4 De Rham cohomology

4.1 Definition. *A differential graded-commutative $\mathbb{R}$-algebra is an $\mathbb{R}$-algebra A with a grading $A = \oplus_{i\in\mathbb{Z}} A_i$, which is graded-commutative (i.e., for x in A_i and y in A_j one has $yx = (-1)^{ij}xy$), and which is equipped with a differential d of degree one (i.e., an $\mathbb{R}$-linear map d from A to A such that $d^2 = 0$, $d(A_i) \subset A_{i+1}$ and $d(xy) = (dx)y + (-1)^i x dy$ for x in A_i and y in A).*

We have a contravariant functor $X \mapsto (\Omega_X(X), d)$ from the category of C^∞-manifolds to that of differential graded-commutative $\mathbb{R}$-algebras. Note that the vector spaces $\Omega_X(X)$ tend to be very big (the typical dimension is $|\mathbb{R}|$). Although this functor does transform morphisms of manifolds into $\mathbb{R}$-linear maps, it does not really simplify the study of the category of C^∞-manifolds. For example, a compact manifold X can be reconstructed from $\Omega^0_X(X)$ alone (the points of X correspond to the maximal ideals, the topology is the Zariski topology, etc.). But, if one composes the functor $X \mapsto (\Omega_X(X), d)$ with the functor that takes homology of differential graded-commutative algebras, then a miracle happens. This composed functor, called de Rham cohomology, has reasonable finiteness properties, and, most importantly, is homotopy invariant. Before proving that, let us write down the definitions in detail.

4.2 Lemma. *Let A be a differential graded-commutative $\mathbb{R}$-algebra. Then its homology $\mathrm{H}(A)$, defined as $\ker(d)/\mathrm{im}(d)$, has an induced structure of graded-commutative $\mathbb{R}$-algebra.*

Proof. Details are left to the reader. Show that $\ker(d)$ is a graded-commutative subalgebra of A, in which $\mathrm{im}(d)$ is a homogeneous ideal. $\square$

4.3 Definition. *Let X be a paracompact C^∞-manifold. Then the de Rham cohomology of X is the graded-commutative $\mathbb{R}$-algebra $\mathrm{H}_{\mathrm{dR}}(X) := \mathrm{H}(\Omega_X(X), d)$. Hence one has:*

$$\mathrm{H}^i_{\mathrm{dR}}(X) = \frac{\ker(d\colon \Omega^i_X(X) \to \Omega^{i+1}_X(X))}{\mathrm{im}(d\colon \Omega^{i-1}_X(X) \to \Omega^i_X(X))}.$$

As explained above, $\mathrm{H}_{\mathrm{dR}}(\cdot)$ is a contravariant functor from the category of manifolds to that of graded-commutative $\mathbb{R}$-algebras. For $f\colon X \to Y$ we will write f^* for the morphism $\mathrm{H}_{\mathrm{dR}}(f)$ from $\mathrm{H}_{\mathrm{dR}}(Y)$ to $\mathrm{H}_{\mathrm{dR}}(X)$.

4.4 Remark. Of course we could define de Rham cohomology for arbitrary C^∞-manifolds (i.e., not necessarily paracompact), but that would not give the results we want. In the general case one should consider the so-called *hypercohomology* $\mathbb{H}_{dR}(X) := \mathbb{H}(\Omega_X, d)$ of the complex of sheaves Ω_X.

4.5 Theorem. *The de Rham cohomology of paracompact C^∞-manifolds is homotopy invariant, i.e., if f_0 and f_1 from X to Y are homotopic, then the two maps f_0^* and f_1^* from $\mathrm{H}_{\mathrm{dR}}(Y)$ to $\mathrm{H}_{\mathrm{dR}}(X)$ are equal.*

Proof. By definition of homotopy, we have a morphism of C^∞-manifolds $F\colon X \times I \to Y$ with I an open interval containing 0 and 1, such that $F|_{X\times\{0\}} = f_0$ and $F|_{X\times\{1\}} = f_1$. (It would be enough to have F a C^1-morphism such that its restrictions to all $X \times i$ with i in I are C^k, but let us not bother.) We will construct a homotopy from 0 to $f_0^* - f_1^*$, i.e., a sequence of $\mathbb{R}$-linear maps $K^i\colon \Omega^i_Y(Y) \to \Omega^{i-1}_X(X)$ such that $f_0^* - f_1^* = dK + Kd$. In order to construct this K, we need to consider $\Omega_{X\times I}$ more closely. Let p_X and p_I denote the projections from $X \times I$ to X and

I, respectively. (More generally, one should consider the vector bundle of differential forms on a product of two manifolds.) At a point (x, i) of $X \times I$, the tangent space $\mathrm{T}_{X\times I}(x, i)$ is canonically isomorphic to the direct sum $\mathrm{T}_X(x) \oplus \mathrm{T}_I(i)$. On the level of vector bundles this gives us:

$$\mathrm{T}_{X\times I} = p_X^* \mathrm{T}_X \oplus p_I^* \mathrm{T}_I.$$

Dualizing this gives:

$$\Omega^1_{X\times I} = p_X^* \Omega^1_X \oplus p_I^* \Omega^1_I.$$

We suggest to the reader to prove that for a ring A and A-modules M_1 and M_2 one has

$$\Lambda^i(M_1 \oplus M_2) = \bigoplus_{j+k=i} \Lambda^j(M_1) \otimes_A \Lambda^k(M_2).$$

In fact, $\Lambda(M_1 \oplus M_2)$ and $\Lambda(M_1) \otimes_A \Lambda(M_2)$ with its grading obtained by "convolution" are naturally isomorphic, as they have the same universal property: to give a morphism of graded A-algebras to a graded-commutative A-algebra B is to give morphisms of A-modules $\phi_1\colon M_1 \to B_1$ and $\phi_2\colon M_2 \to B_2$. It follows that we have a natural isomorphism:

$$\Omega^i_{X\times I} = p_X^* \Omega^i_X \oplus \left(p_X^* \Omega^{i-1}_X \otimes p_I^* \Omega^1_I\right).$$

In local coordinates, this means the following. Suppose that $x_1, \ldots, x_n$ are coordinates on $U \subset X$, and let t be the coordinate on I. Then $\Omega^i_{U\times I}$ has the basis:

$$\begin{cases} dx_{j_1} \cdots dx_{j_i}, & 1 \leq j_1 < \cdots < j_i \leq n, \\ dx_{j_1} \cdots dx_{j_{i-1}} dt, & 1 \leq j_1 < \cdots < j_{i-1} \leq n. \end{cases}$$

For ω in $\Omega^i_{X\times I}(X \times I)$ we can write uniquely $\omega = \omega_1 + \omega_2\, dt$, with ω_1 in $p_X^* \Omega^i_X(X \times I)$ and ω_2 in $p_X^* \Omega^{i-1}_X(X \times I)$. This decomposition of the $\Omega^i_{X\times I}$ also induces a decomposition of the differential $d_{X\times I}$ on $\Omega_{X\times I}$: we have, uniquely, $d_{X\times I} = d_X + d_I$, with d_X increasing the degree with respect to X by one, and d_I the same for I. (For an arbitrary product, this displays the complex $\Omega_{X\times Y}$ as the total complex associated to the double complex $p_X^* \Omega_X \otimes p_Y^* \Omega_Y$.)

We can now define our homotopy operators K_1^i from $\Omega^i_{X\times I}(X \times I)$ to $\Omega^{i-1}_X(X)$; the operator K we want to have will be $K_1 \circ F^*$. With the notation as above ($\omega = \omega_1 + \omega_2$) we define:

$$K_1^i \omega := (-1)^i \int_0^1 \omega_2\, dt. \tag{4.5.1}$$

Since the object to be integrated does not look like a function that one usually integrates, let us write it a bit more explicit. Let x be in X. Then $(K_1^i \omega)(x) = (-1)^i \int_0^1 \omega_2(x, t)\, dt$, and $\omega_2(x, t)$ is an element of the vector space $\Omega^{i-1}_X(x)$ that does not depend on t. This means that the integral is exactly of the kind we considered in the previous section: we integrate a vector space valued volume form (namely: $\omega_2\, |dt|$) over a compact subset $[0, 1]$ of the manifold I. Let i_0 and i_1 denote the inclusions of X into $X \times I$ that send x to $(x, 0)$ and $(x, 1)$, respectively. Then we claim that for all ω in $\Omega^i_{X\times I}(X \times I)$ we have:

$$(dK_1^i + K_1^{i+1} d)\omega = i_0^* \omega - i_1^* \omega. \tag{4.5.2}$$

To prove this identity, note that it is a local problem on X, and that both sides are additive in ω. Hence we may assume that $x_1, \ldots, x_n$ are local coordinates on X and that ω is of the form $f dx_1 \cdots dx_i$ or $g dx_1 \cdots dx_{i-1} dt$ with f and g in $C_{X\times I}(X \times I)$. Let us first consider the case $\omega = f dx_1 \cdots dx_i$. Then $K_1^i \omega = 0$ because $\omega_2 = 0$. On the other hand,

$$d\omega = df dx_1 \cdots dx_i = (d_X f)dx_1 \cdots dx_i + (d_I f)dx_1 \cdots dx_i.$$

It follows that:

$$K_1 d\omega = (-1)^{i+i+1} K_1((\partial f/\partial t)dx_1 \cdots dx_i dt) = -\left(\int_0^1 (\partial f/\partial t)dt\right) dx_1 \cdots dx_i = i_0^*\omega - i_1^*\omega.$$

One should note that the last identity is exactly the fundamental theorem of calculus. Let us now consider the second case: $\omega = g dx_1 \cdots dx_{i-1} dt$. Now we have: $\omega = \omega_2$, hence:

$$K_1^i \omega = (-1)^i \left(\int_0^1 g\, dt\right) dx_1 \cdots dx_{i-1} = (-1)^i G dx_1 \cdots dx_{i-1},$$

where G is the function on X defined by $G(x) = \int_0^1 g(x,t)\, dt$. It follows that:

$$dK_1^i \omega = (-1)^i dG dx_1 \cdots dx_{i-1}.$$

On the other hand, $d\omega = (d_X g)dx_1 \cdots dx_{i-1} dt$. Hence:

$$\begin{aligned}(-1)^{i+1} K_1^{i+1} d\omega &= \left(\int_0^1 (d_X g)\, dt\right) dx_1 \cdots dx_{i-1} = d\left(\int_0^1 g\, dt\right) dx_1 \cdots dx_{i-1} = \\ &= (dG)dx_1 \cdots dx_{i-1},\end{aligned}$$

where the middle equality is the theorem in calculus that says that the derivative with respect to parameters of an integral is the integral of the derivative. So we find that $(dK_1 + K_1 d)\omega = 0$. This is just what we need, since $i_0^*(dt) = d(i_0^* t) = d(0) = 0$, and also $i_1^*(dt) = 0$.

To finish the proof of the theorem, define $K^i := K_1^i \circ F^*$ and note that, for ω in $\Omega_Y^i(Y)$, $i_0^* F^* \omega = f_0^* \omega$ and $i_1^* F^* \omega = f_1^* \omega$. $\square$

We can now compute the de Rham cohomology for some manifolds. The empty manifold gives the zero ring, and a one point manifold gives the $\mathbb{R}$-algebra $\mathbb{R}$ itself. Now suppose that X is a contractible C^k-manifold (i.e., there is a point x in X such that the constant map $f\colon X \to X$ that sends every element of X to x is homotopic to the identity morphism id_X of X. Then f^* induces the identity endomorphism of $\mathrm{H}_{\mathrm{dR}}(X)$. But we can write $f = i_x \circ p$, with i_x the inclusion of $\{x\}$ in X and p the unique map $X \to \{x\}$. But then we see that i_x^* and p^* are inverses, hence $\mathrm{H}_{\mathrm{dR}}(X) = \mathbb{R}$.

