n|$ is one for $\mathrm{Vol}_{U'}$. Let u be in U and put $u' := f(u)$. Then $\mathrm{T}_f(u)$ induces an isomorphism from $\mathrm{Vol}_U(u)$ to $\mathrm{Vol}_{U'}(u')$. Using the defitions of $\mathrm{Vol}_U(u)$ and $\mathrm{Vol}_{U'}(u')$, one sees that under this isomorphism $|dx'_1 \cdots dx'_n|$ is mapped to $|\det(\mathrm{T}_f(u)|\,|dx_1 \cdots dx_n|$. By construction, this isomorphism sends $v'(u')|dx'_1 \cdots dx'_n|$ to $v(u)|dx_1 \cdots dx_n|$, hence we get $v'(u')|\det(\mathrm{T}_f(u))| = v(u)$. The "change of variables formula" from vector calculus says:

$$\int_{V'} v' = \int_{V'} v'|dx'_1 \cdots dx'_n| = \int_V (v' \circ f)|df_1 \cdots df_n| = \int_V (v' \circ f)|\det(\mathrm{T}_f)|.$$

So, from what we have just seen, it follows that $\int_{V'} v' = \int_V v$. This equality is also valid for $V := V_{(i_1,j_1),\ldots,(i_r,j_r)}$ and $V' := V'_{(i_1,j_1),\ldots,(i_r,j_r)}$. That means that indeed our two new sets of data for integration give the same result.

It remains now to be shown that two sets of data, one of which is a refinement of the other, give the same integral. In order to see this, let us reconsider what happens for just one set of data m, ϕ_i and V_i. Considering all possible intersections of the V_i and their complements gives us 2^m subsets that partition X. All these subsets are contained in some V_i, except one: the complement of the union of the V_i. Note that on this last set v is zero. So on each of our subsets we can integrate v, and the sum of these integrals equals the integral of v over X relative to the set of data m, ϕ_i, V_i. (To see that, use that $f \mapsto \int_{\mathbb{R}^n} f$ is additive.) Of course, what we are doing here is the standard game with the boolean algebra generated by the characteristic functions of the V_i. Now suppose that we have a refinement m', ϕ'_j, V'_j. The $2^{m'}$ subsets of X obtained from the V'_j give a partition of X that refines the partition obtained from the V_i. Then our claim is clear. $\square$

Now that we know how to integrate volume forms (i.e., sections of Vol_X), let us discuss the relation between volume forms and differential forms of top degree, i.e., sections of $\Omega_X^{\dim X}$.

3.5.9 Definition. *Let X be a manifold and L a rank one vector bundle on X. An orientation of L is a collection of orientations of all $L(x)$, $x \in X$, which is locally constant. If X is C^k with $k \ge 1$, then an orientation of X is an orientation of $\Omega_X^{\dim X}$.*

Some examples. The trivial line bundle $\mathbb{R} \times X$ has a standard orientation. The same is true for Vol_X, but not always for $\Omega_X^{\dim X}$. If ω is a global section of $\Omega_X^{\dim X}$ such that $\omega(x) \neq 0$ for all x in X, then ω is a basis for $\Omega_X^{\dim X}$, hence gives an isomorphism from $\mathbb{R} \times X$ to $\Omega_X^{\dim X}$, hence gives an orientation on $\Omega_X^{\dim X}$.

3.5.10 Remark. It is not true in general that all orientations on a line bundle L come from a trivialization of it (example: take X to be two copies of $\mathbb{R}$, glued via the identity along $\mathbb{R} - \{0\}$, and take an ugly line bundle). If X is paracompact, then all orientations come indeed from trivializations.

3.5.11 Proposition. *Let X be a C^k-manifold with $k \geq 1$. An orientation on X induces a unique isomorphism from Vol_X to $\Omega_X^{\dim_X}$, such that, at each x in X, it coincides with the map from $\Omega_X^{\dim_X}(x)$ to $\mathrm{Vol}_X(x)$ that sends l to $|l|$. Conversely, an isomorphism from Vol_X to $\Omega_X^{\dim_X}$ with this property induces an orientation on X and the two constructions are inverses.*

Proof. Let x be in X, $V := \mathrm{T}_X(x)$, $n := \dim(V)$, $L := \Omega_X^{\dim_X}(x) = (\Lambda^n(V))^\vee$, and finally $L' := \mathrm{Vol}_X(x) = |\Lambda^n(V)|^\vee$. Recall that we have the map $|\cdot| \colon L \to L'$ that sends l to $|l|$. This map is, of course, not linear. We get a linear map as follows: choose l in L^+ non-zero and send λl, for λ in $\mathbb{R}$, to $\lambda|l|$. Check that this map does not depend on the choice of l, that it coincides with $|\cdot|$ on L^+ and that it is the only linear map with that property. $\square$

It is now clear that for X an oriented C^k-manifold we can integrate sections of $\Omega_X^{\dim_X}$ that have quasi-compact support, by using the isomorphism corresponding to the orientation to transform these sections in volume forms. The procedure to integrate a differential form of top degree ω with quasi-compact support is then the same as in Construction 3.5.7, except that one should take charts that are compatible with the orientation on X and the standard orientation on $\mathbb{R}^n$. The standard orientation on $\mathbb{R}^n$ is the one such that $dx_1 \cdots dx_n$ (in this order!) is positive. Of course, if W is an $\mathbb{R}$-vector space, then we can also integrate sections with quasi-compact support of $W \otimes_{\mathbb{R}} \Omega_X^{\dim_X}$.

To finish this section, let us define a canonical volume form on a Riemannian manifold. So suppose that X is a C^k-manifold, with $k \geq 1$, and that $\langle \cdot, \cdot \rangle$ is a metric on T_X. Let x be in X. Suppose that $v_1, \ldots, v_n$ is an orthonormal basis of $\mathrm{T}_X(x)$. Let $v(x)$ be the volume form on $\mathrm{T}_X(x)$ such that $(v(x))(v_1, \ldots, v_n) = 1$. One checks that this does not depend on the orthonormal basis chosen. Hence it defines a volume form v on X.

To give some example where one uses this, note that a submanifold of a Riemannian manifold inherits the structure of Riemannian manifold. For example, consider the group $\mathrm{SO}_3(\mathbb{R})$ as a compact submanifold of $\mathbb{R}^9$ with its standard Riemannian metric. Then one can ask: what is the volume of $\mathrm{SO}_3(\mathbb{R})$?

3.6 Pullback of vector bundles and of differential forms

Let us first discuss pullback of vector bundles. Let $f \colon X \to Y$ be a morphism of manifolds, and let E be a vector bundle on Y. Then we define a vector bundle f^*E on X by: $(f^*E)(x) := E(f(x))$ for all x in X, and the local trivializations of f^*E are induced by those of E. For $U \subset Y$ open, and s in $E(U)$, we get an element f^*s of $(f^*E)(f^{-1}U)$, defined by: $(f^*s)(x) := s(f(x))$ for all x in $f^{-1}U$. (Note the special case $E = \mathbb{R} \times Y$, where s is just a function and $f^*s = s \circ f$.) The sheaf of sections of f^*E is the tensor product $C_X \otimes_{f^{-1}C_Y} f^{-1}E$, where f^{-1} is pullback of sheaves. For an arbitrary C_Y-module $\mathcal{M}$, its pullback as a module is defined as $f^*\mathcal{M} := C_X \otimes_{f^{-1}C_Y} f^{-1}\mathcal{M}$. Hence on the side of locally free sheaves of modules this operation corresponds to the pullback of vector bundles.

Suppose now moreover that F is a vector bundle on X, and that $g \colon F \to E$ is a morphism of vector bundles (see Definition 2.2.8). We claim that such a g corresponds naturally to a morphism from F to f^*E of vector bundles on X. The proof is trivial, because, for all x in X, $(f^*E)(x) = E(f(x))$.

In particular, the morphism T_f from T_X to T_Y corresponds to a morphism, also written T_f, from T_X to $f^*\mathrm{T}_Y$. Dualizing gives us a morphism $f^* \colon f^*\Omega_Y^1 \to \Omega_X^1$. Doing our tensor operations gives us $f^*\colon f^*\Omega_Y \to \Omega_X$. One easily verifies that this f^* is a morphism of sheaves of graded algebras, and that for $U \subset Y$ open and ω in $\Omega_Y(U)$ one has $f^*d\omega = d(f^*\omega)$; it suffices to prove this for forms of degree one, where one has:

$$(f^*dg)\partial = (dg)(\mathrm{T}_f\partial) = (dg)(\partial \circ f^*) = (\partial \circ f^*)g = \partial(g \circ f) = (d(f^*g))\partial.$$

Suppose now that $\dim_X(x) = \dim_Y(f(x))$ for all x in X. Then one has $f^*\colon f^*\mathrm{Vol}_Y \to \mathrm{Vol}_X$ and $f^*\colon \Omega_Y^{\dim_Y} \to \Omega_X^{\dim_X}$. If f is an isomorphism from X to an open subset U of Y, and v is in $\mathrm{Vol}_Y(Y)$ with quasi-compact support contained in U, then one has $\int_Y v = \int_X f^*v$.

3.7 Some exercises

Let G be a Lie group, i.e., G is a C^∞-manifold, with a C^∞-group structure. Let e be its unit element. We consider the following group actions. The (left) action of G on itself by left translations: for x in G we have $l_x\colon G \to G$ sending y to xy. The (right) action of G on itself by right translations: $r_x\colon y \mapsto yx$. The action of G on itself by conjugation: $c_x\colon y \mapsto xyx^{-1}$. The action of $G \times G$ on G by translations on both sides: $b_{x,y}\colon z \mapsto xzy^{-1}$. Let l denote the morphism of groups from G to $\mathrm{Aut}_{\mathbf{Man}}(G)$ given by the action by left translations. Similarly, we have the anti-morphism r from G to $\mathrm{Aut}_{\mathbf{Man}}(G)$ given by the right translations, the morphism b from $G\times G$ to $\mathrm{Aut}_{\mathbf{Man}}(G)$ and the morphism c from G to the group $\mathrm{Aut}_{\mathrm{Lie}}(G)$ of automorphims of G as Lie group.

We denote the tangent space $\mathrm{T}_G(e)$ by $\mathrm{Lie}(G)$ and by L. For every x in G we have the two isomorphisms $\mathrm{T}_{l_x}(e)$ and $\mathrm{T}_{r_x}(e)$ from L to $\mathrm{T}_G(x)$. These two isomorphisms need not be the same. Show that in fact $\mathrm{T}_{r_x}(e)^{-1}\mathrm{T}_{l_x}(e)$ is the automorphism $\mathrm{T}_{c_x}(e)$ of L. (By the way, L is called the Lie algebra of G; we will discuss the structure of Lie algebra on L a bit further.) Show that $x \mapsto \mathrm{T}_{c_x}(e)$ defines an action of G on L by linear maps; this action is called the adjoint representation of G. Show that both l and r define isomorphisms, still denoted l and r, from the trivial vector bundle $L \times G$ to T_G. In particular, all Lie groups have a trivial tangent bundle, and are orientable.

Let us consider the vector space $\mathrm{T}_G(G)$ of vector fields on G. The group G acts on it via l, r and c; $G\times G$ acts via b. A vector field ∂ on G is called left-invariant if it is invariant for the action given by l; similarly, it is called right-invariant if it is invariant under r, and bi-invariant if invariant under b. Explicitly: ∂ is left-invariant if and only if for all x in G one has $\partial(x) = \mathrm{T}_{l_x}(e)\partial(e)$. Show that $\partial \mapsto \partial(e)$ gives an isomorphism from the vector space of left-invariant vector fields on G to L. Show the same with left replaced by right. Show that the space of bi-invariant forms is isomorphic in this way to the subspace of L on which G acts trivially via its adjoint representation.