5 The exact sequence of Mayer–Vietoris

Let X be a Hausdorff paracompact C^∞-manifold. Suppose that X is the union of two open subsets U and V:

$$X = U \cup V.$$

Then we have a diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(X) & \xrightarrow{i} & \Omega(U) \oplus \Omega(V) & \xrightarrow{r} & \Omega(U \cap V) & \longrightarrow & 0 \\ & & \omega & \mapsto & (\omega|_U, \omega|_V) & & & & \\ & & & & (\alpha, \beta) & \mapsto & \alpha|_{U\cap V} - \beta|_{U\cap V} & & \end{array} \tag{5.1}$$

It is clear that this diagram is exact at $\Omega(X)$ and at $\Omega(U) \oplus \Omega(V)$, and that the maps i and r commute with the differentials d on each of the terms. In order to obtain the long exact sequence of Mayer–Vietoris, we need that r is surjective. It is for proving this surjectivity that the hypothesis that X is paracompact and Hausdorff is used. We need a so-called *partition of unity* subject to the cover of X by U and V: elements f and g in $C^\infty(X)$ such that $1 = f + g$ and $\mathrm{Supp}(f) \subset U$ and $\mathrm{Supp}(g) \subset V$. It is a bit technical to show, but on our paracompact Hausdorff C^∞-manifold partitions of unity subject to any open cover do exist (see for example Definition 7.9, Proposition 7.10 and Theorem 7.8 of my syllabus "Géométrie Variable", out of which this course is more or less extracted, except this section and the next). In this course, we will admit the existence of f and g above.

5.2 Proposition. *The map r in (5.1) is surjective.*

Proof. Let f and g be as above. Let ω be in $\Omega(U \cap V)$. Then $g = 0$ on $X - V$ and $f = 0$ on $X - U$. It follows that $g\omega$ can be extended by zero to U because $g\omega$ is a section of Ω on the open part $U \cap V$ of U with support contained in that open part. Explicitly: there is a unique α in $\Omega(U)$ such that $\alpha|_{U\cap V} = g\omega$ and $\alpha(x) = 0$ for all x in $U - U \cap V$. Similarly, $f\omega$ can be extended by zero to an element β of $\Omega(V)$. But then $r((\alpha, -\beta)) = \omega$. $\square$

5.3 Remark. We note that the map $\omega \mapsto (g\omega, -f\omega)$ from $\Omega(U \cap V)$ to $\Omega(U) \oplus \Omega(V)$ does not necessarily commute with d, because df and dg are not necessarily zero. Hence our splitting of the exact sequence (5.1) of vector spaces is not necessarily a splitting of an exact sequence of complexes.

5.4 Proposition. *Let (A, d), (B, d) and (C, d) be complexes (of $\mathbb{R}$-vector spaces, say). Let:*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of complexes. Then we get maps $\partial^i \colon \mathrm{H}^i(C) \to \mathrm{H}^{i+1}(A)$, for all i in $\mathbb{Z}$, characterised by the following property.

> *Let $c \in C^i$ with $dc = 0$, let $b \in B^i$ with $gb = c$, then $g(db) = dc = 0$, hence there is a unique $a \in A^{i+1}$ with $fa = db$. As $f(da) = d(fa) = d^2b = 0$, and f is injective, we have $da = 0$, and a defines an element of $\mathrm{H}^{i+1}(A)$. This element is the image under ∂ of the element in $\mathrm{H}^i(C)$ given by c.*

Together with the morphisms $f\colon \mathrm{H}(A) \to \mathrm{H}(B)$ and $g\colon \mathrm{H}(B) \to \mathrm{H}(C)$ induced by f and g, the boundary maps ∂ give a long exact sequence:

$$\cdots \to \mathrm{H}^{i-1}(B) \to \mathrm{H}^{i-1}(C) \to \mathrm{H}^{i}(A) \to \mathrm{H}^{i}(B) \to \mathrm{H}^{i}(C) \to \mathrm{H}^{i+1}(A) \to \cdots$$

Proof. This is a long but simple verification that we will not write out. Any book on homological algebra or on algebraic topology should contain this statement. A lemma often used as ingredient for this proof is the so-called "Snake Lemma". This lemma is then be applied to the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & A^i/dA^{i-1} & \longrightarrow & B^i/dB^{i-1} & \longrightarrow & C^i/dC^{i-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (A^{i+1})^{d=0} & \longrightarrow & (B^{i+1})^{d=0} & \longrightarrow & (C^{i+1})^{d=0}, & & \end{array}$$

where the notation $(A^{i+1})^{d=0}$ stands for $\ker(d\colon A^{i+1} \to A^{i+2})$. $\square$

5.5 Theorem. (Mayer–Vietoris) *Let X be a paracompact Hausdorff C^∞-manifold, covered by two open subsets U and V. Then the diagram in (5.1) gives, via Proposition 5.4, a long exact sequence:*

$$\cdots \to \mathrm{H}^{i-1}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{i-1}_{\mathrm{dR}}(V) \to \mathrm{H}^{i-1}_{\mathrm{dR}}(U \cap V) \to \mathrm{H}^{i}_{\mathrm{dR}}(X) \to \mathrm{H}^{i}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{i}_{\mathrm{dR}}(V) \to \mathrm{H}^{i}_{\mathrm{dR}}(U \cap V) \to \cdots$$

Proof. This is a direct application of Propositions 5.2 and 5.4. $\square$

Now we can combine our two tools (homotopy invariance and Mayer–Vietoris) in order to determine the de Rham cohomology for some manifolds, such as the spheres S^n, the projective spaces $\mathbb{P}^n(\mathbb{C})$ and $\mathbb{P}^n(\mathbb{R})$, compact surfaces and $\mathbb{C}$ minus a finite set of points.

5.6 Theorem. *Let $n \geq 1$. Then $\mathrm{H}^i_{\mathrm{dR}}(S^n)$ is zero if $i \notin \{0, n\}$, $\mathrm{H}^0_{\mathrm{dR}}(S^n) = \mathbb{R}$, and $\mathrm{H}^n_{\mathrm{dR}}(S^n)$ is one-dimensional, generated by any ω in $\Omega^n(S^n)$ with $\int_{S^n} \omega \neq 0$. In other words: let $1 \leq i \leq n$ and ω in $\Omega^i(S_n)$; then there exists η in $\Omega^{i-1}(S^n)$ such that $\omega = d\eta$ if and only if $d\omega = 0$ and, if $i = n$, $\int_{S^n} \omega = 0$.*

Proof. Induction on n. Let $n \geq 1$. First we prove the statements concerning the dimension of the $\mathrm{H}^i_{\mathrm{dR}}(S^n)$. On S^n we have the two points $N := (0, \ldots, 0, 1)$ (the North pole) and $S := (0, \ldots, 0, -1)$ (the South pole). We let $U := S^n - \{S\}$ and $V := S^n - \{N\}$. Then U and V are isomorphic (as C^∞-manifolds) to $\mathbb{R}^n$, and $U \cap V$ is homeomorphic to $S^{n-1} \times \mathbb{R}$, hence homotopic to S^{n-1}. We note that S^0 is the disjoint union of two points. As S^n is connected, we have $\mathrm{H}^0_{\mathrm{dR}}(S^n) = \mathbb{R}$, by construction. For $2 \leq i < n$ both terms surrounding $\mathrm{H}^i_{\mathrm{dR}}(S^n)$ in the Mayer–Vietoris sequence are zero, hence $\mathrm{H}^i_{\mathrm{dR}}(S^n) = 0$. For $i = 1$ we have the exact sequence:

$$\mathrm{H}^0_{\mathrm{dR}}(U) \oplus \mathrm{H}^0_{\mathrm{dR}}(V) \longrightarrow \mathrm{H}^0_{\mathrm{dR}}(U \cap V) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(S^n) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(U) \oplus \mathrm{H}^1_{\mathrm{dR}}(V),$$

which is the sequence:

$$\begin{array}{ccccccc} \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(S^n) & \longrightarrow & 0 \\ (x, y) & \mapsto & x - y & & & & \end{array}$$

if $n \geq 2$, and the sequence:

$$\begin{array}{ccccccc} \mathbb{R} \oplus \mathbb{R} & \longrightarrow & \mathbb{R}^2 & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(S^n) & \longrightarrow & 0 \\ (x, y) & \mapsto & (x - y, x - y) & & & & \end{array}$$

if $n = 1$. It follows that $\mathrm{H}^i_{\mathrm{dR}}(S^n) = 0$ for $0 < i < n$, and that $\mathrm{H}^1_{\mathrm{dR}}(S^1)$ is one-dimensional. Suppose now that $n \geq 2$. Then the exact sequence:

$$\mathrm{H}^{n-1}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{n-1}_{\mathrm{dR}}(U) \longrightarrow \mathrm{H}^{n-1}_{\mathrm{dR}}(U \cap V) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(S^n) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(U) \oplus \mathrm{H}^n_{\mathrm{dR}}(V)$$

reads:

$$0 \longrightarrow \mathrm{H}^{n-1}_{\mathrm{dR}}(S^{n-1}) \longrightarrow \mathrm{H}^n_{\mathrm{dR}}(S^n) \longrightarrow 0.$$

Hence all the dimensions are as claimed. In the exercises of last week it was shown that for all η in $\Omega^{n-1}(S^n)$ we have $\int_{S^n} d\eta = 0$, hence $d\Omega^{n-1}(S^n)$ is in the kernel of $\int_{S^n}$. As both $\int_{S^n} : \Omega^n(S^n) \to \mathbb{R}$ as $\Omega^n(S^n) \to \mathrm{H}^n_{\mathrm{dR}}(S^n)$ are 1-dimensional quotients, they are equal. □

5.7 Theorem. *Let $n \geq 0$. Then $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C}))$ is of dimension one if i is even and $0 \leq i \leq 2n$, and zero otherwise.*

Proof. Let $n \geq 0$. Let $P := (0 : 0 \cdots : 1)$ in $\mathbb{P}^n(\mathbb{C})$. Let

$$U := \mathbb{P}^n(\mathbb{C}) - \{P\}, \quad V := \{(a_0 : \cdots : a_{n-1} : 1) \in \mathbb{P}^n(\mathbb{C}) \mid |a_i| < 1 \text{ for all } i\}.$$

Then V is contractible hence homotopic to a point, and $V \cap U = V - \{P\}$ is homotopic to S^{2n-1}. The map:

$$F \colon U \times \mathbb{R} \to U, \quad ((a_0 : \cdots : a_{n-1} : a_n), t) \mapsto (a_0 : \cdots : a_{n-1} : ta_n)$$

gives a homotopy from the identity on U to the projection to the subspace $\mathbb{P}^{n-1}(\mathbb{C})$ (the hyperplane at infinity) of $\mathbb{P}^n(\mathbb{C})$. Summarizing:

$$\mathrm{H}_{\mathrm{dR}}(U) = \mathrm{H}_{\mathrm{dR}}(\mathbb{P}^{n-1}(\mathbb{C})), \quad \mathrm{H}_{\mathrm{dR}}(V) = \mathrm{H}_{\mathrm{dR}}(\cdot), \quad \mathrm{H}_{\mathrm{dR}}(U \cap V) = \mathrm{H}_{\mathrm{dR}}(S^{2n-1}).$$

Hence the Mayer–Vietoris sequence reads:

$$\cdots \to \mathrm{H}^{i-1}_{\mathrm{dR}}(S^{2n-1}) \to \mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C})) \to \mathrm{H}^i_{\mathrm{dR}}(\cdot) \oplus \mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^{n-1}(\mathbb{C})) \to \mathrm{H}^i_{\mathrm{dR}}(S^{2n-1}) \to \cdots$$

It follows that for $i \neq 2n$ the map $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C})) \to \mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^{n-1}(\mathbb{C}))$ is an isomorphism, and that the map $\mathrm{H}^{2n-1}_{\mathrm{dR}}(S^{2n-1}) \to \mathrm{H}^{2n}_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{C}))$ is an isomorphism. □

5.8 Theorem. *Let $n \geq 0$. Then $\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^n(\mathbb{R}))$ is $\mathbb{R}$ if $i = 0$, one-dimensional if $i = n$ and n is odd, and zero otherwise.*

Proof. Let $\iota \colon S^n \to S^n$ be the antipodal map. Then the obvious map $S^n \to \mathbb{P}^n(\mathbb{R})$ is the quotient for the action of the group $\langle \iota \rangle$. Hence the de Rham complex $(\Omega(\mathbb{P}^n(\mathbb{R})), d)$ is the subcomplex of ι-invariants of the de Rham complex $(\Omega(S^n), d)$ of S^n. The direct sum decomposition:

$$(\Omega(S^n), d) = (\Omega(S^n), d)^+ \oplus (\Omega(S^n), d)^-,$$

shows that in the direct sum decomposition $\mathrm{H}_{\mathrm{dR}}(S^n) = \mathrm{H}_{\mathrm{dR}}(S^n)^+ \oplus \mathrm{H}_{\mathrm{dR}}(S^n)^-$ we have:

$$\mathrm{H}_{\mathrm{dR}}(S^n)^+ = \mathrm{H}((\Omega(S^n), d)^+), \quad \mathrm{H}_{\mathrm{dR}}(S^n)^- = \mathrm{H}((\Omega(S^n), d)^-).$$

It remains to show that $\iota^* \colon \mathrm{H}^n_{\mathrm{dR}}(S^n) \to \mathrm{H}^n_{\mathrm{dR}}(S^n)$ is multiplication by $(-1)^{n-1}$ if $n \geq 1$. This is left as an exercise to the reader. Of course, one can also obtain the result of this theorem using appropriate Mayer–Vietoris arguments. □

Without proof we state two results that are easily proved using the techniques above.

5.9 Theorem. *Let $r \geq 1$ and $P_1, \ldots, P_r$ be r distinct points of S^2. Put $X := S^2 - \{P_1, \ldots, P_r\}$. Then $\mathrm{H}^0_{\mathrm{dR}}(X) = \mathbb{R}$, $\dim \mathrm{H}^1_{\mathrm{dR}}(X) = r - 1$, $\mathrm{H}^i_{\mathrm{dR}}(X) = 0$ for $i \geq 2$.*

5.10 Theorem. *Let X be a compact orientable connected C^∞-manifold of dimension two. The classification of such surfaces says that X can be obtained by adding a certain number of handles to S^2. This number of handles, g say, is called called the genus of X. One has: $\mathrm{H}^0_{\mathrm{dR}}(X) = \mathbb{R}$, $\dim \mathrm{H}^1_{\mathrm{dR}}(X) = 2g$, $\dim \mathrm{H}^2_{\mathrm{dR}}(X) = 1$, and $\mathrm{H}^i_{\mathrm{dR}}(X) = 0$ for $i > 2$.*

We end this section with some naive experiment that turns out to be very successful. We recall that the $\mathbb{C}$-algebra of meromorphic functions on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ is the field of fractions $\mathbb{C}(t)$ of $\mathbb{C}[t]$.

Let $r \geq 0$ be an integer, and $a_1, \ldots, a_r$ be r distincs elements of $\mathbb{C}$. We want to consider $X := \mathbb{P}^1(\mathbb{C}) - \{\infty, a_1, \ldots, a_r\}$ as a *complex algebraic* variety, which, for us, at this moment, means that we consider the $\mathbb{C}$-algebra $\mathcal{O}(X)$ of global functions on it to be the $\mathbb{C}$-algebra of rational functions $f \in \mathbb{C}(t)$ that have no pole outside $\{\infty, a_1, \ldots, a_r\}$. We have:

$$\mathcal{O}(X) = \mathbb{C}[t, 1/(t - a_1), \ldots, 1/(t - a_r)].$$

A $\mathbb{C}$-basis of $\mathcal{O}(X)$ is given by the t^n with $n \geq 0$ together with the $(t - a_i)^{n_i}$ with $1 \leq i \leq r$ and $n_i < 0$. The algebraic de Rham complex $\Omega(X)$ is then:

$$0 \longrightarrow \mathcal{O}(X) \xrightarrow{d} \mathcal{O}(X) \cdot dt \longrightarrow 0,$$

with f in $\mathcal{O}(X)$ mapping to $f' \cdot dt$, with f' the usual derivative of f. Using the $\mathbb{C}$-basis given above, it is easy to compute that $\mathrm{H}^0(\Omega(X)) = \mathbb{C}$ and $\mathrm{H}^1(\Omega(X))$ is of dimension r, and represented by the forms $(t - a_i)^{-1} dt$. Surprisingly, we have $\mathrm{H}(\Omega(X)) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}_{\mathrm{dR}}(X)$, with $\mathrm{H}_{\mathrm{dR}}(X)$ the de Rham cohomology of X viewed as a (real) C^∞-manifold.

This example is no coincidence. It is in fact true for any affine non-singular complex algebraic variety X that $\mathrm{H}(\Omega(X)) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{H}_{\mathrm{dR}}(X)$ (a theorem due to Grothendieck). The proof of that is quite complicated, and uses resolution of singularities.

6 Cohomology with compact supports, and Poincaré duality

One of the fundamental results concerning de Rham cohomology is Poincaré duality. In some examples that we have seen (the S^n and the $\mathbb{P}^n(\mathbb{C})$) one notes that $\mathrm{H}^i_{\mathrm{dR}}(X)$ and $\mathrm{H}^j_{\mathrm{dR}}(X)$ have the same dimension if $i+j=\dim_X$. On the other hand, this is not so for $\mathbb{R}^n$. We will see that this comes from the non-compactness of $\mathbb{R}^n$.

In order to set up Poincaré duality, we need a version of de Rham cohomology: the one with compact supports.

6.1 Definition. *Let $k \geq 0$, X a C^k-manifold and E a C^k-vector bundle on X. Then, for $U \subset X$ open, we let $E_c(U)$ denote the sub-$\mathbb{R}$-vector space of $E(U)$ consisisting of the sections with compact support. In particular, on a C^∞-manifold we have the complex $(\Omega_c(X), d)$.*

6.2 Definition. *Let X be a paracompact Hausdorff C^∞-manifold. Then we define $\mathrm{H}_{\mathrm{dR,c}}(X)$, the de Rham cohomology with compact supports, to be the homology of the complex $(\Omega_c(X), d)$. Explicitly, for $i \geq 0$ one has:*

$$\mathrm{H}^i_{\mathrm{dR,c}}(X) = \frac{\ker(d\colon \Omega^i_c(X) \to \Omega^{i+1}_c(X))}{\mathrm{im}(d\colon \Omega^{i-1}_c(X) \to \Omega^i_c(X))}.$$

Of course, if X is compact, then $\Omega_c(X) = \Omega(X)$, and $\mathrm{H}_{\mathrm{dR,c}}(X) = \mathrm{H}_{\mathrm{dR}}(X)$.

We note that a pullback of a differential form with compact support need not have compact support. Hence $X \mapsto (\Omega_c(X), d)$ is not a contravariant functor via pullback for arbitrary morphisms between C^∞-manifolds. Of course, if $f\colon X \to Y$ is such that the inverse image of a compact subset of Y is compact, then f does induce a morphism f^* of differential graded-commutative $\mathbb{R}$-algebras from $\Omega_c(Y)$ to $\Omega_c(X)$.

6.3 Definition. *A morphism $f\colon X \to Y$ of topological spaces is called* proper *if for every compact subset K of Y the inverse image $f^{-1}K$ is compact.*

With this definition it is obvious that:

> $\mathrm{H}_{\mathrm{dR,c}}$ is a contravariant functor for proper morphisms.

As often happens, when you lose something (such as pullback functoriality for arbitrary morphisms), you gain something else. Here it is:

> on a Hausdorff space, taking sections with compact support is covariant for open immersions.

Explicitly, let $U \subset V$ be open, with V a Hausdorff manifold, let E be a vector bundle on V and s in $E_c(U)$. Then the extension of s by zero is in $E_c(V)$ (note that $\mathrm{Supp}(s)$ is closed in V hence that s is zero in the neighborhood $V - \mathrm{Supp}(s)$ of $V - U$). We will denote the extension by zero of s again by s.

We will now establish the analogs for de Rham cohomology with compact supports of the Mayer–Vietoris sequence and the homotopy invariance. We start with the Mayer-Vietoris sequence.

Let X be a paracompact Hausdorff C^∞-manifold. Suppose that X is the union of two open subsets U and V:

$$X = U \cup V.$$

Then we have a diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_c(U\cap V) & \stackrel{i}{\longrightarrow} & \Omega_c(U)\oplus\Omega_c(V) & \stackrel{s}{\longrightarrow} & \Omega_c(X) & \longrightarrow & 0 \\
 & & & & (\alpha,\beta) & \longmapsto & \alpha+\beta & & \\
 & & \omega & \longmapsto & (\omega,-\omega) & & & &
\end{array}
\tag{6.4}
$$

It is clear that this sequence is exact, except possibly at $\Omega_c(X)$. The next proposition takes care of that.

6.5 Proposition. *The map s (for sum) in (6.4) is surjective.*

Proof. Let f and g be a partition of 1 subject to the cover of X by U and V, as in the previous section. Let ω be in $\Omega_c(X)$. Then $f\omega$ has support in U, hence is in $\Omega_c(U)$. Likewise, $g\omega$ is in $\Omega_c(V)$. It follows that $(f\omega, g\omega)$ is mapped to ω under s. $\square$

As i and s above are morphisms of complexes, (6.4) is a short exact sequence of complexes, hence by the results of the previous section, we get a long exact sequence (Mayer–Vietoris for de Rham cohomology with compact supports).

6.6 Theorem. *Let X be a paracompact Hausdorff C^∞-manifold, covered by two open subsets U and V. Then the diagram in (6.4) gives, via Proposition 5.4, a long exact sequence:*

$$\to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(U\cap V) \to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(U)\oplus\mathrm{H}^{i-1}_{\mathrm{dR,c}}(V) \to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(X) \to \mathrm{H}^{i}_{\mathrm{dR,c}}(U\cap V) \to \mathrm{H}^{i}_{\mathrm{dR,c}}(U)\oplus\mathrm{H}^{i}_{\mathrm{dR,c}}(V) \to$$

Let us now give the analog of homotopy invariance. We should say first of all that $\mathrm{H}_{\mathrm{dR,c}}$ is *not* homotopy invariant. For example, one computes directly that $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{R}) = \mathbb{R}$ (via integration), and $\mathrm{H}^0_{\mathrm{dR,c}}(\mathbb{R}) = 0$. Let X be a paracompact Hausdorff C^∞-manifold. Let $p_X\colon \mathbb{R}\times X \to X$ be the projection. Then p_X^* does not map $\Omega_c(X)$ to $\Omega_c(\mathbb{R}\times X)$, but *integration along the fibers*, as we will explain, does send $\Omega_c(\mathbb{R}\times X)$ to $\Omega_c(X)$, and *lowers the degree by one.* We recall from section 4 that for every i we have a direct sum decomposition:

$$\Omega^i_{\mathbb{R}\times X} = p_X^*\Omega^i_X \oplus \left(p_X^*\Omega^{i-1}_X \otimes p_{\mathbb{R}}^*\Omega^1_{\mathbb{R}}\right).$$

In local coordinates, this means the following. Suppose that $x_1,\ldots,x_n$ are coordinates on $U\subset X$, and let t be the coordinate on $\mathbb{R}$. Then $\Omega^i_{\mathbb{R}\times U}$ has the basis:

$$\begin{cases} dx_{j_1}\cdots dx_{j_i}, & 1\le j_1<\cdots<j_i\le n,\\ dx_{j_1}\cdots dx_{j_{i-1}}dt, & 1\le j_1<\cdots<j_{i-1}\le n.\end{cases}$$

For ω in $\Omega^i_c(\mathbb{R}\times X)$ we can write uniquely $\omega = \omega_1 + \omega_2\,dt$, with ω_1 in $p_X^*\Omega^i_c(\mathbb{R}\times X)$ and ω_2 in $p_X^*\Omega^{i-1}_c(\mathbb{R}\times X)$. With this notation, we define:

$$p_{X,!}\colon \Omega_c(\mathbb{R}\times X) \longrightarrow \Omega_c(X), \quad \omega \longmapsto \int_{\mathbb{R}} \omega_2\,dt. \tag{6.7}$$

6.8 Theorem. *Let X be a paracompact Hausdorff C^∞-manifold. The map $p_{X,!}$ above is a morphism of complexes, and induces isomorphisms $p_{X,!}\colon \mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{R}\times X) \to \mathrm{H}^{i-1}_{\mathrm{dR,c}}(X)$. An inverse is induced by the morphism of complexes $i\colon \Omega_c(X)\to\Omega_c(\mathbb{R}\times X)$, $\omega\mapsto\omega e$, with e in $\Omega^1_c(\mathbb{R})$ such that $\int_{\mathbb{R}} e = 1$.*

Proof. It is clear that $p_{X,!}i$ is the identity on $\Omega_c(X)$. So it suffices now to show that $ip_{X,!}$ is homotopic to the identity on $\Omega_c(\mathbb{R}\times X)$. The homotopy operator is the following:

$$\omega = \omega_1 + \omega_2\,dt \longmapsto \left((a,x)\mapsto \int_\infty^a i_x^*\omega_2\,dt - \left(\int_{-\infty}^a e\right)\left(\int_{\mathbb{R}} i_x^*\omega_2\,dt\right)\right),$$

where $i_x\colon \mathbb{R}\to\mathbb{R}\times X$ is the map $a\mapsto(a,x)$. (I take this on faith from Bott and Tu.) The verification of this is left to the reader. □

6.9 Theorem. *Let $n\geq 0$. Then $\mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$ if $i\neq n$ and $\mathrm{H}^n_{\mathrm{dR,c}}(\mathbb{R}^n) = \mathbb{R}$ via the map $\Omega^n_c(\mathbb{R}^n)\to\mathbb{R}$, $\omega\mapsto\int_{\mathbb{R}^n}\omega$.*

Now we have the right prerequisites to formulate and prove Poincaré duality.