Before we define the Lie algebra structure on L we need a general result on derivations. Let k be a field and A a k-algebra. Then we have the A-module of k-derivations $\mathrm{Der}_k(A) := \mathrm{Der}_k(A, A)$. One verifies immediately that for ∂_1 and ∂_2 in $\mathrm{Der}_k(A)$, the commutator $[\partial_1, \partial_2] := \partial_1\partial_2 - \partial_2\partial_1$ is in $\mathrm{Der}_k(A)$. We apply this to vector fields. Let X be a C^k-manifold (with $k \geq \infty$, remember?), $U \subset X$ open, ∂_1 and ∂_2 vector fields on U. Then we may view ∂_1 and ∂_2 as elements of $\mathrm{Der}_{\mathbb{R}}(C_U^k)$ (recall that we have a canonical isomorphism between T_X and $\mathbf{Der}_{\mathbb{R}}(C_X)$). Hence we get $[\partial_1, \partial_2]$ in $\mathrm{T}_X(U)$. (Assuming that U is an open subset $\mathbb{R}^n$, compute explicitly what this operation looks like.)

We go back to our Lie group G. Show that for ∂_1 and ∂_2 two left-invariant vector fields on G, $[\partial_1, \partial_2]$ is left-invariant too. (Of course the same holds for right-invariance; one should in fact prove a lemma concerning ∂_1 and ∂_2 on a manifold X that are invariant under an automorphism

σ of X.) Because the space of left-invariant vector fields is just L (via $\partial \mapsto \partial(e)$), we get a map $[\cdot,\cdot]$ from $L \times L \to L$. This map is called the Lie bracket. Show that it is bilinear, alternating and that it satisfies Jacobi's identity:

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

A vector space L with such an operation is called a Lie algebra. We will compute the Lie algebras of the Lie groups that we have seen in §1. We start with the group $G := \mathrm{GL}_n(\mathbb{R})$ (some $n \geq 0$). Since G is an open subset of $\mathrm{M}_n(\mathbb{R})$, we identify the $\mathrm{T}_G(x)$ with $\mathrm{M}_n(\mathbb{R})$. Let a be in $L = \mathrm{T}_e(G) = \mathrm{M}_n(\mathbb{R})$. We wish to describe explicitly the left-invariant vector field ∂_a on G such that $\partial_a(e) = a$. Verify that for g in G we have $\partial_a(g) = ga$. Now we compute $\partial_a x_{i,j}$, where the $x_{i,j}$ are the coordinate functions on $\mathrm{M}_n(\mathbb{R})$. For g in G, $(\partial_a x_{i,j})(g)$ is by definition the derivative of $x_{i,j}$ at g in the direction given by $\partial_a(g)$, i.e., in the direction ga. So we compute:

$$x_{i,j}(g + \varepsilon ga) = (g + \varepsilon ga)_{i,j} = g_{i,j} + \varepsilon (ga)_{i,j} = x_{i,j}(g) + \varepsilon \sum_k g_{i,k} a_{k,j}.$$

It follows that $\partial_a(x_{i,j}) = \sum_k x_{i,k} a_{k,j}$. Applying this formula twice gives:

$$(\partial_a \partial_b) x_{i,j} = \partial_a (\sum_k x_{i,k} b_{k,j}) = \sum_{k,k'} x_{i,k'} a_{k',k} b_{k,j} = \sum_{k'} x_{i,k'} (ab)_{k',j} = \partial_{ab} x_{i,j}.$$

From this we get:

$$[\partial_a, \partial_b] x_{i,j} = \partial_{[a,b]} x_{i,j}, \text{ and } [\partial_a, \partial_b] = \partial_{[a,b]},$$

since the $x_{i,j}$ are linearly independent over $\mathbb{R}$. So the Lie bracket for $\mathrm{GL}_n(\mathbb{R})$ is just the ordinary commutator of matrices. The reader should check that if we had used right-invariant vector fields to define the Lie bracket, we would have found the opposite result (use that $x \mapsto x^{-1}$ induces multiplication by -1 on L). Let us now reconsider the subgroups of $\mathrm{GL}_n(\mathbb{R})$ that we considered in §1. From the computations we did there, it follows that $\mathrm{Lie}(\mathrm{SL}_n(\mathbb{R}))$ is the subspace of $\mathrm{M}_n(\mathbb{R})$ consisting of the elements with trace zero, that $\mathrm{Lie}(\mathrm{SO}_n(\mathbb{R})) = \mathrm{M}_n(\mathbb{R})^-$, the space of anti-symmetric matrices, and that $\mathrm{Lie}(\mathrm{Sp}_{2n}(\mathbb{R}))$ is the space of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c^t = c$, $b^t = b$ and $d = -a^t$. From the construction of the Lie bracket it follows that the Lie bracket for any of these subgroups is just the restriction of the one for $\mathrm{M}_n(\mathbb{R})$ in the first two cases and $\mathrm{M}_{2n}(\mathbb{R})$ in the last.

Let us now look at differential forms on Lie groups. Just as for vector fields, we have the notions of left-invariant, right-invariant and bi-invariant elements of Ω^i_G. Of particular importance are bi-invariant differential forms of top degree, since those give us bi-invariant volume forms. Show, by pure thought, that $\mathrm{SO}_n(\mathbb{R})$ has such a non-zero bi-invariant form (use that $\mathrm{SO}_n(\mathbb{R})$ is compact and connected). Show that $\mathrm{O}_2(\mathbb{R})$ has a non-zero bi-invariant volume form, but not a non-zero bi-invariant 1-form. Show that $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$ both have non-zero bi-invariant forms of top degree (use that the commutator subgroup of $\mathrm{SL}_n(\mathbb{R})$ is $\mathrm{SL}_n(\mathbb{R})$ itself).

Of particular fun should be the following exercise. Compute explicitly the bi-invariant volume form v on $G := \mathrm{SO}_3(\mathbb{R})$ for which G has volume one. Compute the distribution g, with respect to v, of the angles of rotation, say in the interval $[0,\pi]$, of the elements of G. More precisely, let f be the function $G \to [-1,1]$ that sends x to $(\mathrm{tr}(x) - 1)/2$; determine the continuous function g on $]-1,1[$ such that for every continuous $h \colon [-1,1] \to \mathbb{R}$ one has:

$$\int_G f^*(h) v = \int_{-1}^{1} gh.$$

In the same way, compute the distribution g_2 for the function f_2 from G to $[-1,1]$ that sends x to $(\mathrm{tr}(x^2)-1)/2$. Why is the result so remarkable? (Hint: it might be useful to use the following chart for G; let $U := \{x \in \mathbb{R}^3 \mid 0 < \|x\| < \pi\}$ and let $\psi \colon U \to G$ be the map that sends a to the rotation of angle $|a|$ with (oriented) axis $\mathbb{R}a$.)

4 De Rham cohomology

4.1 Definition. *A differential graded-commutative $\mathbb{R}$-algebra is an $\mathbb{R}$-algebra A with a grading $A = \oplus_{i \in \mathbb{Z}} A_i$, which is graded-commutative (i.e., for x in A_i and y in A_j one has $yx = (-1)^{ij} xy$), and which is equipped with a differential d of degree one (i.e., an $\mathbb{R}$-linear map d from A to A such that $d^2 = 0$, $d(A_i) \subset A_{i+1}$ and $d(xy) = (dx)y + (-1)^i x dy$ for x in A_i and y in A).*

We have a contravariant functor $X \mapsto (\Omega_X(X), d)$ from the category of C^∞-manifolds to that of differential graded-commutative $\mathbb{R}$-algebras. Note that the vector spaces $\Omega_X(X)$ tend to be very big (the typical dimension is $|\mathbb{R}|$). Although this functor does transform morphisms of manifolds into $\mathbb{R}$-linear maps, it does not really simplify the study of the category of C^∞-manifolds. For example, a compact manifold X can be reconstructed from $\Omega_X^0(X)$ alone (the points of X correspond to the maximal ideals, the topology is the Zariski topology, etc.). But, if one composes the functor $X \mapsto (\Omega_X(X), d)$ with the functor that takes homology of differential graded-commutative algebras, then a miracle happens. This composed functor, called de Rham cohomology, has reasonable finiteness properties, and, most importantly, is homotopy invariant. Before proving that, let us write down the definitions in detail.

4.2 Lemma. *Let A be a differential graded-commutative $\mathbb{R}$-algebra. Then its homology $\mathrm{H}(A)$, defined as $\ker(d)/\mathrm{im}(d)$, has an induced structure of graded-commutative $\mathbb{R}$-algebra.*

Proof. Details are left to the reader. Show that $\ker(d)$ is a graded-commutative subalgebra of A, in which $\mathrm{im}(d)$ is a homogeneous ideal. □

4.3 Definition. *Let X be a paracompact C^∞-manifold. Then the de Rham cohomology of X is the graded-commutative $\mathbb{R}$-algebra $\mathrm{H}_{\mathrm{dR}}(X) := \mathrm{H}(\Omega_X(X), d)$. Hence one has:*

$$\mathrm{H}^i_{\mathrm{dR}}(X) = \frac{\ker(d\colon \Omega^i_X(X) \to \Omega^{i+1}_X(X))}{\mathrm{im}(d\colon \Omega^{i-1}_X(X) \to \Omega^i_X(X))}.$$

As explained above, $\mathrm{H}_{\mathrm{dR}}(\cdot)$ is a contravariant functor from the category of manifolds to that of graded-commutative $\mathbb{R}$-algebras. For $f\colon X \to Y$ we will write f^* for the morphism $\mathrm{H}_{\mathrm{dR}}(f)$ from $\mathrm{H}_{\mathrm{dR}}(Y)$ to $\mathrm{H}_{\mathrm{dR}}(X)$.

4.4 Remark. Of course we could define de Rham cohomology for arbitrary C^∞-manifolds (i.e., not necessarily paracompact), but that would not give the results we want. In the general case one should consider the so-called *hypercohomology* $\mathbb{H}_{\mathrm{d}R}(X) := \mathbb{H}(\Omega_X, d)$ of the complex of sheaves Ω_X.

4.5 Theorem. *The de Rham cohomology of paracompact C^∞-manifolds is homotopy invariant, i.e., if f_0 and f_1 from X to Y are homotopic, then the two maps f_0^* and f_1^* from $\mathrm{H}_{\mathrm{dR}}(Y)$ to $\mathrm{H}_{\mathrm{dR}}(X)$ are equal.*

Proof. By definition of homotopy, we have a morphism of C^∞-manifolds $F\colon X \times I \to Y$ with I an open interval containing 0 and 1, such that $F|_{X\times\{0\}} = f_0$ and $F|_{X\times\{1\}} = f_1$. (It would be enough to have F a C^1-morphism such that its restrictions to all $X \times i$ with i in I are C^k, but let us not bother.) We will construct a homotopy from 0 to $f_0^* - f_1^*$, i.e., a sequence of $\mathbb{R}$-linear maps $K^i\colon \Omega^i_Y(Y) \to \Omega^{i-1}_X(X)$ such that $f_0^* - f_1^* = dK + Kd$. In order to construct this K, we need to consider $\Omega_{X\times I}$ more closely. Let p_X and p_I denote the projections from $X \times I$ to X and

I, respectively. (More generally, one should consider the vector bundle of differential forms on a product of two manifolds.) At a point (x, i) of $X \times I$, the tangent space $\mathrm{T}_{X\times I}(x, i)$ is canonically isomorphic to the direct sum $\mathrm{T}_X(x) \oplus \mathrm{T}_I(i)$. On the level of vector bundles this gives us:

$$\mathrm{T}_{X\times I} = p_X^*\mathrm{T}_X \oplus p_I^*\mathrm{T}_I.$$

Dualizing this gives:

$$\Omega^1_{X\times I} = p_X^*\Omega^1_X \oplus p_I^*\Omega^1_I.$$

We suggest to the reader to prove that for a ring A and A-modules M_1 and M_2 one has

$$\Lambda^i(M_1 \oplus M_2) = \bigoplus_{j+k=i} \Lambda^j(M_1) \otimes_A \Lambda^k(M_2).$$