6.10 Theorem. (Poincaré duality) *Let X be a paracompact Hausdorff C^∞-manifold with a given orientation, of dimension n. Then the pairings:*

$$\Omega^i(X)\times\Omega^{n-i}_c(X)\longrightarrow\mathbb{R},\quad (\omega,\eta)\longmapsto\int_X\omega\eta$$

induce isomorphisms:

$$\mathrm{H}^i_{\mathrm{dR}}(X)\longrightarrow\mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)^\vee.$$

Proof. First we note that, as X is oriented, we have $\int_X\colon\Omega^n_c(X)\to\mathbb{R}$. Secondly, for $\omega\in\Omega^i(X)$ and $\eta\in\Omega^{n-i}_c(X)$ we have $\omega\eta$ in $\Omega^n_c(X)$. Hence the pairings exist. As $d(\omega\eta) = (d\omega)\eta + (-1)^i\omega d\eta$, $\omega\eta$ is closed if ω and η are closed. If $\omega = d\omega'$ and η is closed, then $d(\omega'\eta) = \omega\eta$ and $\int_X\omega\eta = 0$. If ω is closed and $\eta = d\eta'$, then $d(\omega\eta') = (-1)^i\omega\eta$ and $\int_X\omega\eta = 0$. Hence we get induced pairings:

$$\mathrm{H}^i_{\mathrm{dR}}(X)\times\mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)\longrightarrow\mathbb{R}.$$

These pairings induce morphisms:

$$\mathrm{H}^i_{\mathrm{dR}}(X)\longrightarrow\mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)^\vee.$$

It remains to show that these morphisms are isomorphisms. We will only prove this for X that have a finite *good cover*. For the general case, the reader is referred to the reference given in Bott and Tu: Greub, Halperin and Vanstone, "Connections, curvature, and cohomology", vol. 1, Academic Press, New York, 1972. We also note that a Poincaré duality statement for singular homology and singular cohomology with compact support that does not need the hypotheses Hausdorff and paracompact can be found in Massey's book on singular homology.

A good cover of X is a family U_i, $i\in I$ (with I an arbitrary set), such that for all non-empty finite subsets J of I, the intersection $U_J := \cap_{i\in J}U_i$ is diffeomorphic to $\mathbb{R}^n$ or empty. Each paracompact Hausdorff manifold admits a good cover. The idea to prove this is to choose a Riemannian metric on X (via a partition of 1 subject to a cover by charts), and to use geodesically convex neighborhoods of points. For some details, see Bott and Tu. Compact X have finite good covers. Good covers are useful for computing cohomology using Mayer-Vietoris sequences.

Now assume that X has a finite good cover $U_1, \ldots, U_r$. Then we will prove by induction on r that the Poincaré duality morphisms are isomorphisms. For $r = 0$ or 1 this follows from our knowledge of the cohomology of $\emptyset$ and of $\mathbb{R}^n$. Now let $U := U_1 \cup \cdots \cup U_{r-1}$, and let $V := U_r$. Then we know that the Poincaré duality morphisms for U and V are isomorphisms. We have a diagram as follows:

$$\begin{array}{ccccccc}
\mathrm{H}^{i-1}_{\mathrm{dR}}(U \cap V) & \to & \mathrm{H}^{i}_{\mathrm{dR}}(X) & \to & \mathrm{H}^{i}_{\mathrm{dR}}(U) \oplus \mathrm{H}^{i}_{\mathrm{dR}}(V) & \to & \mathrm{H}^{i}_{\mathrm{dR}}(U \cap V) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^{n-i+1}_{\mathrm{dR,c}}(U \cap V)^\vee & \to & \mathrm{H}^{n-i}_{\mathrm{dR,c}}(X)^\vee & \to & \mathrm{H}^{n-i}_{\mathrm{dR,c}}(U)^\vee \oplus \mathrm{H}^{n-i}_{\mathrm{dR,c}}(V)^\vee & \to & \mathrm{H}^{n-i}_{\mathrm{dR,c}}(U \cap V)^\vee
\end{array}$$

This diagram is commutative up a sign $(-1)^{i+1}$ at the square whose upper left corner is $\mathrm{H}^{i}_{\mathrm{dR}}(U \cap V)$ (verification left to the reader; however, below we will do the computation that leads to the $(-1)^{i+1}$). Hence, by appropriately changing the signs of the boundary maps, we get a morphism of long exact sequences in which the vertical maps are the Poincaré duality morphisms. The four vertical arrows surrounding the Poincaré duality morphisms for X are isomorphisms. The *five lemma* below finishes the proof.

As promised, we will show that the square whose upper left corner is $\mathrm{H}^{i}_{\mathrm{dR}}(U \cap V)$ is commutative up to a sign $(-1)^{i+1}$. We start with an element in $\mathrm{H}^{i}_{\mathrm{dR}}(U \cap V)$, represented by some ω in $\Omega^i(U \cap V)$ with $d\omega = 0$. Let ω_U and ω_V be elements of $\Omega^i(U)$ and $\Omega^i(V)$, respectively, such that $\omega = \omega_U - \omega_V$, as in the construction of the Mayer-Vietoris sequence. Then $d\omega_U - d\omega_V = 0$ on $U \cap V$, hence we have a unique $\widetilde{\omega}$ in $\Omega^{i+1}(X)$ whose restrictions to U and V are $d\omega_U$ and $d\omega_V$, respectively. This $\widetilde{\omega}$ is closed, because it is so on U and on V, and its class $[\widetilde{\omega}]$ in $\mathrm{H}^{i+1}_{\mathrm{dR}}(X)$ is the image of $[\omega]$ under the boundary map. Similarly, we should take an element $[\eta]$ in $\mathrm{H}^{n-i-1}_{\mathrm{dR,c}}(X)$, with η in $\Omega^{n-i-1}_c(X)$ and $d\eta = 0$. Let η_U and η_V be in $\Omega^{n-i-1}_c(U)$ and $\Omega^{n-i-1}_c(V)$, respectively, such that $\eta = \eta_U + \eta_V$. Then $d\eta_U + d\eta_V = 0$, hence both terms are supported in $U \cap V$, and there is a unique $\widetilde{\eta}$ in $\Omega^{n-i}_c(U \cap V)$ such that $\widetilde{\eta} = d\eta_U = -d\eta_V$. We have $d\widetilde{\eta} = 0$, and the image of $[\eta]$ under the boundary map is then $[\widetilde{\eta}]$.

The image of $[\widetilde{\omega}]$ in $\mathrm{H}^{n-i-1}_{\mathrm{dR,c}}(X)^\vee$, applied to $[\eta]$, gives the real number $\int_X \widetilde{\omega}\eta$. On the other hand, the image of $[\omega]$ in $\mathrm{H}^{n-i}_{\mathrm{dR,c}}(U \cap V)^\vee$, applied to $[\widetilde{\eta}]$, is $\int_{U \cap V} \omega\widetilde{\eta}$. We compute:

$$\begin{aligned}
\int_X \widetilde{\omega}\eta &= \int_X \widetilde{\omega}(\eta_U + \eta_V) = \int_U \widetilde{\omega}\eta_U + \int_V \widetilde{\omega}\eta_V = \int_U (d\omega_U)\eta_U + \int_V (d\omega_V)\eta_V = \\
&= (-1)^{i+1}\left(\int_U \omega_U d\eta_U + \int_V \omega_V d\eta_V\right) = (-1)^{i+1}\left(\int_U \omega_U\widetilde{\eta} - \int_V \omega_V\widetilde{\eta}\right) = (-1)^{i+1}\int_{U\cap V} \omega\widetilde{\eta},
\end{aligned}$$

where we have used that

$$d(\omega_U\eta_U) = (d\omega_U)\eta_U + (-1)^i\omega_U(d\eta_U), \quad \text{and} \quad \int_U d(\omega_U\eta_U) = 0,$$

and the same for V. □

6.11 Lemma. *Suppose we have a commutative diagram (of modules over some ring, say) with exact rows:*

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & J
\end{array}$$

such that the four outer vertical arrows are isomorphisms, then $C \to H$ is an isomorphism.

Proof. An ordinary diagram chase. $\square$

The induction argument on the number of elements of a finite good cover in the proof of Theorem 6.10 also proves the following result.

6.12 Theorem. *Let X be a Hausdorff manifold with a finite good cover. Then the vector spaces $\mathrm{H}_{\mathrm{dR}}(X)$ and $\mathrm{H}_{\mathrm{dR,c}}(X)$ are finite dimensional. In particular, this holds for compact X.*

6.13 Example. Let $X := \mathbb{C} - \mathbb{Z}$. Then $\mathrm{H}^1_{\mathrm{dR,c}}(X)$ has a basis indexed by $\mathbb{Z}$ (think of small loops around the integers). But $\mathrm{H}^1_{\mathrm{dR}}(X)$ is the dual of this space, and hence of dimension $|\mathbb{R}|$.

6.14 Remark. We note that Poincaré duality can be formulated on a non-orientable manifold if one uses the *orientation sheaf* $\Omega_X^{\dim X} \otimes \mathrm{Vol}_X^\vee$. See the book "Cohomology of sheaves" by Iversen, or also section 7 of Chapter I of the book by Bott and Tu.

7 The Künneth formula

Let X and Y be paracompact Hausdorff C^∞-manifolds. The projections p_X and p_Y from $X \times Y$ to X and Y induce morphisms of graded-commutative $\mathbb{R}$-algebras: $p_X^*\colon \mathrm{H}_{\mathrm{dR}}(X) \to \mathrm{H}_{\mathrm{dR}}(X \times Y)$ and $p_Y^*\colon \mathrm{H}_{\mathrm{dR}}(Y) \to \mathrm{H}_{\mathrm{dR}}(X \times Y)$. The universal property of the tensor product of algebras then gives us a morphism of graded-commutative $\mathbb{R}$-algebras:

$$(p_X^*, p_Y^*)\colon \mathrm{H}_{\mathrm{dR}}(X) \otimes_{\mathbb{R}} \mathrm{H}_{\mathrm{dR}}(Y) \longrightarrow \mathrm{H}_{\mathrm{dR}}(X \times Y).$$

We will call this morphism the morphism induced by the projections p_X and p_Y.

In the arguments below, we will again use the notion of good cover, but now for a manifold of which the dimension is not necessary constant. In that case, we ask that any finite intersection of the elements of the cover is empty or isomorphic to $\mathbb{R}^n$ for some $n \geq 0$, where, of course, n may vary.

7.1 Theorem. (Künneth) *Let X and Y be paracompact C^∞-manifolds, and suppose that X or Y admits a finite good cover. Then the map induced by the projections p_X and p_Y:*

$$\mathrm{H}_{\mathrm{dR}}(X) \otimes \mathrm{H}_{\mathrm{dR}}(Y) \longrightarrow \mathrm{H}_{\mathrm{dR}}(X \times Y)$$

is an isomorphism of graded-commutative algebras.

Proof. By permuting X and Y, if necessary, we may and do suppose that Y has a finite good cover $V_1, \ldots, V_r$. We use induction on r.

Suppose first that $r = 1$. Then Y is empty or isomorphic to $\mathbb{R}^n$ for some $n \geq 0$. If Y is empty, the statement is clearly true, as $\mathrm{H}_{\mathrm{dR}}(Y)$ and $\mathrm{H}_{\mathrm{dR}}(X \times Y)$ are both the zero $\mathbb{R}$-algebra. So we suppose that Y is isomorphic to $\mathbb{R}^n$, with $n \geq 0$. In fact, we may suppose that $Y = \mathbb{R}^n$. We know that $\mathrm{H}_{\mathrm{dR}}(\mathbb{R}^n) = \mathbb{R}$ (with $\mathbb{R}$ in degree zero), and that $p_X^*\colon \mathrm{H}_{\mathrm{dR}} X \to \mathrm{H}_{\mathrm{dR}}(X \times \mathbb{R}^n)$ is an isomorphism because the inclusion $X \to X \times \mathbb{R}^n$, $x \mapsto (x, 0)$, is an inverse up to homotopy. This means that we have proved the statement for $r = 1$.

Now we suppose that $r \geq 2$. Put $U := V_1 \cup \cdots \cup V_{r-1}$ and $V := V_r$. Then U and V cover Y, and $X \times U$ and $X \times V$ cover $X \times Y$. Using the Mayer-Vietoris sequences associated to these two covers we get a diagram involving two long exact sequences (of which, because of typographical constraints, we only show three terms):

$$\begin{array}{ccccc}
\bigoplus\limits_{j+l=i-1} \mathrm{H}^j(X) \otimes \mathrm{H}^l(U \cap V) & \to & \bigoplus\limits_{j+k=i} \mathrm{H}^j(X) \otimes \mathrm{H}^k(Y) & \to & \bigoplus\limits_{j+k=i} \mathrm{H}^j(X) \otimes (\mathrm{H}^k(U) \oplus \mathrm{H}^k(V)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^{i-1}(X \times (U \cap V)) & \to & \mathrm{H}^i(X \times Y) & \to & \mathrm{H}^i(X \times U) \oplus \mathrm{H}^i(X \times V)
\end{array}$$

The first line is obtained by tensoring the Mayer-Vietoris sequence for the cover (U, V) of Y with each of the $\mathrm{H}^j(X)$, and combining these suitable in a direct sum. We claim that the diagram is actually a morphism of complexes, i.e., that all the squares in it are commutative.

Let us for example do the most difficult verification, namely, that the left square is commutative. So let i, j and l be given so that $j + l = i - 1$, let $[\omega]$ be in $\mathrm{H}^j(X)$ and $[\eta]$ in $\mathrm{H}^k(U \cap V)$, and let us calculate the two images in $\mathrm{H}^i(X \times Y)$. Let η_U and η_V be in $\Omega^l(U)$ and $\Omega^l(V)$, respectively, such that the difference of their restrictions to $U \cap V$ satisfy: $\eta_U - \eta_V = \eta$. Then, we have on $U \cap V$: $0 = d\eta = d\eta_U - d\eta_V$. Let $\widetilde{\eta}$ be the unique element in $\Omega^{l+1}(Y)$ such that $\widetilde{\eta}|_U = d\eta_U$ and $\widetilde{\eta}|_V = d\eta_V$. Then the image of $[\omega] \otimes [\eta]$ in $\mathrm{H}^j(X) \otimes \mathrm{H}^{l+1}(Y)$ is $[\omega] \otimes [\widetilde{\eta}]$, which in turn maps to

$[\omega\widetilde{\eta}]$ in $\mathrm{H}^i(X \times Y)$. Now $\omega\eta_U$ and $\omega\eta_V$ are in $\Omega^{i-1}(X \times U)$ and $\Omega^{i-1}(X \times V)$, respectively, such that $\omega\eta_U - \omega\eta_V = \omega\eta$ on $X \times U \cap V$, and $\omega\widetilde{\eta}|_{X\times U} = d(\omega\eta_U)$, and $\omega\widetilde{\eta}|_{X\times V} = d(\omega\eta_V)$. It follows that $[\omega] \otimes [\eta]$ maps to $[\omega\eta]$ in $\mathrm{H}^{i-1}(X \times U \cap V)$, and then to $[\omega\widetilde{\eta}]$ in $\mathrm{H}^i(X \times Y)$. $\square$

7.2 Remark. Let both X and Y be the zero-dimensional manifold $\mathbb{N}$. Then $\mathrm{H}^0_{\mathrm{dR}}(X \times Y)$ is $\mathbb{R}^{\mathbb{N}\times\mathbb{N}}$, whereas $\mathrm{H}^0_{\mathrm{dR}}(X) \otimes_{\mathbb{R}} \mathrm{H}^0_{\mathrm{dR}}(Y) = \mathbb{R}^{\mathbb{N}} \otimes_{\mathbb{R}} \mathbb{R}^{\mathbb{N}}$. This shows that the natural map from $\mathrm{H}^0_{\mathrm{dR}}(X) \otimes_{\mathbb{R}} \mathrm{H}^0_{\mathrm{dR}}(Y)$ to $\mathrm{H}^0_{\mathrm{dR}}(X \times Y)$ is injective but not surjective. Hence at least some finiteness condition is necessary in order for the Künneth formula to hold.

8 Vector bundles

Let X be a manifold. Then we have, for each x in X, the tangent space $\mathrm{T}_X(x)$. We want to combine all these tangent spaces into one single object, itself a manifold, called the *tangent bundle*, and denoted T_X. As a set, T_X is just the disjoint union of the $\mathrm{T}_X(x)$, for x in X. We let p be the map from T_X to X such that $p^{-1}x = \mathrm{T}_X(x)$ for all x. Let (U, ϕ, U') be a chart, with U' open in $\mathbb{R}^n$. Then we get a bijection:

$$\Phi\colon \mathbb{R}^n \times U' \longrightarrow p^{-1}U, \quad (v,x) \mapsto [(v,\phi)],$$

where $[(v,\phi)]$ in $\mathrm{T}_X(\phi x)$ denotes the *physical* tangent vector at ϕx corresponding to v in the chart ϕ. If (U_i, ϕ_i, U_i'), $i \in I$ is an atlas for X, with U_i open in $\mathbb{R}^{n_i}$, we have, for i and j in I:

$$\Phi_j^{-1} \circ \Phi_i \colon \mathbb{R}^{n_i} \times \phi_i^{-1}(U_i \cap U_j) \longrightarrow \mathbb{R}^{n_j} \times \phi_j^{-1}(U_i \cap U_j), \quad (v,x) \mapsto (((D(\phi_j^{-1}\phi_i))(x))v, (\phi_j^{-1}\phi_i)x).$$

First of all, we note that the $\Phi_j^{-1} \circ \Phi_i$ are diffeomorphisms, hence the Φ_i give T_X the structure of a manifold (and this structure does not depend on the choice of an atlas). In the charts Φ_i the map p corresponds to the projection $\mathbb{R}^{n_i} \times U_i' \to U_i'$, which is differentiable, hence the map $p\colon \mathrm{T}_X \to X$ is a morphism of manifolds, and in fact a submersion. Secondly, the charts Φ_i have the property that for each x in U_i' the induced map from $\mathbb{R}^{n_i}$ to $T_x(\phi_i x)$ is an isomorphism of $\mathbb{R}$-vector spaces. As an example, we note that the tangent bundle of the circle is isomorphic to the cylinder.

For $f\colon X \to Y$ a morphism of manifolds, we get a map (of sets, say) $\mathrm{T}_f\colon T_X \to \mathrm{T}_Y$, that sends an element v of $\mathrm{T}_X(x)$ to its image $(T_f x)v$ in $\mathrm{T}_Y(fx)$ under the tangent map of f at x. This map T_f is differentiable with respect to the structures of manifold on T_X and T_Y; it is called the *tangent map* of f, and it sits in a commutative diagram:

$$\begin{array}{ccc}
\mathrm{T}_X & \xrightarrow{\mathrm{T}_f} & \mathrm{T}_Y \\
\downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\
X & \xrightarrow{f} & Y
\end{array}$$

Now that we have seen the tangent bundle, it is good to define the general notion of vector bundle on a manifold.

8.1 Definition. *Let X be a manifold. A (real) vector bundle on X is a five tuple $(E, p, 0, +, \cdot)$, where $p\colon E \to X$ is a morphism of manifolds, $(0, +, \cdot)$ the structure of $\mathbb{R}$-vector space on all fibres $p^{-1}\{x\}$ for x in X, such that X can be covered by open subsets U_i for which there exists an isomorphism (of manifolds) $\Phi_i\colon \mathbb{R}^{n_i} \times U_i \to p^{-1}U_i$ such that $p \circ \Phi_i$ is the projection $\mathbb{R}^{n_i} \times U_i \to U_i$, and such that for all x in U_i the induced map $\Phi_{i,x}$ from $\mathbb{R}^{n_i}$ to $p^{-1}\{x\}$ is an isomorphism of $\mathbb{R}$-vector spaces. The integer n_i is called the rank of E over U_i, and Φ_i is called a trivialisation of E over U_i. In what follows we will only consider vector bundles of constant rank, i.e., such that all n_i are equal.*

For $f\colon X \to Y$ a morphism of manifolds, $p\colon E \to X$ and $q\colon F \to Y$ vector bundles, a morphism of vector bundles from E to F is a morphism of manifolds $g\colon E \to F$ with $q \circ g = f \circ p$, that is $\mathbb{R}$-linear on the fibres.

With these definitions, it is clear that for X a manifold, the tangent bundle T_X is a vector bundle over X, and that, for $f\colon X \to Y$ a morphism of manifolds, the tangent map $\mathrm{T}_f\colon T_X \to \mathrm{T}_Y$ is a morphism of vector bundles.

The most trivial example of a vector bundle is the *trivial bundle* $\mathbb{R}^n_X := \mathbb{R}^n \times X$. The Moebius band gives an example of a non-trivial vector bundle of rank 1 over the circle.

8.2 Construction. Let X be a manifold, and let $p\colon E \to X$ be a vector bundle. Suppose that U_i, $i \in I$, is an open cover of X, and that $\Phi_i : \mathbb{R}^n \times U_i \to p^{-1}U_i$ are local trivialisations. Then for every i and j in I we have an isomorphism $g_{j,i} := \Phi_j^{-1} \circ \Phi_i\colon \mathbb{R}^n \times (U_i \cap U_j) \to \mathbb{R}^n \times (U_i \cap U_j)$. By definition these $g_{i,j}$ are smooth maps from $U_i \cap U_j$ to $\mathrm{GL}_n(\mathbb{R})$; they are called the *transition maps* of E with respect to the local trivialisations Φ_i. The $g_{i,j}$ satisfy the so-called *cocycle condition*: for all i, j and k in I one has $g_{k,i} = g_{k,j} \circ g_{j,i}$ on $U_i \cap U_j \cap U_k$. Vice versa, given an open covering U_i, i in I, and smooth maps $g_{j,i}$ from $U_i \cap U_j$ to $\mathrm{GL}_n(\mathbb{R})$ satisfying the cocycle condition, one obtains (by "glueing" the $\mathbb{R}^n \times U_i$) a vector bundle E over X, with given trivialisations over the U_i, with transition maps the $g_{j,i}$.

8.3 Constructions on vector bundles

Let X be a manifold. For $p\colon E \to X$ and $q\colon F \to X$ two vector bundles, one obtains their *direct sum* (also sometimes archaically called Whitney sum by some) $E \oplus F$ as follows: for each x in X one puts $(E \oplus F)_x := E_x \oplus F_x$, and for local trivialisations $\Phi\colon \mathbb{R}^n \times U \to E|_U$ and $\Psi\colon \mathbb{R}^m \times U \to F|_U$ one lets $\Phi \oplus \Psi\colon \mathbb{R}^{n+m} \times U \to (E \oplus F)|_U$ given by $(\Phi \oplus \Psi)_x = \Phi_x \oplus \Psi_x$ be a local trivialisation (this defines uniquely a structure of manifold on $E \oplus F$).