In fact, $\Lambda(M_1 \oplus M_2)$ and $\Lambda(M_1) \otimes_A \Lambda(M_2)$ with its grading obtained by "convolution" are naturally isomorphic, as they have the same universal property: to give a morphism of graded A-algebras to a graded-commutative A-algebra B is to give morphisms of A-modules $\phi_1 \colon M_1 \to B_1$ and $\phi_2 \colon M_2 \to B_2$. It follows that we have a natural isomorphism:

$$\Omega^i_{X\times I} = p_X^*\Omega^i_X \oplus \left(p_X^*\Omega^{i-1}_X \otimes p_I^*\Omega^1_I\right).$$

In local coordinates, this means the following. Suppose that $x_1, \ldots, x_n$ are coordinates on $U \subset X$, and let t be the coordinate on I. Then $\Omega^i_{U\times I}$ has the basis:

$$\begin{cases} dx_{j_1} \cdots dx_{j_i}, & 1 \leq j_1 < \cdots < j_i \leq n, \\ dx_{j_1} \cdots dx_{j_{i-1}} dt, & 1 \leq j_1 < \cdots < j_{i-1} \leq n. \end{cases}$$

For ω in $\Omega^i_{X\times I}(X \times I)$ we can write uniquely $\omega = \omega_1 + \omega_2\, dt$, with ω_1 in $p_X^*\Omega^i_X(X \times I)$ and ω_2 in $p_X^*\Omega^{i-1}_X(X \times I)$. This decomposition of the $\Omega^i_{X\times I}$ also induces a decomposition of the differential $d_{X\times I}$ on $\Omega_{X\times I}$: we have, uniquely, $d_{X\times I} = d_X + d_I$, with d_X increasing the degree with respect to X by one, and d_I the same for I. (For an arbitrary product, this displays the complex $\Omega_{X\times Y}$ as the total complex associated to the double complex $p_X^*\Omega_X \otimes p_Y^*\Omega_Y$.)

We can now define our homotopy operators K_1^i from $\Omega^i_{X\times I}(X \times I)$ to $\Omega^{i-1}_X(X)$; the operator K we want to have will be $K_1 \circ F^*$. With the notation as above ($\omega = \omega_1 + \omega_2$) we define:

$$K_1^i\omega := (-1)^i \int_0^1 \omega_2\, dt. \tag{4.5.1}$$

Since the object to be integrated does not look like a function that one usually integrates, let us write it a bit more explicit. Let x be in X. Then $(K_1^i\omega)(x) = (-1)^i \int_0^1 \omega_2(x, t)\, dt$, and $\omega_2(x, t)$ is an element of the vector space $\Omega^{i-1}_X(x)$ that does not depend on t. This means that the integral is exactly of the kind we considered in the previous section: we integrate a vector space valued volume form (namely: $\omega_2\,|dt|$) over a compact subset $[0, 1]$ of the manifold I. Let i_0 and i_1 denote the inclusions of X into $X \times I$ that send x to $(x, 0)$ and $(x, 1)$, respectively. Then we claim that for all ω in $\Omega^i_{X\times I}(X \times I)$ we have:

$$(dK_1^i + K_1^{i+1}d)\omega = i_0^*\omega - i_1^*\omega. \tag{4.5.2}$$

To prove this identity, note that it is a local problem on X, and that both sides are additive in ω. Hence we may assume that $x_1, \ldots, x_n$ are local coordinates on X and that ω is of the form $fdx_1\cdots dx_i$ or $gdx_1\cdots dx_{i-1}dt$ with f and g in $C_{X\times I}(X\times I)$. Let us first consider the case $\omega = fdx_1\cdots dx_i$. Then $K_1^i\omega = 0$ because $\omega_2 = 0$. On the other hand,

$$d\omega = df dx_1\cdots dx_i = (d_X f)dx_1\cdots dx_i + (d_I f)dx_1\cdots dx_i.$$

It follows that:

$$K_1 d\omega = (-1)^{i+i+1}K_1((\partial f/\partial t)dx_1\cdots dx_i dt) = -\left(\int_0^1 (\partial f/\partial t)dt\right)dx_1\cdots dx_i = i_0^*\omega - i_1^*\omega.$$

One should note that the last identity is exactly the fundamental theorem of calculus. Let us now consider the second case: $\omega = gdx_1\cdots dx_{i-1}dt$. Now we have: $\omega = \omega_2$, hence:

$$K_1^i\omega = (-1)^i\left(\int_0^1 g\,dt\right)dx_1\cdots dx_{i-1} = (-1)^i G dx_1\cdots dx_{i-1},$$

where G is the function on X defined by $G(x) = \int_0^1 g(x,t)\,dt$. It follows that:

$$dK_1^i\omega = (-1)^i dG dx_1\cdots dx_{i-1}.$$

On the other hand, $d\omega = (d_X g)dx_1\cdots dx_{i-1}dt$. Hence:

$$\begin{aligned}(-1)^{i+1}K_1^{i+1}d\omega &= \left(\int_0^1 (d_X g)\,dt\right)dx_1\cdots dx_{i-1} = d\left(\int_0^1 g\,dt\right)dx_1\cdots dx_{i-1} = \\ &= (dG)dx_1\cdots dx_{i-1},\end{aligned}$$

where the middle equality is the theorem in calculus that says that the derivative with respect to parameters of an integral is the integral of the derivative. So we find that $(dK_1 + K_1 d)\omega = 0$. This is just what we need, since $i_0^*(dt) = d(i_0^*t) = d(0) = 0$, and also $i_1^*(dt) = 0$.

To finish the proof of the theorem, define $K^i := K_1^i \circ F^*$ and note that, for ω in $\Omega_Y^i(Y)$, $i_0^*F^*\omega = f_0^*\omega$ and $i_1^*F^*\omega = f_1^*\omega$. □

We can now compute the de Rham cohomology for some manifolds. The empty manifold gives the zero ring, and a one point manifold gives the $\mathbb{R}$-algebra $\mathbb{R}$ itself. Now suppose that X is a contractible C^k-manifold (i.e., there is a point x in X such that the constant map $f\colon X \to X$ that sends every element of X to x is homotopic to the identity morphism id_X of X. Then f^* induces the identity endomorphism of $\mathrm{H}_{\mathrm{dR}}(X)$. But we can write $f = i_x \circ p$, with i_x the inclusion of $\{x\}$ in X and p the unique map $X \to \{x\}$. But then we see that i_x^* and p^* are inverses, hence $\mathrm{H}_{\mathrm{dR}}(X) = \mathbb{R}$.

5 The exact sequence of Mayer–Vietoris

Let X be a Hausdorff paracompact C^∞-manifold. Suppose that X is the union of two open subsets U and V:

$$X = U \cup V.$$

Then we have a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(X) & \xrightarrow{i} & \Omega(U)\oplus\Omega(V) & \xrightarrow{r} & \Omega(U\cap V) & \longrightarrow & 0 \\ & & \omega & \mapsto & (\omega|_U, \omega|_V) & & & & \\ & & & & (\alpha,\beta) & \mapsto & \alpha|_{U\cap V} - \beta|_{U\cap V} & & \end{array} \tag{5.1}$$

It is clear that this diagram is exact at $\Omega(X)$ and at $\Omega(U)\oplus\Omega(V)$, and that the maps i and r commute with the differentials d on each of the terms. In order to obtain the long exact sequence of Mayer–Vietoris, we need that r is surjective. It is for proving this surjectivity that the hypothesis that X is paracompact and Hausdorff is used. We need a so-called *partition of unity* subject to the cover of X by U and V: elements f and g in $C^\infty(X)$ such that $1 = f + g$ and $\mathrm{Supp}(f) \subset U$ and $\mathrm{Supp}(g) \subset V$. It is a bit technical to show, but on our paracompact Hausdorff C^∞-manifold partitions of unity subject to any open cover do exist (see for example Definition 7.9, Proposition 7.10 and Theorem 7.8 of my syllabus "Géométrie Variable", out of which this course is more or less extracted, except this section and the next). In this course, we will admit the existence of f and g above.

5.2 Proposition. *The map r in (5.1) is surjective.*

Proof. Let f and g be as above. Let ω be in $\Omega(U\cap V)$. Then $g = 0$ on $X - V$ and $f = 0$ on $X - U$. It follows that $g\omega$ can be extended by zero to U because $g\omega$ is a section of Ω on the open part $U\cap V$ of U with support contained in that open part. Explicitly: there is a unique α in $\Omega(U)$ such that $\alpha|_{U\cap V} = g\omega$ and $\alpha(x) = 0$ for all x in $U - U\cap V$. Similarly, $f\omega$ can be extended by zero to an element β of $\Omega(V)$. But then $r((\alpha, -\beta)) = \omega$. $\square$

5.3 Remark. We note that the map $\omega \mapsto (g\omega, -f\omega)$ from $\Omega(U\cap V)$ to $\Omega(U)\oplus\Omega(V)$ does not necessarily commute with d, because df and dg are not necessarily zero. Hence our splitting of the exact sequence (5.1) of vector spaces is not necessarily a splitting of an exact sequence of complexes.

5.4 Proposition. *Let (A,d), (B,d) and (C,d) be complexes (of $\mathbb{R}$-vector spaces, say). Let:*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of complexes. Then we get maps $\partial^i\colon \mathrm{H}^i(C) \to \mathrm{H}^{i+1}(A)$, for all i in $\mathbb{Z}$, characterised by the following property.

> *Let $c \in C^i$ with $dc = 0$, let $b \in B^i$ with $gb = c$, then $g(db) = dc = 0$, hence there is a unique $a \in A^{i+1}$ with $fa = db$. As $f(da) = d(fa) = d^2b = 0$, and f is injective, we have $da = 0$, and a defines an element of $\mathrm{H}^{i+1}(A)$. This element is the image under ∂ of the element in $\mathrm{H}^i(C)$ given by c.*

Together with the morphisms $f \colon \mathrm{H}(A) \to \mathrm{H}(B)$ and $g \colon \mathrm{H}(B) \to \mathrm{H}(C)$ induced by f and g, the boundary maps ∂ give a long exact sequence:

$$\cdots \to \mathrm{H}^{i-1}(B) \to \mathrm{H}^{i-1}(C) \to \mathrm{H}^i(A) \to \mathrm{H}^i(B) \to \mathrm{H}^i(C) \to \mathrm{H}^{i+1}(A) \to \cdots$$

Proof. This is a long but simple verification that we will not write out. Any book on homological algebra or on algebraic topology should contain this statement. A lemma often used as ingredient for this proof is the so-called "Snake Lemma". This lemma is then be applied to the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & A^i/dA^{i-1} & \longrightarrow & B^i/dB^{i-1} & \longrightarrow & C^i/dC^{i-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (A^{i+1})^{d=0} & \longrightarrow & (B^{i+1})^{d=0} & \longrightarrow & (C^{i+1})^{d=0}, & & \end{array}$$

where the notation $(A^{i+1})^{d=0}$ stands for $\ker(d \colon A^{i+1} \to A^{i+2})$. $\square$

5.5 Theorem. (Mayer–Vietoris) *Let X be a paracompact Hausdorff C^∞-manifold, covered by two open subsets U and V. Then the diagram in (5.1) gives, via Proposition 5.4, a long exact sequence:*

$$\cdots \to \mathrm{H}^{i-1}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{i-1}_{\mathrm{dR}}(V) \to \mathrm{H}^{i-1}_{\mathrm{dR}}(U \cap V) \to \mathrm{H}^i_{\mathrm{dR}}(X) \to \mathrm{H}^i_{\mathrm{dR}}(U) \oplus \mathrm{H}^i_{\mathrm{dR}}(V) \to \mathrm{H}^i_{\mathrm{dR}}(U \cap V) \to \cdots$$

Proof. This is a direct application of Propositions 5.2 and 5.4. $\square$

Now we can combine our two tools (homotopy invariance and Mayer–Vietoris) in order to determine the de Rham cohomology for some manifolds, such as the spheres S^n, the projective spaces $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{P}^n(\mathbb{R})$, compact surfaces and $\mathbb{C}$ minus a finite set of points.

5.6 Theorem. *Let $n \geq 1$. Then $\mathrm{H}^i_{\mathrm{dR}}(S^n)$ is zero if $i \notin \{0, n\}$, $\mathrm{H}^0_{\mathrm{dR}}(S^n) = \mathbb{R}$, and $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is one-dimensional, generated by any ω in $\Omega^n(S^n)$ with $\int_{S^n} \omega \neq 0$. In other words: let $1 \leq i \leq n$ and ω in $\Omega^i(S_n)$; then there exists η in $\Omega^{i-1}(S^n)$ such that $\omega = d\eta$ if and only if $d\omega = 0$ and, if $i = n$, $\int_{S^n} \omega = 0$.*

Proof. Induction on n. Let $n \geq 1$. First we prove the statements concerning the dimension of the $\mathrm{H}^i_{\mathrm{dR}}(S^n)$. On S^n we have the two points $N := (0, \ldots, 0, 1)$ (the North pole) and $S := (0, \ldots, 0, -1)$ (the South pole). We let $U := S^n - \{S\}$ and $V := S^n - \{N\}$. Then U and V are isomorphic (as C^∞-manifolds) to $\mathbb{R}^n$, and $U \cap V$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, hence homotopic to S^{n-1}. We note that S^0 is the disjoint union of two points. As S^n is connected, we have $\mathrm{H}^0_{\mathrm{dR}}(S^n) = \mathbb{R}$, by construction. For $2 \leq i < n$ both terms surrounding $\mathrm{H}^i_{\mathrm{dR}}(S^n)$ in the Mayer–Vietoris sequence are zero, hence $\mathrm{H}^i_{\mathrm{dR}}(S^n) = 0$. For $i = 1$ we have the exact sequence:

$$\mathrm{H}^0_{\mathrm{dR}}(U) \oplus \mathrm{H}^0_{\mathrm{dR}}(V) \longrightarrow \mathrm{H}^0_{\mathrm{dR}}(U \cap V) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(S^n) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(U) \oplus \mathrm{H}^1_{\mathrm{dR}}(V),$$

which is the sequence:

$$\begin{array}{ccccccc} \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(S^n) & \longrightarrow & 0 \\ (x, y) & \mapsto & x - y & & & & \end{array}$$

if $n \geq 2$, and the sequence:

$$\begin{array}{ccccccc} \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(S^n) & \longrightarrow & 0 \\ (x, y) & \mapsto & (x - y, x - y) & & & & \end{array}$$

if $n = 1$. It follows that $\mathrm{H}^i_{\mathrm{dR}}(S^n) = 0$ for $0 < i < n$, and that $\mathrm{H}^1_{\mathrm{dR}}(S^1)$ is one-dimensional. Suppose now that $n \geq 2$. Then the exact sequence:

$$\mathrm{H}^{n-1}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{n-1}_{\mathrm{dR}}(U) \longrightarrow \mathrm{H}^{n-1}_{\mathrm{dR}}(U \cap V) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(S^n) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(U) \oplus \mathrm{H}^n_{\mathrm{dR}}(V)$$

reads:

$$0 \longrightarrow \mathrm{H}^{n-1}_{\mathrm{dR}}(S^{n-1}) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(S^n) \longrightarrow 0.$$

Hence all the dimensions are as claimed. In the exercises of last week it was shown that for all η in $\Omega^{n-1}(S^n)$ we have $\int_{S^n} d\eta = 0$, hence $d\Omega^{n-1}(S^n)$ is in the kernel of $\int_{S^n}$. As both $\int_{S^n} : \Omega^n(S^n) \to \mathbb{R}$ as $\Omega^n(S^n) \to \mathrm{H}^n_{\mathrm{dR}}(S^n)$ are 1-dimensional quotients, they are equal. $\square$

5.7 Theorem. *Let $n \geq 0$. Then $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C}))$ is of dimension one if i is even and $0 \leq i \leq 2n$, and zero otherwise.*

Proof. Let $n \geq 0$. Let $P := (0 : 0 \cdots : 1)$ in $\mathbb{P}^n(\mathbb{C})$. Let

$$U := \mathbb{P}^n(\mathbb{C}) - \{P\}, \quad V := \{(a_0 : \cdots : a_{n-1} : 1) \in \mathbb{P}^n(\mathbb{C}) \mid |a_i| < 1 \text{ for all } i\}.$$

Then V is contractible hence homotopic to a point, and $V \cap U = V - \{P\}$ is homotopic to S^{2n-1}. The map:

$$F \colon U \times \mathbb{R} \to U, \quad ((a_0 : \cdots : a_{n-1} : a_n), t) \mapsto (a_0 : \cdots : a_{n-1} : ta_n)$$

gives a homotopy from the identity on U to the projection to the subspace $\mathbb{P}^{n-1}(\mathbb{C})$ (the hyperplane at infinity) of $\mathbb{P}^n(\mathbb{C})$. Summarizing:

$$\mathrm{H}_{\mathrm{dR}}(U) = \mathrm{H}_{\mathrm{dR}}(\mathbb{P}^{n-1}(\mathbb{C})), \quad \mathrm{H}_{\mathrm{dR}}(V) = \mathrm{H}_{\mathrm{dR}}(\cdot), \quad \mathrm{H}_{\mathrm{dR}}(U \cap V) = \mathrm{H}_{\mathrm{dR}}(S^{2n-1}).$$

Hence the Mayer–Vietoris sequence reads:

$$\cdots \to \mathrm{H}^{i-1}_{\mathrm{dR}}(S^{2n-1}) \to \mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C})) \to \mathrm{H}^i_{\mathrm{dR}}(\cdot) \oplus \mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^{n-1}(\mathbb{C})) \to \mathrm{H}^i_{\mathrm{dR}}(S^{2n-1}) \to \cdots$$

It follows that for $i \neq 2n$ the map $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C})) \to \mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^{n-1}(\mathbb{C}))$ is an isomorphism, and that the map $\mathrm{H}^{2n-1}_{\mathrm{dR}}(S^{2n-1}) \to \mathrm{H}^{2n}_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C}))$ is an isomorphism. $\square$

5.8 Theorem. *Let $n \geq 0$. Then $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{R}))$ is $\mathbb{R}$ if $i = 0$, one-dimensional if $i = n$ and n is odd, and zero otherwise.*

Proof. Let $\iota \colon S^n \to S^n$ be the antipodal map. Then the obvious map $S^n \to \mathbb{P}^n(\mathbb{R})$ is the quotient for the action of the group $\langle \iota \rangle$. Hence the de Rham complex $(\Omega(\mathbb{P}^n(\mathbb{R})), d)$ is the subcomplex of ι-invariants of the de Rham complex $(\Omega(S^n), d)$ of S^n. The direct sum decomposition:

$$(\Omega(S^n), d) = (\Omega(S^n), d)^+ \oplus (\Omega(S^n), d)^-,$$

shows that in the direct sum decomposition $\mathrm{H}_{\mathrm{dR}}(S^n) = \mathrm{H}_{\mathrm{dR}}(S^n)^+ \oplus \mathrm{H}_{\mathrm{dR}}(S^n)^-$ we have:

$$\mathrm{H}_{\mathrm{dR}}(S^n)^+ = \mathrm{H}((\Omega(S^n), d)^+), \quad \mathrm{H}_{\mathrm{dR}}(S^n)^- = \mathrm{H}((\Omega(S^n), d)^-).$$

It remains to show that $\iota^* \colon \mathrm{H}^n_{\mathrm{dR}}(S^n) \to \mathrm{H}^n_{\mathrm{dR}}(S^n)$ is multiplication by $(-1)^{n-1}$ if $n \geq 1$. This is left as an exercise to the reader. Of course, one can also obtain the result of this theorem using appropriate Mayer–Vietoris arguments. $\square$

Without proof we state two results that are easily proved using the techniques above.

5.9 Theorem. *Let $r \geq 1$ and $P_1, \ldots, P_r$ be r distinct points of S^2. Put $X := S^2 - \{P_1, \ldots, P_r\}$. Then $\mathrm{H}^0_{\mathrm{dR}}(X) = \mathbb{R}$, $\dim \mathrm{H}^1_{\mathrm{dR}}(X) = r - 1$, $\mathrm{H}^i_{\mathrm{dR}}(X) = 0$ for $i \geq 2$.*

5.10 Theorem. *Let X be a compact orientable connected C^∞-manifold of dimension two. The classification of such surfaces says that X can be obtained by adding a certain number of handles to S^2. This number of handles, g say, is called called the genus of X. One has: $\mathrm{H}^0_{\mathrm{dR}}(X) = \mathbb{R}$, $\dim \mathrm{H}^1_{\mathrm{dR}}(X) = 2g$, $\dim \mathrm{H}^2_{\mathrm{dR}}(X) = 1$, and $\mathrm{H}^i_{\mathrm{dR}}(X) = 0$ for $i > 2$.*

We end this section with some naive experiment that turns out to be very successful. We recall that the $\mathbb{C}$-algebra of meromorphic functions on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is the field of fractions $\mathbb{C}(t)$ of $\mathbb{C}[t]$.

Let $r \geq 0$ be an integer, and $a_1, \ldots, a_r$ be r distincs elements of $\mathbb{C}$. We want to consider $X := \mathbb{P}^1(\mathbb{C}) - \{\infty, a_1, \ldots, a_r\}$ as a *complex algebraic* variety, which, for us, at this moment, means that we consider the $\mathbb{C}$-algebra $\mathcal{O}(X)$ of global functions on it to be the $\mathbb{C}$-algebra of rational functions $f \in \mathbb{C}(t)$ that have no pole outside $\{\infty, a_1, \ldots, a_r\}$. We have:

$$\mathcal{O}(X) = \mathbb{C}[t, 1/(t - a_1), \ldots, 1/(t - a_r)].$$

A $\mathbb{C}$-basis of $\mathcal{O}(X)$ is given by the t^n with $n \geq 0$ together with the $(t - a_i)^{n_i}$ with $1 \leq i \leq r$ and $n_i < 0$. The algebraic de Rham complex $\Omega(X)$ is then:

$$0 \longrightarrow \mathcal{O}(X) \xrightarrow{d} \mathcal{O}(X)\cdot dt \longrightarrow 0,$$

with f in $\mathcal{O}(X)$ mapping to $f'\cdot dt$, with f' the usual derivative of f. Using the $\mathbb{C}$-basis given above, it is easy to compute that $\mathrm{H}^0(\Omega(X)) = \mathbb{C}$ and $\mathrm{H}^1(\Omega(X))$ is of dimension r, and represented by the forms $(t - a_i)^{-1}dt$. Surprisingly, we have $\mathrm{H}(\Omega(X)) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}_{\mathrm{dR}}(X)$, with $\mathrm{H}_{\mathrm{dR}}(X)$ the de Rham cohomology of X viewed as a (real) C^∞-manifold.

This example is no coincidence. It is in fact true for any affine non-singular complex algebraic variety X that $\mathrm{H}(\Omega(X)) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}_{\mathrm{dR}}(X)$ (a theorem due to Grothendieck). The proof of that is quite complicated, and uses resolution of singularities.

6 Cohomology with compact supports, and Poincaré duality

One of the fundamental results concerning de Rham cohomology is Poincaré duality. In some examples that we have seen (the S^n and the $\mathbb{P}^n(\mathbb{C})$) one notes that $\mathrm{H}^i_{\mathrm{dR}}(X)$ and $\mathrm{H}^j_{\mathrm{dR}}(X)$ have the same dimension if $i + j = \dim_X$. On the other hand, this is not so for $\mathbb{R}^n$. We will see that this comes from the non-compactness of $\mathbb{R}^n$.

In order to set up Poincaré duality, we need a version of de Rham cohomology: the one with compact supports.