Another way to describe $E \oplus F$ is to let it be the subset of (e, f) in $E \times F$ such that $pe = qf$, and equip it with the structure of a vector bundle (exercise). This is maybe the first example of a fibered product that one encounters in ones life.

Let E be a vector bundle over X. A *subbundle* of E is an injective morphism $F \to E$ of vector bundles over X. A *quotient* of E is a surjective morphism $E \to F$ of vector bundles over X. Clearly a subbundle $F \to E$ has a *quotient* or *cokernel* $E \to E/F$, and a quotient $q\colon E \to F$ has a *kernel* $\ker q \to E$ (a subbundle).

For E a vector bundle we have its *dual* $E^\vee$ such that $(E^\vee)_x = (E_x)^\vee$. For E and F vector bundles on X we have the *tensor product* $E \otimes F$, given by $(E \otimes F)_x = E_x \otimes F_x$. For E a vector bundle, we have its *symmetric powers* $\mathrm{Sym}^d E$ given by $(\mathrm{Sym}^d E)_x = \mathrm{Sym}^d(E_x)$, and its *alternating (or exterior) powers* $\bigwedge^d E$ given by $(\bigwedge^d E)_x = \bigwedge^d(E_x)$. An important case of the last one is the *maximal exterior power* $\det E = \bigwedge^n E$, where n is the rank of E.

In each of the constructions above, one can consider what happens with the transition maps of the vector bundles involved. For the direct sum, one gets $g_{j,i} \oplus h_{j,i}$, matrices composed of blocks, with the blocks $g_{j,i}$ and $h_{j,i}$ on the diagonal. In the case of a subbundle or a quotient, one can always choose the local trivialisation such that, in the trivialisation, the injection is given by the first so many coordinates, and the surjection is a projection on the last so many coordinates; the transition maps are then given by matrices in a two by two upper triangular block form (in fact, in this setting (differentiable manifolds) one can always get a two by two upper diagonal block form). For the dual, one replaces $g_{j,i}$ by $g_{j,i}^{-1,\vee}$ (contragredient representation). For the tensor product, one gets $g_{j,i} \otimes h_{j,i}$. For the symmetric and exterior powers, one gets $\mathrm{Sym}^d g_{j,i}$ and $\bigwedge^d g_{j,i}$. Finally, for $\det E$ one gets $\det g_{j,i}$. Let us remark that any smooth representation $\rho\colon \mathrm{GL}_n \to \mathrm{GL}_m$ gives a construction on vector bundles that maps a vector bundle of rank n with transition maps $g_{j,i}$ to one of rank m with $\rho g_{j,i}$.

A *section* of a vector bundle $p\colon E \to X$ is a map (smooth, of course) $s\colon X \to E$ such that $p \circ s = \mathrm{id}_X$; the set of sections $E(X)$ is an $\mathbb{R}$-vector space, and moreover a $\mathcal{O}(X)$-module (here $\mathcal{O}(X)$ denotes the $\mathbb{R}$-algebra of smooth $\mathbb{R}$-valued functions on X). The simplest example of a section is the *zero section* 0 given by $0(x) = 0_x$, where 0_x denotes the zero element of E_x.

We have already seen other examples: for $d \geq 0$ we have $\Omega^d(X) = \Omega^d_X(X)$, with $\Omega^d_X = \bigwedge^d \mathrm{T}^\vee_X$. In particular, $\mathcal{O}(X) = \mathbb{R}_X(X)$.

For $U \subset X$ open, we let $E(U)$ be the set of smooth maps $s\colon U \to E$ such that $p \circ s = \mathrm{id}_U$; then $E(U)$ is called the space (or $\mathcal{O}(U)$-module) of sections of E over U. To give a trivialisation of E on U is equivalent to giving $s_1, \ldots, s_n$ in $E(U)$ such that for every u in U the n-tuple $(s_1 u, \ldots, s_n u)$ is a basis of E_u. Such an n-tuple $(s_1, \ldots, s_n)$ will be called a *basis* or a trivialisation of E_U.

A *orientation* on a vector bundle E of rank n is a decomposition of $\det E - 0(X)$ into two open subsets, called the positive and negative parts. This is compatible with our notion of orientation of X: it coincides with orientations on T_X. We stress that an orientation on the vector bundle E is not the same as an orientation on the manifold E: the first notion is only about (continuous) orientations of the E_x, whereas the second involves more. As an exercise we propose that the reader shows that T_X as a manifold is orientable for all X.

8.4 Restrictions, pullbacks, the normal bundle of a subvariety

Let X be a manifold, $Y \subset X$ a submanifold, and E a vector bundle on X. Then the *restriction* E_Y of E to Y is the vector bundle on Y obtained by putting $(E_Y)_y = E_y$, and taking the restrictions of local trivialisations of E on U_i to $Y \cap U_i$.

More generally, for $f\colon Y \to X$ and E a vector bundle on X, we define the *pullback* f^*E (or also $f^{-1}E$) of E to Y (via f) by: $(f^*E)_y := E_{fy}$, and if $s_1, \ldots, s_n$ in $E(U)$ is a trivialisation, then $f^*s_1, \ldots, f^*s_n$ defined by $(f^*s_i)y = s_i(fy)$ form a trivialisation of f^*E on $f^{-1}U$. Restriction of a vector bundle is then the same as pullback via the inclusion map.

Let now X be a manifold, and Y a submanifold. Then T_Y is a subbundle of $(\mathrm{T}_X)_Y$. The *normal bundle* N_Y (or $N_{Y,X}$ if we want to stress the dependance on X) of Y in X is then the quotient $(\mathrm{T}_X)_Y/\mathrm{T}_Y$. By construction, we have a short exact sequence of vector bundles on Y:

$$0 \longrightarrow \mathrm{T}_Y \longrightarrow (\mathrm{T}_X)_Y \longrightarrow \mathrm{N}_{Y,X} \longrightarrow 0.$$

8.5 Metrics on vector bundles

Let X be a manifold, and let E be a vector bundle over X. A *metric*, or also inner product, on E is smoothly varying set of inner products b_x on the E_x. The precise meaning of "smoothly varying" here is that for any open $U \subset X$ and elements s and t in $E(U)$ the function $x \mapsto b_x(s(x), t(x))$, from U to $\mathbb{R}$, is smooth, i.e., in $\mathcal{O}(U)$. For example, the standard inner product on $\mathbb{R}^n$ gives the standard innner product on the trivial bundle $\mathbb{R}^n_X$, such that $b(f, g) = f_1 g_1 + \cdots + f_n g_n$, for all U and f and g in $\mathbb{R}^n_X(U)$.

We want to show that E admits a metric (far from unique, of course, if X is non-empty). Let U_i, $i \in I$ be a cover of X such that the E_{U_i} are trivial. We fix a choice of trivialisations $\phi_i\colon \mathbb{R}^n_{U_i} \to E_{U_i}$. Let f_i, $i \in I$, be a partition of unity: $f_i\colon X \to [0,1] \subset \mathbb{R}$ is in $\mathcal{O}(X)$, $\mathrm{Supp}\, f_i \subset U_i$, each x in X has a neighborhood that meets only finitely many of the $\mathrm{Supp}\, f_i$, and $\sum_i f_i = 1$. Let b_i be the inner product on E_{U_i} corresponding to the standard inner product on $\mathbb{R}^n_{U_i}$ via ϕ_i. Then $b := \sum_i f_i b_i$ is an inner product on E. (Here it is important that the b_i are positive definite, and that $f_i x \geq 0$ for all i and x.)

Suppose now that b is a metric on E. Let U be open in X and $s_1, \ldots, s_n$ a basis of E on U. Then the Gramm-Schmidt process gives us an orthonormal basis $(t_1, \ldots, t_n)$ of E_U. Using only local orthonormal trivialisations gives transition maps $g_{j,i}\colon U_i \cap U_j \to \mathrm{O}_n(\mathbb{R})$. If moreover E is oriented, using oriented orthonormal bases give $g_{j,i}$ with values in $\mathrm{SO}_n(\mathbb{R})$. (This phenomenon is called *reduction of the structure group.*)

8.6 Tubular neighborhoods and cohomology classes of closed submanifolds.

Let X be an oriented manifold, and let Y be a closed oriented submanifold of X, of codimension d. We recall that the class $[Y]$ in $\mathrm{H}_{\mathrm{dR}}(X)$ is characterised as follows: $[Y]$ is the class of an element η_Y in $\Omega^d(X)$ with $d\eta_Y = 0$ such that for every ω in $\Omega_c(X)$ with $d\omega = 0$ one has $\int_Y \omega = \int_X \omega\eta_Y$. The aim of this subsection is to give a useful description of $[Y]$, in terms of the normal bundle $\mathrm{N}_{Y,X}$ of Y in X. The following result says that $\mathrm{N}_{Y,X}$ can seen as an open neighborhood of Y in X.

8.6.1 Theorem. *Let Y be a closed submanifold of a manifold X. Then there is an open immersion $j\colon \mathrm{N}_{Y,X} \to X$ such that $j \circ 0\colon Y \to X$ is the inclusion of Y in X, where 0 denotes the zero-section of $\mathrm{N}_{Y,X}$.*

Such an embedding j is called a *tubular neighborhood* of Y in X, and the theorem is called the tubular neighborhood theorem. For the proof of this result we refer to the references given in the book by Bott and Tu. We remark that such tubular neighborhoods can be easily constructed using the exponential map from Riemannian geometry, as follows (the exponential map sends a sufficiently small tangent vector v at x to $\gamma_v 1$, where γ_v is the geodesic that starts at x and has derivative v). One chooses a Riemannian metric on X, i.e., a metric on T_X. Then, for each y in Y, we have the orthogonal direct sum decomposition $\mathrm{T}_X(y) = \mathrm{T}_Y(y) \oplus \mathrm{N}_Y(y)$, and we let $\exp_y$ be the exponential map from $\mathrm{T}_X(y)$ to X, on the largest open ball in $\mathrm{T}_X(y)$ on which it is defined. The restriction of $\exp_y$ to $\mathrm{N}_Y(y)$ then defines a map from an open neighborhood of $0(Y)$ in N_Y to X. As the derivative of this map is the identity on $0(Y)$, it induces an open immersion on a neighborhood of $0(Y)$.

Tubular neighborboods should be thought of as "tubes", or "sausages", around closed submanifolds. Such tubes can be given the structure of a vector bundle over Y, hence are locally a product of Y and some $\mathbb{R}^d$, but usually are not globally such a product. Tubular neighborhoods can be shrunken easily: if j is as in the theorem, and if U is any open neighborhood of Y in X then there exists a tubular neighborhood of the form $j' = j \circ f$ with $f\colon \mathrm{N}_{Y,X} \to \mathrm{N}_{Y,X}$ such that j' factors through U.

8.6.2 Definition. *Let $p\colon E \to Y$ be a vector bundle over a manifold Y. Let S be a subset of E. Then S is said to be* vertically compact *if $p|_S\colon S \to Y$ is proper, and S is said to be* horizontally compact *if $\overline{pS}$ is compact. An element ω of $\Omega(E)$ is said to have vertically (or horizontally)compact support if $\mathrm{Supp}\,\omega$ is vertically (or horizontally) compact.*

8.6.3 Theorem. *Let Y be a closed orientable submanifold of codimension n in an orientable manifold X, and let E denote the normal bundle of Y in X. Let $j\colon E \to X$ be a tubular* *neighborhood of Y. Let η be in $\Omega^n(E)$ with $d\eta = 0$, with* $\operatorname{Supp}\eta$ *vertically compact, and such that $\int_{E_y} \eta = 1$ for all y in Y. Then $j_*\eta$ in $\Omega^n(X)$ represents $[Y]$. Such forms η do exist.*

Proof. Let us suppose that η in $\Omega^n(E)$ satisfies the three requirements of the theorem. We shrink $j\colon E \to X$ outside $\operatorname{Supp}\eta$ such that the projection $p\colon jE \to Y$ extends continuously to the closure $\overline{jE}$. This does not change $j_*\eta$. Let now ω be in $\Omega_c^{\dim(Y)}(X)$, with $d\omega = 0$. Then $(\operatorname{Supp}\omega) \cap \overline{jE}$ is compact, hence its image in Y under p is compact. It follows that $j^*\omega$ has horizontally compact support. From now on we work in E, and write ω for $j^*\omega$.