6.1 Definition. *Let $k \geq 0$, X a C^k-manifold and E a C^k-vector bundle on X. Then, for $U \subset X$ open, we let $E_c(U)$ denote the sub-$\mathbb{R}$-vector space of $E(U)$ consisisting of the sections with compact support. In particular, on a C^∞-manifold we have the complex $(\Omega_c(X), d)$.*

6.2 Definition. *Let X be a paracompact Hausdorff C^∞-manifold. Then we define $\mathrm{H}_{\mathrm{dR,c}}(X)$, the de Rham cohomology with compact supports, to be the homology of the complex $(\Omega_c(X), d)$. Explicitly, for $i \geq 0$ one has:*

$$\mathrm{H}^i_{\mathrm{dR,c}}(X) = \frac{\ker(d\colon \Omega^i_c(X) \to \Omega^{i+1}_c(X))}{\mathrm{im}(d\colon \Omega^{i-1}_c(X) \to \Omega^i_c(X))}.$$

Of course, if X is compact, then $\Omega_c(X) = \Omega(X)$, and $\mathrm{H}_{\mathrm{dR,c}}(X) = \mathrm{H}_{\mathrm{dR}}(X)$.

We note that a pullback of a differential form with compact support need not have compact support. Hence $X \mapsto (\Omega_c(X), d)$ is not a contravariant functor via pullback for arbitrary morphisms between C^∞-manifolds. Of course, if $f\colon X \to Y$ is such that the inverse image of a compact subset of Y is compact, then f does induce a morphism f^* of differential graded-commutative $\mathbb{R}$-algebras from $\Omega_c(Y)$ to $\Omega_c(X)$.

6.3 Definition. *A morphism $f\colon X \to Y$ of topological spaces is called* proper *if for every compact subset K of Y the inverse image $f^{-1}K$ is compact.*

With this definition it is obvious that:

> $\mathrm{H}_{\mathrm{dR,c}}$ is a contravariant functor for proper morphisms.

As often happens, when you lose something (such as pullback functoriality for arbitrary morphisms), you gain something else. Here it is:

> on a Hausdorff space, taking sections with compact support is covariant for open immersions.

Explicitly, let $U \subset V$ be open, with V a Hausdorff manifold, let E be a vector bundle on V and s in $E_c(U)$. Then the extension of s by zero is in $E_c(V)$ (note that $\mathrm{Supp}(s)$ is closed in V hence that s is zero in the neighborhood $V - \mathrm{Supp}(s)$ of $V - U$). We will denote the extension by zero of s again by s.

We will now establish the analogs for de Rham cohomology with compact supports of the Mayer–Vietoris sequence and the homotopy invariance. We start with the Mayer-Vietoris sequence.

Let X be a paracompact Hausdorff C^∞-manifold. Suppose that X is the union of two open subsets U and V:

$$X = U \cup V.$$

Then we have a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_c(U\cap V) & \xrightarrow{i} & \Omega_c(U)\oplus\Omega_c(V) & \xrightarrow{s} & \Omega_c(X) & \longrightarrow & 0 \\ & & & & (\alpha,\beta) & \longmapsto & \alpha+\beta & & \\ & & \omega & \longmapsto & (\omega,-\omega) & & & & \end{array} \tag{6.4}$$

It is clear that this sequence is exact, except possibly at $\Omega_c(X)$. The next proposition takes care of that.

6.5 Proposition. *The map s (for sum) in (6.4) is surjective.*

Proof. Let f and g be a partition of 1 subject to the cover of X by U and V, as in the previous section. Let ω be in $\Omega_c(X)$. Then $f\omega$ has support in U, hence is in $\Omega_c(U)$. Likewise, $g\omega$ is in $\Omega_c(V)$. It follows that $(f\omega, g\omega)$ is mapped to ω under s. □

As i and s above are morphisms of complexes, (6.4) is a short exact sequence of complexes, hence by the results of the previous section, we get a long exact sequence (Mayer–Vietoris for de Rham cohomology with compact supports).

6.6 Theorem. *Let X be a paracompact Hausdorff C^∞-manifold, covered by two open subsets U and V. Then the diagram in (6.4) gives, via Proposition 5.4, a long exact sequence:*

$$\to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(U\cap V) \to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(U)\oplus\mathrm{H}^{i-1}_{\mathrm{dR,c}}(V) \to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(X) \to \mathrm{H}^{i}_{\mathrm{dR,c}}(U\cap V) \to \mathrm{H}^{i}_{\mathrm{dR,c}}(U)\oplus\mathrm{H}^{i}_{\mathrm{dR,c}}(V) \to$$

Let us now give the analog of homotopy invariance. We should say first of all that $\mathrm{H}_{\mathrm{dR,c}}$ is *not* homotopy invariant. For example, one computes directly that $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R}) = \mathbb{R}$ (via integration), and $\mathrm{H}^0_{\mathrm{dR,c}}(\mathbb{R}) = 0$. Let X be a paracompact Hausdorff C^∞-manifold. Let $p_X\colon \mathbb{R}\times X \to X$ be the projection. Then p_X^* does not map $\Omega_c(X)$ to $\Omega_c(\mathbb{R}\times X)$, but *integration along the fibers*, as we will explain, does send $\Omega_c(\mathbb{R}\times X)$ to $\Omega_c(X)$, and *lowers the degree by one.* We recall from section 4 that for every i we have a direct sum decomposition:

$$\Omega^i_{\mathbb{R}\times X} = p_X^*\Omega^i_X \oplus \left(p_X^*\Omega^{i-1}_X \otimes p_{\mathbb{R}}^*\Omega^1_{\mathbb{R}}\right).$$

In local coordinates, this means the following. Suppose that $x_1,\ldots,x_n$ are coordinates on $U\subset X$, and let t be the coordinate on $\mathbb{R}$. Then $\Omega^i_{\mathbb{R}\times U}$ has the basis:

$$\begin{cases} dx_{j_1}\cdots dx_{j_i}, & 1\le j_1<\cdots<j_i\le n, \\ dx_{j_1}\cdots dx_{j_{i-1}}dt, & 1\le j_1<\cdots<j_{i-1}\le n. \end{cases}$$

For ω in $\Omega^i_c(\mathbb{R}\times X)$ we can write uniquely $\omega = \omega_1 + \omega_2\,dt$, with ω_1 in $p_X^*\Omega^i_c(\mathbb{R}\times X)$ and ω_2 in $p_X^*\Omega^{i-1}_c(\mathbb{R}\times X)$. With this notation, we define:

$$p_{X,!}\colon \Omega_c(\mathbb{R}\times X) \longrightarrow \Omega_c(X), \quad \omega \longmapsto \int_{\mathbb{R}} \omega_2\,dt. \tag{6.7}$$

6.8 Theorem. *Let X be a paracompact Hausdorff C^∞-manifold. The map $p_{X,!}$ above is a morphism of complexes, and induces isomorphisms $p_{X,!}\colon \mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{R}\times X) \to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(X)$. An inverse is induced by the morphism of complexes $i\colon \Omega_c(X)\to\Omega_c(\mathbb{R}\times X)$, $\omega\mapsto\omega e$, with e in $\Omega^1_c(\mathbb{R})$ such that $\int_{\mathbb{R}} e = 1$.*

Proof. It is clear that $p_{X,!}i$ is the identity on $\Omega_c(X)$. So it suffices now to show that $ip_{X,!}$ is homotopic to the identity on $\Omega_c(\mathbb{R}\times X)$. The homotopy operator is the following:

$$\omega=\omega_1+\omega_2\,dt \longmapsto \left((a,x)\mapsto \int_\infty^a i_x^*\omega_2\,dt - \left(\int_{-\infty}^a e\right)\left(\int_{\mathbb{R}} i_x^*\omega_2\,dt\right)\right),$$

where $i_x\colon \mathbb{R}\to\mathbb{R}\times X$ is the map $a\mapsto(a,x)$. (I take this on faith from Bott and Tu.) The verification of this is left to the reader. $\square$

6.9 Theorem. *Let $n\geq 0$. Then $\mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{R}^n)=0$ if $i\neq n$ and $\mathrm{H}^n_{\mathrm{dR,c}}(\mathbb{R}^n)=\mathbb{R}$ via the map $\Omega^n_c(\mathbb{R}^n)\to\mathbb{R}$, $\omega\mapsto\int_{\mathbb{R}^n}\omega$.*

Now we have the right prerequisites to formulate and prove Poincaré duality.

6.10 Theorem. (Poincaré duality) *Let X be a paracompact Hausdorff C^∞-manifold with a given orientation, of dimension n. Then the pairings:*

$$\Omega^i(X)\times\Omega^{n-i}_c(X)\longrightarrow\mathbb{R},\quad (\omega,\eta)\longmapsto\int_X\omega\eta$$

induce isomorphisms:

$$\mathrm{H}^i_{\mathrm{dR}}(X)\longrightarrow\mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)^\vee.$$

Proof. First we note that, as X is oriented, we have $\int_X\colon\Omega^n_c(X)\to\mathbb{R}$. Secondly, for $\omega\in\Omega^i(X)$ and $\eta\in\Omega^{n-i}_c(X)$ we have $\omega\eta$ in $\Omega^n_c(X)$. Hence the pairings exist. As $d(\omega\eta)=(d\omega)\eta+(-1)^i\omega d\eta$, $\omega\eta$ is closed if ω and η are closed. If $\omega=d\omega'$ and η is closed, then $d(\omega'\eta)=\omega\eta$ and $\int_X\omega\eta=0$. If ω is closed and $\eta=d\eta'$, then $d(\omega\eta')=(-1)^i\omega\eta$ and $\int_X\omega\eta=0$. Hence we get induced pairings:

$$\mathrm{H}^i_{\mathrm{dR}}(X)\times\mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)\longrightarrow\mathbb{R}.$$

These pairings induce morphisms:

$$\mathrm{H}^i_{\mathrm{dR}}(X)\longrightarrow\mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)^\vee.$$

It remains to show that these morphisms are isomorphisms. We will only prove this for X that have a finite *good cover*. For the general case, the reader is referred to the reference given in Bott and Tu: Greub, Halperin and Vanstone, "Connections, curvature, and cohomology", vol. 1, Academic Press, New York, 1972. We also note that a Poincaré duality statement for singular homology and singular cohomology with compact support that does not need the hypotheses Hausdorff and paracompact can be found in Massey's book on singular homology.

A good cover of X is a family U_i, $i\in I$ (with I an arbitrary set), such that for all non-empty finite subsets J of I, the intersection $U_J:=\cap_{i\in J}U_i$ is diffeomorphic to $\mathbb{R}^n$ or empty. Each paracompact Hausdorff manifold admits a good cover. The idea to prove this is to choose a Riemannian metric on X (via a partition of 1 subject to a cover by charts), and to use geodesically convex neighborhoods of points. For some details, see Bott and Tu. Compact X have finite good covers. Good covers are useful for computing cohomology using Mayer-Vietoris sequences.

Now assume that X has a finite good cover $U_1, \ldots, U_r$. Then we will prove by induction on r that the Poincaré duality morphisms are isomorphisms. For $r = 0$ or 1 this follows from our knowledge of the cohomology of $\emptyset$ and of $\mathbb{R}^n$. Now let $U := U_1 \cup \cdots \cup U_{r-1}$, and let $V := U_r$. Then we know that the Poincaré duality morphisms for U and V are isomorphisms. We have a morphism between long exact sequences (checking the commutativity is left to the reader):

$$\begin{array}{ccccccc}
\mathrm{H}^{i-1}_{\mathrm{dR}}(U\cap V) & \to & \mathrm{H}^{i}_{\mathrm{dR}}(X) & \to & \mathrm{H}^{i}_{\mathrm{dR}}(U)\oplus \mathrm{H}^{i}_{\mathrm{dR}}(V) & \to & \mathrm{H}^{i}_{\mathrm{dR}}(U\cap V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^{n-i+1}_{\mathrm{dR,c}}(U\cap V)^\vee & \to & \mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)^\vee & \to & \mathrm{H}^{n-i}_{\mathrm{dR,c}}(U)^\vee\oplus \mathrm{H}^{n-i}_{\mathrm{dR,c}}(V)^\vee & \to & \mathrm{H}^{n-i}_{\mathrm{dR,c}}(U\cap V)^\vee
\end{array}$$

The four vertical arrows surrounding the Poincaré duality morphisms for X are isomorphisms. The *five lemma* below finishes the proof. $\square$

6.11 Lemma. *Suppose we have a commutative diagram (of modules over some ring, say) with exact rows:*

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & J
\end{array}$$

such that the four outer vertical arrows are isomorphisms, then $C \to H$ is an isomorphism.

Proof. An ordinary diagram chase. $\square$

The proof of Theorem 6.10 also proves the following result.

6.12 Theorem. *Let X be a Hausdorff manifold with a finite good cover. Then the vector spaces $\mathrm{H}_{\mathrm{dR}}(X)$ and $\mathrm{H}_{\mathrm{dR,c}}(X)$ are finite dimensional. In particular, this holds for compact X.*

6.13 Example. Let $X := \mathbb{R}^2 - \mathbb{Z}$. Then $\mathrm{H}^1_{\mathrm{dR,c}}(X)$ has a basis indexed by $\mathbb{Z}$ (think of small loops around the integers). But $\mathrm{H}^1_{\mathrm{dR}}(X)$ is the dual of this space, and hence of dimension $|\mathbb{R}|$.

6.14 Remark. We note that Poincaré duality can be formulated on a non-orientable manifold if one uses the *orientation sheaf* $\Omega_X^{\dim X} \otimes \mathrm{Vol}_X^\vee$. See the book "Cohomology of sheaves" by Iversen.

There is some space left on this page. We use it to state one last theorem about de Rham cohomology: the Künneth formula for the cohomology of a product.

6.15 Theorem. (Künneth) *Let X and Y be paracompact C^∞-manifolds. Then the maps induced by the projections p_X and p_Y:*

$$\mathrm{H}_{\mathrm{dR}}(X)\otimes \mathrm{H}_{\mathrm{dR}}(Y) \longrightarrow \mathrm{H}_{\mathrm{dR}}(X\times Y), \quad \mathrm{H}_{\mathrm{dR,c}}(X)\otimes \mathrm{H}_{\mathrm{dR,c}}(Y) \longrightarrow \mathrm{H}_{\mathrm{dR,c}}(X\times Y)$$

are isomorphisms of graded-commutative algebras (possibly without unit in the second case).

For a proof we refer to Bott and Tu.

Opgaven week 1 (2002/09/09)

Maak tenminste vier van de volgende sommen, en lever die in op maandag 09/16 (of eerder).

1. Bewijs de uitspraken over een product $X \times Y$ die in de tekst gedaan worden, d.w.z., geef een atlas, en laat zien dat $(X \times Y, \mathrm{pr}_X, \mathrm{pr}_Y)$ de genoemde universele eigenschap heeft.
2. Laat zien dat de cirkel S^1 niet homeomorf is met een open deelverzameling van een $\mathbb{R}^n$.
3. Laat zien dat $\mathbb{R}$ niet homeomorf is met $\mathbb{R}^n$, voor n verschillend van 1.
4. Opgave 1.17 uit de tekst.
5. Opgave 1.21 uit de tekst.
6. Zij $n \geq 1$. Laat zien dat $\mathrm{SL}_n(\mathbb{R})$ samenhangend is.
7. Zij $n \geq 1$. Laat zien dat $\mathrm{SO}_n(\mathbb{R})$ samenhangend is.
8. Zij $n \geq 1$. Laat zien dat $\mathrm{Sp}_{2n}(\mathbb{R})$ samenhangend is.

Opgaven week 2 (2002/09/16)

Maak tenminste vier van de volgende sommen, en lever die in op maandag 09/23 (of eerder).

1. Laat f de afbeelding van $\mathbb{R}^3$ naar $\mathbb{R}^2$ zijn gegeven door: $f(x, y, z) = (xy, xz)$. Bepaal expliciet bij welke (x, y, z) in $\mathbb{R}^3$ f een submersie is. Geef een atlas voor de $f^{-1}\{(a, b)\}$ met $(a, b) \neq (0, 0)$ in $\mathbb{R}^2$. Is $f^{-1}\{(0, 0)\}$ (met de van $\mathbb{R}^3$ geïnduceerde topologie) een topologisch manifold?

2. (Inverse functie stelling.) Laat $k \geq 1$, $n \geq 0$, $U \subset \mathbb{R}^n$ open, en $f\colon U \to \mathbb{R}^n$ een C^k-afbeelding. Laat x in U zijn, en neem aan dat $(Df)x\colon \mathbb{R}^n \to \mathbb{R}^n$ bijectief is. Bewijs dat er open deelverzamelingen $V \subset U$ en $W \subset \mathbb{R}^n$ zijn, en een C^k-afbeelding $g\colon W \to V$, zodat x in V is, en $f\colon V \to W$ en g inversen van elkaar zijn. Hint: pas de impliciete functie stelling toe (op de afbeelding $F\colon \mathbb{R}^n \times U \to \mathbb{R}^n$, $(b, a) \mapsto f(a) - b$).

3. (Lokaal model van submersies.) Laat $n, m \geq 0$, $k \geq 1$, $U \subset \mathbb{R}^n$ open, en $f\colon U \to \mathbb{R}^m$ een C^k-afbeelding. Laat x in U zijn, en neem aan dat $(Df)x\colon \mathbb{R}^n \to \mathbb{R}^m$ surjectief is, d.w.z., f is een submersie in x.

   Laat zien dat er een C^k-isomorfisme $\phi\colon W \to V$ is met W open in $\mathbb{R}^n$ en V een open omgeving van x in U, zodat $f \circ \phi\colon W \to \mathbb{R}^m$ de projectie op de eerste m coordinaten is. (Een submersie ziet er lokaal, in geschikte coordinaten, uit als een lineaire projectie.)

   Hint: na een geschikte hernummering van de coordinaten van $\mathbb{R}^n$ is het zo dat de afbeelding $F\colon U \to \mathbb{R}^n$, $y \mapsto (f(y), y_{m+1}, \ldots, y_n)$ bijectieve afgeleide heeft in x (gebruik lineaire algebra). Neem voor ϕ een locale inverse van F zoals gegeven door de vorige opgave.

4. (Lokaal model van immersies.) Laat $k \geq 1$, $n, m \geq 0$, $U \subset \mathbb{R}^n$ open en $f\colon U \to \mathbb{R}^m$ een C^k-afbeelding. Laat x in U zijn, en neem aan dat $(Df)x\mathbb{R}^n \to \mathbb{R}^m$ injectief is, d.w.z., f is een *immersie* in x. Bewijs dat er open deelverzamelingen V en W in $\mathbb{R}^m$ zijn, met $f(x) \in V$, en een C^k-isomorfisme $\phi\colon V \to W$, zodat voor alle y in $f^{-1}V$ geldt:

   $$(\phi \circ f)(y_1, \ldots, y_n) = (y_1, \ldots, y_n, 0, \ldots, 0).$$

   In geschikte lokale coordinaten is een immersie dus een injectieve lineaire afbeelding.

   Hint: na een geschikte hernummering van de coordinaten van $\mathbb{R}^m$ heeft de afbeelding $F\colon U \times \mathbb{R}^{m-n} \to \mathbb{R}^m$, $(y, z) \mapsto (f(y), z)$ bijectieve afgeleide in $(x, 0)$.

5. Laat $n, m \geq 0$. Laat $U \subset \mathbb{R}^n$ open zijn, en $f\colon U \to \mathbb{R}^m$ een C^1-afbeelding met $(Df)(x)$ surjectief voor alle x in U (d.w.z., f is een submersie). Laat zien dat f een *open afbeelding* is, d.w.z., dat voor alle $V \subset U$ open fV open is in $\mathbb{R}^m$.

6. Laat $f\colon X \to Y$ een surjectief open morfisme van topologische ruimten zijn, met Y samenhangend, en zodat de vezels van f, d.w.z. de $f^{-1}\{y\}$ met y in Y (met hun geïnduceerde topologie) samenhangend zijn. Bewijs dat X samenhangend is. Hint: laat U een open en gesloten deelverzameling van X zijn, bewijs dat fU en $f(X - U)$ lege doorsnede hebben.

7. Laat $k \geq 0$. Laat X een verzameling zijn met een C^k-atlas (I, n, U, ϕ). Bewijs dat de collectie deelverzamelingen V van X met de eigenschap dat, voor alle i in I, $\phi_i^{-1}V$ open in $\mathbb{R}^{n_i}$ is, een topologie is. Laat zien dat dit de fijnste topologie is waarvoor alle kaarten ϕ_i continu zijn.

Opgaven week 3 (2002/09/30)

Maak tenminste drie van de volgende vier sommen, en lever die in op donderdag 10/10 (of eerder). (Deze week is anders vanwege het Leidens ontzet; donderdag 3 october is er geen college.)

1. Laat $k \geq 0$. Geef een functie van $\mathbb{R}$ naar $\mathbb{R}$ die C^k is, maar niet C^{k+1}.

2. Som 1.19 van de tekst.

3. Som 1.20 van de tekst.

4. Laat $\mathbb{F}$ een eindig lichaam zijn, en $q := |\mathbb{F}|$. Bereken $|\mathbb{P}^n(\mathbb{F})|$ voor alle $n \geq 0$, en $|\mathrm{Gr}_2(\mathbb{F}^4)|$. Hint: gebruik de bijectie $\mathrm{GL}_4(\mathbb{F})/P \to \mathrm{Gr}_2(\mathbb{F}^4)$ uit de tekst, en tel aantallen elementen in deze twee groepen.

Opgaven week 4 (2002/10/07)

Maak de volgende vier sommen, en lever die in op maandag 10/14 (of eerder).

1. Som 1.14 uit de tekst, in de context van topologische manifolds. Dat wil zeggen, laat zien dat $\mathbb{P}^1(\mathbb{R})$ homeomorf is met S^1, en $\mathbb{P}^1(\mathbb{C})$ met S^2. Gebruik bijvoorbeeld één-punts compacitifaties. In de volgende twee opgaven maken we in het laatste geval een mooie afbeelding (zelfs algebraïsch).

2. (Quaternionen algebra.) Laat $\mathbb{H}$ de deelverzameling van $\mathrm{M}_2(\mathbb{C})$ bestaande uit de elementen van de vorm $\left(\begin{smallmatrix} a & -\overline{b} \\ b & \overline{a} \end{smallmatrix}\right)$.

   (a) Laat zien dat $\mathbb{H}$ een $\mathbb{R}$-deel-algebra van $\mathrm{M}_2(\mathbb{C})$ is. Dus $\mathbb{H}$ is een associatieve $\mathbb{R}$-algebra met eenheidselement. Merk op dat $\mathbb{H}$ niet commutatief is.

   (b) Laat zien dat $h \mapsto h^* := \overline{h}^t$ (getransponeerde van de complex geconjugeerde) een involutie van $\mathbb{H}$ is (dus: $(h^*)^* = h$, $(\lambda_1 h_1 + \lambda_2 h_2)^* = \lambda_1 h_1^* + \lambda_2 h_2^*$, $(h_1 h_2)^* = h_2^* h_1^*$, voor alle h, h_1, h_2 in $\mathbb{H}$, λ_1, λ_2 in $\mathbb{R}$). Voor $h \in \mathbb{H}$ definiëren we $\mathrm{tr}(h) := h + h^*$ (het spoor) en $\mathrm{N}(h) := hh^*$ (de norm). Laat zien dat dit elementen van $\mathbb{R}$ zijn, dat $h^2 - \mathrm{tr}(h)h + \mathrm{N}(h) = 0$, en dat $N(h_1 h_2) = N(h_1)N(h_2)$ voor alle h_1 en h_2 in $\mathbb{H}$.