We have to show that $\int_E \omega\eta = \int_Y 0^*\omega$. We note that $0 \circ p$ is homotopic to id_E (via multiplication by $t \in \mathbb{R}$ in the fibres of E). Hence there is an element α in $\Omega^{\dim(Y)-1}(E)$ such that $\omega = p^*0^*\omega + d\alpha$. Hence $\omega\eta = (p^*0^*\omega)\eta + (d\alpha)\eta$. We note that both $\operatorname{Supp}\omega\eta$ and $\operatorname{Supp}(p^*0^*\omega)\eta$ are compact as ω has horizontally compact support and η has vertically compact support, but it is not clear that $\operatorname{Supp}\alpha\eta$ is compact. However, it follows from the construction of α from the homotopy that α has horizontally compact support (in fact, $p\operatorname{Supp}\alpha$ is contained in $p\operatorname{Supp}\omega$). As $d\eta = 0$, we have $(d\alpha)\eta = d(\alpha\eta)$, hence, by Stokes, $\int_E (d\alpha)\eta = 0$. Hence we have to show that $\int_E (p^*0^*\omega)\eta = \int_{0(Y)} \omega$. Integration over E can be done by first integrating over the fibres of $E \to Y$, and then over Y. For y in Y, $p^*0^*\omega$ is constant on E_y, hence we have: $\int_{E_y} (p^*0^*\omega)\eta = (0^*\omega)_y \int_{E_y} \eta$. Hence $\int_E (p^*0^*\omega)\eta = \int_Y 0^*\omega$.

For the proof that such forms η do exist, we refer to Thm. 6.17 of Bott and Tu, where they show, using the Mayer-Vietoris argument, that integration along the fibres of E induces an isomorphism $\mathrm{H}^*_{\mathrm{dR,cv}}(E) \longrightarrow \mathrm{H}^{*-n}_{\mathrm{dR}}(X)$, where the $\mathrm{H}^i_{\mathrm{dR,cv}}(E)$ are constructed from the subcomplex of $\Omega(E)$ consisting of forms whose support is proper over X.

$\square$

9 Transversal intersections

Let X be a manifold, Y and Z closed submanifolds. Then Y and Z are called *transversal* if for every x in $Y \cap Z$ one has $\mathrm{T}_Y(x) + \mathrm{T}_Z(x) = \mathrm{T}_X(x)$. For example, if $\dim(Y) + \dim(Z) < \dim(X)$, then Y and Z are transversal if and only if they do not intersect at all. An equivalent way to express transversality is in terms of the codimension of subspaces of $\mathrm{T}_X(x)$:

$$\mathrm{codim}(\mathrm{T}_Y(x) \cap \mathrm{T}_Z(x)) = \mathrm{codim}(\mathrm{T}_Y(x)) + \mathrm{codim}(\mathrm{T}_Z(x)),$$

or as:

$$\mathrm{T}_X(x)/(\mathrm{T}_Y(x) \cap \mathrm{T}_Z(x)) = \mathrm{T}_X(x)/\mathrm{T}_Y(x) \oplus \mathrm{T}_X(x)/\mathrm{T}_Z(x).$$

Suppose now that Y and Z are transversal, and that x is in $Y \cap Z$. We choose, in a suitable open neighborhood U x, a submersion $f\colon U \to \mathbb{R}^d$ such that $Y \cap U = f^{-1}0$ and a submersion $g\colon U \to \mathbb{R}^e$ such that $Z \cap U = g^{-1}0$. Of course, d and e are the codimensions of Y and Z in X at x, respectively. Then the map $h\colon U \to \mathbb{R}^{d+e}$ given by $u \mapsto (fu, gu)$ is a submersion at x because the kernel of its tangent map at x is $\mathrm{T}_Y(x) \cap \mathrm{T}_Z(x)$, hence of codimension $d + e$. After shrinking U, if necessary, h is a submersion on U, and it follows that $Y \cap Z \cap U = h^{-1}0$ is a closed submanifold of U, and hence that $Y \cap Z$ is a closed submanifold of X. A good way of thinking about transversality is that Y and Z are locally given by independent equations: for each x in $Y \cap Z$ there are local coordinates $h_1, \ldots, h_n$ at x such that Y is given by the equations $h_i = 0$ for $1 \leq i \leq d$, and Z is given by the equations $h_i = 0$ for $d < i \leq d + e$.

By our direct sum description we have:

$$\mathrm{N}_{Y\cap Z} = (\mathrm{N}_Y)_{Y\cap Z} \oplus (\mathrm{N}_Z)_{Y\cap Z}.$$

If moreover X, Y and Z are all oriented, we get an orientation on $Y \cap Z$ as follows. The orientations on X and Y give an orientation on N_Y by requiring the short exact sequence:

$$0 \longrightarrow \mathrm{T}_Y \longrightarrow (\mathrm{T}_X)_Y \longrightarrow \mathrm{N}_Y \longrightarrow 0$$

to be compatible with the orientations. Likewise, we get an orientation on N_Z, hence an orientation on $\mathrm{N}_{Y\cap Z}$ by demanding the direct sum above (in that order) to be compatible with orientations. Finally, we get an orientation on $\mathrm{T}_{Y\cap Z}$ from the exact sequence:

$$0 \longrightarrow \mathrm{T}_{Y\cap Z} \longrightarrow (\mathrm{T}_X)_{Y\cap Z} \longrightarrow \mathrm{N}_{Y\cap Z} \longrightarrow 0.$$

In order to remember how things are oriented, we should keep in mind that the principle is that the standard short exact sequences and the direct sum decomposition above are to be compatible with the orientations.

9.1 Theorem. *Let X be an oriented manifold, Y and Z be two transversal oriented submanifolds. Let $i_Z\colon Z \to X$ denote the inclusion, and let $[Y \cap Z; Z]$ be the class of $Y \cap Z$ in Z. Then we have:*

$$[Y \cap Z; Z] = i_Z^*[Y] \quad \textit{in } \mathrm{H}_{\mathrm{dR}}(Z),$$

and:

$$[Y \cap Z] = [Y]\cdot[Z] \quad \textit{in } \mathrm{H}_{\mathrm{dR}}(X).$$

Proof. We first show that the second claim follows from the first. Let η_Y and η_Z in $\Omega(X)$ represent $[Y]$ and $[Z]$, respectively. Then, according to the first claim, $i_Z^*\eta_Y$ represents $[Y\cap Z; Z]$. Let ω be in $\Omega_c(X)$. Then we have:

$$\int_X \omega\eta_Y\eta_Z = \int_Z \omega\eta_Y = \int_{Y\cap Z}\omega.$$

Hence it remains to prove the first claim. We do this by constructing tubular neighborhoods of Y in X and of $Y\cap Z$ in Z that are suitably related. First we choose a metric on T_X. For x in $Y\cap Z$ we have $\mathrm{N}_{Y\cap Z}(x) = \mathrm{N}_Y(x)\oplus\mathrm{N}_Z(x)$, inside $\mathrm{T}_X(x)$, where now $\mathrm{N}_Y(x) = \mathrm{T}_Y(x)^\perp$, and $\mathrm{N}_Z(x) = \mathrm{T}_Z(x)^\perp$, and $\mathrm{N}_{Y\cap Z}(x) = \mathrm{T}_{Y\cap Z}(x)^\perp$. Let $\mathrm{N}_{Y\cap Z;Z}$ be the normal bundle of $Y\cap Z$ in $(\mathrm{T}_Z)|_{Y\cap Z}$. Then we have

$$\mathrm{N}_Y(x)\oplus\mathrm{N}_Z(x) = \mathrm{N}_{Y\cap Z}(x) = \mathrm{N}_Z(x)\oplus\mathrm{N}_{Y\cap Z;Z}(x).$$

This gives us an isomorphism:

$$f\colon (\mathrm{N}_Y)|_{Y\cap Z} \longrightarrow \mathrm{N}_{Y\cap Z;Z}.$$

Let U be a sufficiently small tubular neighborhood of the zero section in N_Y such that the exponential map $\phi\colon U\to X$ is defined on it, as well as the exponential map $\psi\colon f(U\cap(\mathrm{N}_Y)|_{Y\cap Z})\to Z$. Then ψ is a tubular neighborhood of Y in X, and ψ is one for $Y\cap Z$ in Z. If U is sufficiently small, we have, for each v in $U\cap(\mathrm{N}_Y)|_{Y\cap Z}$, a unique shortest geodesic from ϕv to $\psi(fv)$, giving us a homotopy from ϕ to ψf.

Now we choose η in $\Omega(U)$ of degree $\mathrm{codim}(Y)$, closed, with support proper over Y, and such that $\int_{\mathrm{N}_Y(y)}\eta = 1$ for each y in Y. The homotopy from ϕ to ψf and the closedness of η then show that η has integral one over the fibres of the tubular neighborhood ψ.

I admit that this proof should be done better (either a better method, or more details). $\square$

9.2 Corollary. *Let X, Y and Z be as in the Theorem. If moreover X is compact and of constant dimension n and $\dim(Y)+\dim(Z)=\dim(X)$, then $Y\cap Z$ is finite, and we have:*

$$\sum_{x\in Y\cap Z}\varepsilon_{Y,Z}(x) = \int_X [Y]{\cdot}[Z],$$

where $\varepsilon_{Y,Z}(x)=1$ if the orientations on $\mathrm{T}_X(x)=\mathrm{N}_Y(x)\oplus\mathrm{N}_Z(x)$ given by the orientations of X, Y and Z agree, and $\varepsilon_{Y,Z}(x)=-1$ if they do not agree. Hence the right hand side is an integer, called the intersection number*, that we will denote by $Y{\cdot}Z$.*

Proof. The dimension of $Y\cap Z$ is zero. As X is compact, $Y\cap Z$ is, hence $Y\cap Z$ is a finite set of points, $x_1,\ldots,x_n$, say. Note that $[Y\cap Z]$ is represented by a sum $\omega_1+\cdots+\omega_n$, with each ω_i a volume form concentrated around x_i, of mass 1 if the orientations match, and of mass -1 if they don't. $\square$

9.3 Example. Let $X := \mathrm{S}^1\times\mathrm{S}^1$, the torus, and let us consider one-dimensional closed submanifolds. For such a submanifold Y, we have its class $[Y]$ in $\mathrm{H}^1_{\mathrm{dR}}(X)$. We want to show that we can determine $[Y]$ from two intersection numbers. We write $\mathrm{H}_{\mathrm{dR}}(\mathrm{S}^1) = \mathbb{R}[x]/(x^2) = \mathbb{R}\oplus\mathbb{R}{\cdot}x$, with x the class of a point, hence in degree 1 (we have $x=[d\phi/2\pi]$). Then, by Künneth, we have $\mathrm{H}_{\mathrm{dR}}(X) = \mathbb{R}[x,y]/(x^2,y^2)$, with basis $(1,x,y,xy)$ (and $yx=-xy$). Now x and y are the classes of (any) fibers F_1 and F_2 of the two projections p_1 and p_2; they form a basis of $\mathrm{H}^1_{\mathrm{dR}}(X)$, hence $[Y]=ax+by$ for unique a and b in $\mathbb{R}$.

Consider the restriction of p_1 to Y. If t in S^1 is a regular value, then Y and $p_1^{-1}\{t\}$ are transversal, hence we have:

$$[Y \cap p_1^{-1}\{t\}] = [Y] \cdot [p_1^{-1}\{t\}] = (ax + by)x = -b{\cdot}xy$$

It follows that $b = -Y{\cdot}p_1^{-1}\{t\}$. Likewise, a is the intersection number $Y{\cdot}p_2^{-1}\{u\}$ for any regular value u of p_2. Hence we have seen that $[Y]$ can be determined from the intersections with the fibers of p_1 and p_2. This information can then be used to compute other intersection classes, or integrals.

10 The Lefschetz formula

Let $f\colon X \to X$ be an endomorphism of a manifold. We let Γ_f denote the *graph* of f in $X \times X$, i.e, Γ_f is the image of the map $X \to X \times X$, $x \mapsto (x, fx)$; it is a closed submanifold of $X \times X$ (to see this, take local coordinates around x and fx). We are interested in counting the x in X that are fixed by f, i.e., the x such that $fx = x$, with multiplicities. These so-called *fixed points* are precisely the points in $\Gamma_f \cap \Delta$, where Δ is the diagonal in $X \times X$, i.e., the set of (x, x); we note that $\Delta = \Gamma_{\mathrm{id}_X}$, the graph of the identity morphism of X. The formula of the title of this section is the one in the following theorem.