   (c) Laat zien dat $\mathbb{H}$ een delingsalgebra is: ieder element ongelijk aan nul is inverteerbaar. Geef een formule voor de inverse van een element $h \neq 0$ in termen van h^* en $\mathrm{N}(h)$.

   (d) Laat zien dat $(1, i, j, k)$ een $\mathbb{R}$-basis van $\mathbb{H}$ is, met $1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $i = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$, $j = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $k = ij$. Ga na dat $ji = -ij$ en $i^2 = j^2 = -1$. Bereken k^2, ik, ki, jk, kj door de associativiteit in $\mathbb{H}$ te gebruiken.

3. We bekijken de actie van de vermenigvuldigingsgroep $\mathbb{H}^*$ van $\mathbb{H}$ op $\mathbb{H}$ door conjugatie: $(h, x) \mapsto hxh^{-1}$.

   (a) Laat zien dat de baan $\mathbb{H}^* \cdot i$ van i de verzameling van h is met $\mathrm{tr}(h) = 0$ en $\mathrm{N}(h) = 1$. (Hint: dit kan door lineaire algebra te gebruiken (diagonalisatie eigenschappen van unitaire matrices).) Concludeer dat $\mathbb{H}^* \cdot i$ de verzameling van $bi + cj + dk$ is met b, c, d in $\mathbb{R}$ en $b^2 + c^2 + d^2 = 1$. Kortom, $\mathbb{H}^* \cdot i$ is S^2, en $h \mapsto hih^{-1}$ is een afbeelding van $\mathbb{H}^*$ naar S^2. Schrijf deze afbeelding expliciet uit voor $h = a + bi + cj + dk$ met a, b, c, d in $\mathbb{R}$.

   (b) We bekijken de afbeelding $\mathbb{C} \to \mathbb{H}$, $a + bi \mapsto a + bi$ (a en b in $\mathbb{R}$). Laat zien dat dit een injectief morfisme van $\mathbb{R}$-algebras is, en dat het beeld gelijk is aan de centralisator van i in $\mathbb{H}$, d.w.z., de verzameling van h zodat $hi = ih$.

   (c) De afbeelding $\mathbb{C} \to \mathbb{H}$ maakt van $\mathbb{H}$ een 2-dimensionale $\mathbb{C}$-vectorruimte, zeg met basis $(1, j)$. Concludeer dat de afbeelding $h \mapsto hih^{-1}$ van $\mathbb{H} - \{0\}$ naar S^2 een bijectie induceert van $\mathbb{P}^1(\mathbb{C})$ naar S^2.

4. (a) Laat zien dat de bijectie $\mathbb{P}^1(\mathbb{C}) \to S^2$ van de vorige opgave een homeomorfisme is.

   (b) Laat zien dat die bijectie een isomorfisme van C^∞-manifolds is. Hint: gebruik een kaart van $\mathbb{P}^1(\mathbb{C})$, en laat zien dat de afgeleide bijectief is. Als je weinig wilt rekenen, kun je de actie van $\mathbb{H}^*$ op $\mathbb{P}^1(\mathbb{C}) = \mathbb{H}^*/\mathbb{C}^*$ via links vermenigvuldigingen gebruiken; dan is het genoeg om in één punt te laten zien dat de afgeleide bijectief is.

Opgaven week 5 (2002/10/14)

Maak de volgende vier sommen, en lever die in op maandag 10/21 (of eerder).

1. Laat $k \geq 1$, en $f\colon X \to Y$ een morfisme van C^k-manifolds. Dan noemen we f een submersie als voor alle x in X geldt dat $\mathrm{T}_f(x)\colon \mathrm{T}_X(x) \to \mathrm{T}_Y(fx)$ surjectief is. Laat zien dat een submersie een open afbeelding is.

   Laat nu $n \geq 2$, $G := \mathrm{SO}_n(\mathbb{R})$. Dan werkt G op $\mathbb{R}^n$, en de baan van de standaard eerste basis vector e_1 is de eenheidsbol S^{n-1}. We bekijken de afbeelding $f\colon G \to S^{n-1}$, $g \mapsto ge_1$. Dan is f een morfisme van C^∞-manifolds, compatibel met de G-actie: $f(gx) = gf(x)$ voor alle g en x in G. Laat zien dat hieruit volgt dat $\mathrm{T}_f(g)$ surjectief is voor alle g dan en slechts dan als $\mathrm{T}_f(1)$ surjectief is (maak een mooi commutatief diagram met allemaal raakafbeeldingen). Bereken $\mathrm{T}_{S^{n-1}}(e_1)$ en $\mathrm{T}_G(e_1)$ als deelruimten van $\mathbb{R}^n$ en $\mathrm{M}_n(\mathbb{R})$ (respectievelijk), en laat zien dat $\mathrm{T}_f(1)$ surjectief is.

   Laat zien dat de vezels van f allemaal isomorf zijn met $\mathrm{SO}_{n-1}(\mathbb{R})$ (gebruik weer de actie van G), en concludeer dat $\mathrm{SO}_n(\mathbb{R})$ samenhangend is.

2. Laat $k \geq 1$, en laat X en Y beide C^k-manifolds zijn. Laat x in X en y in Y. Dan hebben we een morfisme $p\colon \mathrm{T}_{X\times Y}(x,y) \to \mathrm{T}_X(x) \times \mathrm{T}_Y(y)$ geïnduceerd door de projecties $p_X\colon X \times Y \to X$ en $p_Y\colon X \times Y \to Y$. Aan de andere kant is er een morfisme $i\colon \mathrm{T}_X(x)\times\mathrm{T}_Y(y) \to \mathrm{T}_{X\times Y}(x,y)$ geïnduceerd door de injecties $i_{X,y}\colon X \to X\times Y$, $x' \mapsto (x',y)$ en $i_{Y,x}\colon Y \to X \times Y$, $y' \mapsto (x,y')$. Laat zien dat p en i inversen zijn. (Gebruik een kaart te (x,y), bijvoorbeeld.)

3. Laat $k \geq 1$. Laat G een C^k-Liegroep zijn (d.w.z., G is een C^k-manifold met compatibele groepsstructuur). Dan hebben we morfismen van C^k-manifolds $\mu\colon G \times G \to G$, $(x,y) \mapsto xy$, en $\iota\colon G \to G$, $x \mapsto x^{-1}$. Laat zien dat $\mathrm{T}_\mu\colon \mathrm{T}_G(e) \times \mathrm{T}_G(e) = \mathrm{T}_{G\times G}(e) \to \mathrm{T}_G(e)$ simpelweg de optelling is, en dat $\mathrm{T}_\iota(e)$ vermenigvuldiging met -1 is.

4. Laat $k \geq 1$, en G een C^k-Liegroep zijn. Laat H een gesloten C^k-ondergroep van G zijn. Dan werkt H van rechts op G door translaties. We gaan het quotiënt G/H de structuur van C^k-manifold geven, en laten zien dat de quotiënt afbeelding $p\colon G \to G/H$ een vezeling is, met vezel H.

   We volgen het procédé dat we hebben gezien bij de $\mathrm{Gr}_{d,n}(\mathbb{R})$. Dus G/H krijgt de quotiënt topologie (U is open dan en slechts dan als $p^{-1}U$ open is), en we definiëren voor U open dat $C^k_{G/H}(U) := C^k_G(p^{-1}U)^H$.

   Omdat p compatibel is met de linksactie van G ($p(gx) = g{\cdot}p(x)$ voor alle x en g), is het voldoende aan te tonen dat $(G/H, C^k_{G/H})$ een C^k-manifold is in een omgeving van $p(e)$. Laat zien dat er een kaart $\phi\colon U \to G$ bestaat, met $\phi(0) = e$ en $U \subset \mathbb{R}^n \times \mathbb{R}^m$, zodat $\phi^{-1}H = U \cap (\mathbb{R}^n \times \{0\})$ (gebruik som 4 van week 2). Laat $T \subset U \cap (\{0\} \times \mathbb{R}^m)$ een open omgeving van 0 zijn, en bekijk de afbeelding:

   $$f\colon T \times H \longrightarrow G, \quad (t,h) \mapsto \phi(t)h.$$

   Laat zien dat $\mathrm{T}_f(0)$ bijectief is en concludeer dat dit zo is voor alle $(t,0)$ met t in een omgeving van 0. Door T klein genoeg te nemen mogen we aannemen dat $\mathrm{T}_f(t,0)$ bijectief is voor alle t. Concludeer dat $\mathrm{T}_f(t,h)$ bijectief is voor alle (t,h), aangezien f compatibel is met de rechts-actie van H. Laat zien dat als T voldoende klein is, dat $\phi(t_1)^{-1}\phi(t_2) \in H$ impliceert dat $t_1 = t_2$. Neem aan dat T voldoende klein is, en concludeer dat f injectief is, en dus een isomorfisme naar een open deel van G, van de vorm $p^{-1}U$. Laat zien dat $\overline{f} := pf \colon T \to G/H$ een isomorfisme is van (T, C_T^k) naar $(U, C_{G/H}^k|_U)$. Dus $(G/H, C_{G/H}^k)$ is een C^k-manifold.

Concludeer dat $p \colon G \to G/H$ een vezeling is met vezel H (dit volgt direct uit het voorgaande in een omgeving van $p(e)$; gebruik dan de links-G-actie).

Opgaven week 6 (2002/10/28)

Maak van de volgende vier sommen de eerste drie, en lever die in op maandag 11/4 (of eerder) (de vierde som wordt donderdag op college behandeld).

1. Maak de som die impliciet in het bewijs van Propositie 3.1.1 beschreven is.

2. Laat i in $\mathbb{Z}_{\geq 0}$, A een ring, en M een A-moduul.

   (a) Laat $n \geq 0$ en neem aan dat M vrij is van rang n. Bewijs dat de rang van het vrije A-moduul $\mathrm{S}^i(M)$ gelijk is aan $\binom{i+n-1}{n-1}$ met een combinatorisch argument (d.w.z. laat zien dat het aantal monomen van graad i in de variabelen $x_1, \ldots, x_n$ gelijk is aan het aantal keuzen van $n-1$ elementen uit een verzameling met $i+n-1$ elementen).

   (b) Laat zien dat de afbeelding $M^i \to \mathrm{S}^i(M)$, $(m_1, \ldots, m_i) \mapsto m_1 \cdots m_i$ een universele symmetrische i-lineaire afbeelding is. (Hint: gebruik dat $M^i \to \mathrm{T}^i(M)$ een universele i-lineaire afbeelding is, en laat zien dat de kern van $\mathrm{T}^i(M) \to \mathrm{S}^i(M)$ voortgebracht wordt door elementen van de vorm $u_1 \otimes \cdots \otimes u_j \otimes (x \otimes y - y \otimes x) \otimes v_1 \otimes \cdots \otimes v_k$ met $j \geq 0$, $k \geq 0$, en $2 + j + k = i$.)

   (c) Laat zien dat de afbeelding $M^i \to \Lambda^i(M)$, $(m_1, \ldots, m_i) \mapsto m_1 \cdots m_i$ een universele alternerende i-lineaire afbeelding is.

3. Laat k een lichaam zijn, V een k-vectorruimte en d in $\mathbb{Z}_{\geq 0}$. We hebben de Grassman variëteit $\mathrm{Gr}_d(V)$ van d-dimensionale deelruimten van V gedefiniëerd, met de structuur van C^∞-manifold als k gelijk is aan $\mathbb{R}$ en V eindig dimensionaal is.

   Laat W een d-dimensionale deelruimte van V zijn, en laat $w_1, \ldots, w_d$ een basis zijn van W. Dan krijgen we een element $w_1 \cdots w_d$ van $\Lambda^d(V)$. Laat zien dat dit element ongelijk aan nul is, en dat zijn beeld in $\mathbb{P}(\Lambda^d(V))$ alleen van W afhangt. Dus we krijgen een afbeelding ϕ van $\mathrm{Gr}_d(V)$ naar $\mathbb{P}(\Lambda^d(V))$. Laat zien dat deze afbeeling injectief is. Deze afbeelding heet de Plücker inbedding.

   Laat dan zien dat, voor $k = \mathbb{R}$, ϕ een C^∞-morfisme is, met overal injectieve afgeleide. (Hier is het handig om de actie van $\mathrm{GL}(V)$ te gebruiken.)

4. Laat A een $\mathbb{Q}$-algebra zijn, M een A-moduul, en $n \geq 0$. De groep $G := \mathrm{S}_n$ van permutaties van $\{1, \ldots, n\}$ werkt (van rechts) op het A-moduul M^n door $(\sigma m)_i = m_{\sigma(i)}$, en dus (ook van rechts) op $\mathrm{T}^n(M)$: $m_1 \cdots m_n$ wordt naar $m_{\sigma(1)} \cdots m_{\sigma(n)}$ gestuurd. Laat nu p_1 en p_ε de endomorfismen van $\mathrm{T}^n(M)$ zijn gegeven door:

   $$p_1(x) = \frac{1}{n!} \sum_\sigma x{\cdot}\sigma, \qquad p_\varepsilon(x) = \frac{1}{n!} \sum_\sigma \varepsilon(\sigma) x{\cdot}\sigma,$$

   waarbij de sommen over alle σ in S^n zijn. Laat zien dat p_1 en p_ε orthogonale idempotenten zijn: $p_1^2 = p_1$, $p_\varepsilon^2 = p_\varepsilon$, en $p_1 p_\varepsilon = p_\varepsilon p_1 = 0$. Laat $\mathrm{S}^n(M)' := p_1 \mathrm{T}^n(M)$, en $\Lambda^n(M)' := p_\varepsilon \mathrm{T}^n(M)$. Dan heten $\mathrm{S}^n(M)'$ en $\Lambda^n(M)'$ de A-modulen van *symmetrische* en *anti-symmetrische* tensoren van graad n van M. Laat zien dat de natuurlijke afbeeldingen $\mathrm{S}^n(M)' \to \mathrm{S}^n(M)$ en $\Lambda^n(M)' \to \Lambda^n(M)$ isomorfismen zijn. (Hierbij is het handig te gebruiken dat S_n ook op $\mathrm{S}^n(M)$ en $\Lambda_n(M)$ werkt, compatibel met de actie op $\mathrm{T}^n(M)$.)

Opgaven week 8 (2002/11/18)

Maak de volgende sommen, en lever die in op maandag 11/25 (of eerder).

Laat $X := S^2$ de eenheidsbol in $\mathbb{R}^3$ zijn, dus gegeven door de vergelijking $x_1^2 + x_2^2 + x_3^2 = 1$. We geven X de Riemannse metriek geïnduceerd door de standaardmetriek op $\mathbb{R}^3$.

1. Bereken de gradiënt van de functie x_1 op X. Is deze gelijk aan de orthogonale projectie van de gradiënt van x_1 gezien als functie op $\mathbb{R}^3$? En zo ja, kun je dat ook inzien zonder te rekenen?

2. Geef een orthogonale basis (e_1, e_2) van T_X op $X - \{(0,0,1),(0,0,-1)\}$ (d.w.z. op het complement van de noord en zuidpool), zoals je die op wereldbollen ziet. Teken de twee vectorvelden e_1 en e_2 in een omgeving van de noordpool.

3. Geef een orthogonale basis (e_1, e_2) van T_X op $X - \{(0,0,1)\}$ (d.w.z. op het complement van de noordpool). Hint: je kunt een stereografische projectie gebruiken. Teken de twee vectorvelden (e_1, e_2) in de buurt van de noordpool.

Opgaven week 9 (2002/11/25)

Maak de volgende sommen, en lever die in op maandag 12/02 (of eerder).

1. Laat Y een gesloten deelverzameling van $\mathbb{R}$ zijn, en U zijn complement. Laat X het C^∞-manifold zijn gekregen door twee copiën van $\mathbb{R}$ aan elkaar te plakken via de identiteitsafbeelding van U naar zichzelf (dus X is de reële lijn met Y verdubbeld). Laat g_1 en g_2 de bijbehorende open immersies van $\mathbb{R}$ in X zijn. Laat v een volume vorm op X zijn, met quasicompacte support.

   Druk $\int_X v$ uit in de integralen over $\mathbb{R}$ en over U van $g_1^* v$ en $g_2^* v$.

   Druk $\int_X v$ uit in de integralen over Y en over U van $g_1^* v$ en $g_2^* v$.

   Geef een voorbeeld waar al deze deze integralen over $\mathbb{R}$, U en Y ongelijk aan nul zijn.

2. Laat $n \geq 1$ en $G := \mathrm{GL}_n(\mathbb{R})$. Laat $x_{i,j} \colon G \to \mathbb{R}$ de functie zijn gegeven door: $g \mapsto g_{i,j}$. Bewijs dat de differentiaalvorm

$$\frac{dx_{1,1}dx_{2,1}\cdots dx_{n,1}\cdot dx_{1,2}\cdots dx_{n,2}\cdots\cdots dx_{n,1}\cdots dx_{n,n}}{\det^n}$$

   van graad n^2 invariant is onder alle links en rechtstranslaties van G naar zichzelf. Hier is $\det \colon G \to \mathbb{R}$ de determinant-functie.

   Aanwijzing: de definities toepassen. Laat bijvoorbeeld zien dat $(g\cdot)^* x_{i,j} = \sum_k g_{i,k} x_{k,j}$ door $(gh)_{i,j} = \sum_k g_{i,j} h_{k,j}$ te gebruiken.

Supplement gradiënt

In het college is, voor $k \geq 1$, een *pseudo-Riemannse C^k-manifold* gedefiniëerd als een paar (X, b), met X een C^k-manifold en b een metriek op T_X. Voor wat zo'n b is: zie het begin van sectie 3.3. Als b positief is dan heet (X, b) een *Riemannse C^k-manifold*.

Als (X, b) en (X', b') pseudo-Riemannse C^k-manifolds zijn, dan heet een C^k-morfisme f van X naar X' een *isometrie* als voor iedere x in X de $\mathbb{R}$-lineaire afbeelding $\mathrm{T}_f(x)\colon \mathrm{T}_X(x) \to \mathrm{T}_{X'}(f(x))$ een isometrie is, d.w.z., voor alle v en w in $\mathrm{T}_X(x)$ geldt $b'((\mathrm{T}_f(x))v, (\mathrm{T}_f(x))w) = b(v, w)$.

Laat (X, b) een pseudo-Riemannse manifold zijn. We noteren de metriek b soms ook als $\langle\cdot,\cdot\rangle$, en we gebruiken ook dat b een isomorfisme $\phi\colon T_X \to \Omega^1_X$ geeft zodat voor alle x in X, en voor alle v en w in $\mathrm{T}_X(x)$: $((\phi x)v)w = \langle v, w\rangle_x$. We definiëren we nu een morfisme "gradiënt" als volgt:

$$\mathrm{grad}\colon C^k_X \xrightarrow{d} \Omega^1_X \xrightarrow{\phi^{-1}} \mathrm{T}_X.$$

Het is duidelijk dat voor $\mathbb{R}^n$ met de standaardmetriek dit begrip van gradiënt overeenkomt met het begrip dat we al kennen (want de standaardbases van Ω^1_X en T_X zijn dan compatibel via ϕ). Dit levert ons de informatie dat isometrieën van $\mathbb{R}^n$ compatibel zijn met het nemen van gradiënten.

In het algemeen geldt dat voor $U \subset X$ open, f in $C^k_X(U)$ en ∂ in $\mathrm{T}_X(U)$:

$$\langle \mathrm{grad} f, \partial\rangle = (df)\partial = \partial f.$$

Opgaven week 10 (2002/12/02)

Laat $n \geq 0$, en X een Hausdorffs georiënteerd C^∞-manifold met $\dim_X = n$: voor iedere x in X hebben we een decompositie $\Omega^n_X(x) = \Omega^n_X(x)^+ \coprod \{0\} \coprod \Omega^n_X(x)^-$, die lokaal constant is. Laat nu ω een element zijn van $\Omega^{n-1}_X(X)$, met compacte support. We gaan bewijzen dat $\int_X d\omega = 0$, een speciaal geval van de stelling van Stokes.

1. Laat zien dat voor iedere x in X er een georiënteerde kaart $\phi\colon U \to X$ is met $x \in \phi(U)$, en een C^∞-functie $f\colon X \to [0,1] \subset \mathbb{R}$ met compacte support bevat in $\phi(U)$ en $f = 1$ in een omgeving van x. Waar heb je hierbij nodig dat X Hausdorff is? (Geef eventueel een voorbeeld van een niet Hausdorffse X en een x waarvoor niet zo'n ϕ en f bestaan.)

2. Laat $K \subset X$ een compacte deelverzameling zijn. Laat zien dat er een $r \geq 0$ bestaat, georiënteerde kaarten $\phi_i\colon U_i \to X$, voor $1 \leq i \leq r$, en functies p_i in $C^\infty_X(X)$ met $\mathrm{Supp}(p_i)$ compact en bevat in $\phi_i(U_i)$, en $\sum_i p_i = 1$ op K. Zie eventueel Lemma 12.4 en het bewijs daarvan in het diktaat van Looijenga.

3. Laat $K := \mathrm{Supp}(\omega)$, en laat r, ϕ_i en p_i als hierboven zijn. Dan hebben we:
$$\int_X d\omega = \int_X d(\sum_i p_i\omega) = \sum_i \int_X d(p_i\omega) = \sum_i \int_{\mathbb{R}^n} \phi_i^* d(p_i\omega) = \sum_i \int_{\mathbb{R}^n} d(\phi_i^*(p_i\omega)).$$
Om te bewijzen dat $\int_X d\omega = 0$ mogen we nu dus aannemen dat $X = \mathbb{R}^n$.

4. Nu is X gelijk aan $\mathbb{R}^n$. Schrijf $\omega = \sum_i f_i \prod_{j\neq i} dx_j$, waarbij de producten in de standaard volgorde zijn genomen. We mogen dus ook aannemen, na eventuele permutatie van de coordinaten, dat $\omega = f dx_2 \cdots dx_n$. Schrijf het nu gewoon verder uit (integreer eerst naar de variabele x_1, en gebruik dat voor alle $(a_2, \ldots, a_n)$ in $\mathbb{R}^{n-1}$ de functie $x \mapsto f(x, a_1, \ldots, a_n)$ compacte support heeft).

Opgaven week 11 (2002/12/09)

1. Laat zien dat S^n oriënteerbaar is voor alle $n \geq 0$. Hint: gebruik de standaardoriëntatie van $\mathbb{R}^{n+1}$ en de uitwendige genormaliseerde normaalvector.

2. Laat $n \geq 0$, en laat $\iota\colon S^n \to S^n$ de antipodale afbeelding zijn. Laat zien dat $\mathbb{P}^n(\mathbb{R}) = S^n/\langle\iota\rangle$.

3. Laat $n \geq 1$. Laat zien dat ι als in de vorige opgave de twee oriëntaties van S^n verwisselt precies dan als n even is. Concludeer dat $\iota^*\colon \mathrm{H}^n_{\mathrm{dR}}(S^n) \to \mathrm{H}^n_{\mathrm{dR}}(S^n)$ vermenigvuldiging met $(-1)^{n-1}$ is.

4. (De stelling van de kruin.) Laat $n \geq 2$ een even geheel getal zijn en D een C^∞-vectorveld op S^n zijn. Bewijs dat D een nulpunt heeft.

   Hint: stel dat D geen nulpunt heeft. Laat zien dat er dan een homotopie is tussen de identiteit op S^n en de antipodale afbeelding. Merk op dat dit in tegenspraak is met de vorige opgave.

5. Laat $n \geq 1$ een oneven geheel getal zijn. Geef een C^∞-vectorveld op S^n zonder nulpunt.