10.1 Theorem. (Lefschetz) *Let X be a manifold, compact, orientable, and $f\colon X \to X$ a morphism such that its graph Γ_f is transversal to the diagonal Δ in $X \times X$. Then we have:*

$$\sum_{x=fx} \mathrm{sign}(\det(\mathrm{id} - \mathrm{T}_f x)) = \sum_{i \geq 0} (-1)^i \mathrm{tr}(f^*, \mathrm{H}^i(X)).$$

The proof of this result will involve several steps. First, we will "compute" the classes $[\Gamma_f]$ and $[\Delta]$ in $\mathrm{H}(X)$. Then we will compute $[\Gamma_f]\cdot[\Delta]$ as the right hand side of the Lefschetz formula, and $[\Gamma_f \cap \Delta]$ as the left hand side. During the proof, we will use an orientation of X, but all terms in the theorem do not depend on it.

10.2 Proposition. *Let X be a compact oriented manifold, and $f\colon X \to X$ an endomorphism. Let $[\omega_i]$, $i \in I$, be a basis of $\mathrm{H}(X)$, with each ω_i homogeneous, say of degree $\deg \omega_i$. Let $[\tau_i]$, $i \in I$ be the Poincaré dual basis: $\int_X \omega_i \tau_j = \delta_{i,j}$. Let p_1 and p_2 be the two projections $X \times X \to X$ on the first and second factor. Then the $p_1^*[\omega_i]\cdot p_2^*[\tau_j]$, i, j in I, form a basis of $\mathrm{H}(X \times X)$ by Künneth's theorem. Let the real numbers $f_{i,j}$, $(i,j) \in I^2$, be defined by $f^*[\omega_i] = \sum_j f_{j,i}[\omega_j]$. Then one has:*

$$[\Gamma_f] = \sum_{i,j} (-1)^{\deg \omega_i} f_{i,j} p_1^*[\omega_i]\cdot p_2^*[\tau_j] = \sum_j (-1)^{\deg \omega_j} p_1^* f^*[\omega_j]\cdot p_2^*[\tau_j].$$

In particular, for $f = \mathrm{id}_X$ we have:

$$[\Delta] = \sum_i (-1)^{\deg \omega_i} p_1^*[\omega_i]\cdot p_2^*[\tau_i] = (-1)^{\dim X} \sum_i (-1)^{\deg \omega_i} p_2^*[\omega_i]\cdot p_1^*[\tau_i].$$

Proof. As a small warming up for the tensor calculus involved, we note that for a finite dimensional vector space V, with basis v_i and dual basis $v_i^\vee$, and an endomorphism f, we have the formula $f = \sum_{i,j} (v_j^\vee(fv_i))\cdot v_i^\vee \otimes v_j$.

Let us write Γ for Γ_f, and let $\iota\colon X \to X \times X$ be the map $x \mapsto (x, fx)$. There are unique $c_{i,j}$ in $\mathbb{R}$ such that $[\Gamma] = \sum_{i,j} c_{i,j} p_1^*[\omega_i]\cdot p_2^*[\tau_j]$. By definition we then have, for every closed η in $\Omega(X \times X)$:

$$\text{(10.2.1)} \qquad \int_X \iota^*\eta = \int_\Gamma \eta = \int_{X\times X} \eta\cdot[\Gamma] = \sum_{i,j} c_{i,j} \int_{X \times X} \eta\cdot p_1^*\omega_i\cdot p_2^*\tau_j.$$

The idea is now simply to apply this to η of the form $p_1^*\tau_k\cdot p_2^*\omega_l$, for arbitrary k and l in I. The left hand side of (10.2.1) then becomes:

$$\int_X \iota^*(p_1^*\tau_k\cdot p_2^*\omega_l) = \int_X \tau_k\cdot f^*\omega_l = \sum_m f_{m,l} \int_X \tau_k \omega_m = \sum_m (-1)^{(\deg \omega_m)(\deg \tau_k)} f_{m,l}\delta_{m,k} =$$
$$= (-1)^{(\deg \omega_k)(\deg \tau_k)} f_{k,l}.$$

On the other hand, the right hand side of (10.2.1) becomes:

$$\begin{aligned}\sum_{i,j} c_{i,j}\int_{X\times X} p_1^*\tau_k\cdot p_2^*\omega_l\cdot p_1^*\omega_i\cdot p_2^*\tau_j &= \sum_{i,j} c_{i,j}(-1)^{(\deg\omega_l)(\deg\omega_i)+(\deg\tau_k)(\deg\omega_i)}\int_X \omega_i\tau_k\cdot\int_X\omega_l\tau_j\\ &= \sum_{i,j} c_{i,j}(-1)^{(\deg\omega_l)(\deg\omega_i)+(\deg\tau_k)(\deg\omega_i)}\delta_{i,k}\delta_{j,l}\\ &= c_{k,l}\cdot(-1)^{(\deg\omega_l)(\deg\omega_k)+(\deg\tau_k)(\deg\omega_k)}.\end{aligned}$$

Combining the two we see that $c_{k,l} = (-1)^{\deg\omega_k} f_{k,l}$ (note that $f_{k,l} = 0$ if $\deg\omega_k \neq \deg\omega_l$), which completes the proof of the first identity for $[\Gamma_f]$. The second identity follows from the definition of the $f_{i,j}$. The first expression for $[\Delta]$ follows from the one for $[\Gamma_{\mathrm{id}}]$ because $\mathrm{id}^*\omega_i = \omega_i$. For the second one, one uses that the "flip" $(x,y)\mapsto(y,x)$ on $X\times X$ preserves the orientation on Δ (as it induces the identity on it), but changes the orientation of $X\times X$ by a factor $(-1)^{\dim X}$: in local coordinates:

$$dy_1\cdots dy_n dx_1\cdots dx_n = (-1)^{n^2} dx_1\cdots dx_n dy_1\cdots dy_n,$$

and $n^2 - n$ is even. □

10.3 Proposition. *Notation as in the previous proposition. Then we have:*

$$\int_{X\times X}[\Gamma]\cdot[\Delta] = \sum_i (-1)^i \mathrm{tr}(f^*, \mathrm{H}^i(X)).$$

Proof. We simply compute:

$$\begin{aligned}\int_{X\times X}[\Gamma]\cdot[\Delta] &= \sum_{i,j,k}(-1)^{\dim X+\deg\omega_i+\deg\omega_k} f_{i,j}\int_{X\times X} p_1^*\omega_i p_2^*\tau_j p_2^*\omega_k p_1^*\tau_k\\ &= \sum_{i,j,k}(-1)^{\dim X+\deg\omega_i+\deg\omega_k+\deg\tau_k\deg\omega_k+\deg\tau_k\deg\tau_j+\deg\omega_k\deg\tau_j} f_{i,j}\int_X\omega_i\tau_k\int_X\omega_k\tau_j\\ &= \sum_{i,j,k}(-1)^{\dim X+\deg\omega_i+\deg\omega_k+\deg\tau_k\deg\omega_k+\deg\tau_k\deg\tau_j+\deg\omega_k\deg\tau_j} f_{i,j}\delta_{i,k}\delta_{k,j}\\ &= \sum_i(-1)^{\dim X+\deg\tau_i} f_{i,i}\\ &= \sum_i(-1)^{\deg\omega_i} f_{i,i} = \sum_i(-1)^i\mathrm{tr}(f^*,\mathrm{H}^i(X)).\end{aligned}$$

□

We can now finish the proof of Theorem 10.1.

Proof. (of Theorem 10.1) The result on transversal intersections of the previous section tells us:

$$\int_{X\times X}[\Gamma]\cdot[\Delta] = \sum_{x=fx}\varepsilon_f(x),$$

where $\varepsilon_f(x) = 1$ and $\varepsilon_f(x) = -1$ if the two orientations on $\mathrm{T}_{X\times X}(x,x) = \mathrm{N}_\Gamma(x,x)\oplus\mathrm{N}_\Delta(x,x)$ given by the orientations of $X\times X$, Γ and Δ do or do not agree. Let us recall that the direct sum decomposition of $\mathrm{T}_{X\times X}$ comes from $\mathrm{T}_{X\times X}(x,x) = \mathrm{T}_\Delta(x,x)\oplus\mathrm{T}_\Gamma(x,x)$. So we have to compare the given orientations on the first and last of the following vector spaces:

$$\mathrm{T}_{X\times X}(x,x) = \mathrm{T}_\Delta(x,x)\oplus\mathrm{T}_\Gamma(x,x) = \mathrm{N}_\Gamma(x,x)\oplus\mathrm{N}_\Delta(x,x).$$

So let x be a fixed point, and let $(v_1,\ldots,v_n)$ be a positive basis of $\mathrm{T}_X(x)$. We put $t := \mathrm{T}_f(x)$. Then we have positive bases as follows:

- $\mathrm{T}_{X\times X}(x,x)$: $((v_1,0),\ldots,(v_n,0),(0,v_1),\ldots,(0,v_n))$.
- $\mathrm{T}_\Gamma(x,x)$: $(v_1,tv_1),\ldots,(v_n,tv_n)$ (to be positively completed with $((0,v_1),\ldots,(0,v_n))$).
- $\mathrm{T}_\Delta(x,x)$: $((v_1,v_1),\ldots,(v_n,v_n))$ (to be positively completed with $((0,v_1),\ldots,(0,v_n))$).

The completions of the positive bases of $\mathrm{T}_\Gamma(x,x)$ and $\mathrm{T}_\Delta(x,x)$ to positive bases of $\mathrm{T}_{X\times X}(x,x)$ above show that $((0,v_1),\ldots,(0,v_n))$ induces a positive basis of $\mathrm{N}_\Gamma(x,x)$, and of $\mathrm{N}_\Delta(x,x)$.

Let us now consider the given isomorphism $\mathrm{T}_\Delta(x,x)\to\mathrm{N}_\Gamma(x,x)$ and compute the factor by which it changes the orientations. This is done by computing in $\mathrm{T}_{X\times X}(x,x)/\mathrm{T}_\Gamma(x,x)$. The element (v_i,v_i) of $\mathrm{T}_\Delta(x,x)$ is mapped to $\overline{(v_i,v_i)}=\overline{(0,v_i-tv_i)}$. Hence the factor here for comparing with the basis of the $\overline{(0,v_i)}$ is $\operatorname{sign}\det(\mathrm{id}-t)$.

We do the same for $\mathrm{T}_\Gamma(x,x)\to\mathrm{N}_\Delta(x,x)$. We compute in $\mathrm{T}_{X\times X}(x,x)/\mathrm{T}_\Delta(x,x)$. The element (v_i,tv_i) is mapped to $\overline{(v_i,tv_i)}=\overline{(0,tv_i-v_i)}$. Hence the factor for comparing with the basis of the $\overline{(0,v_i)}$ is $\operatorname{sign}\det(t-\mathrm{id})$.

Combining these, we see that the sign between the orientations on $\mathrm{T}_\Delta(x,x)\oplus\mathrm{T}_\Gamma(x,x)$ and $\mathrm{N}_\Gamma(x,x)\oplus\mathrm{N}_\Delta(x,x)$ is the product of the signs of $\det(\mathrm{id}-t)$ and $\det(t-\mathrm{id})$.

Finally, let us compute the sign between $\mathrm{T}_{X\times X}(x,x)$ and $\mathrm{T}_\Delta(x,x)\oplus\mathrm{T}_\Gamma(x,x)$. We start with our positive basis of the last one, and apply elementary operations to it that preserve the orientation:

$$\begin{aligned}
&((v_1,v_1),\ldots,(v_n,v_n),(v_1,tv_1),\ldots,(v_n,tv_n))\equiv\\
&\qquad\equiv((v_1,v_1),\ldots,(v_n,v_n),(0,tv_1-v_1),\ldots,(0,tv_n-v_n))\equiv\\
&\qquad\qquad\equiv((v_1,0),\ldots,(v_n,0),(0,tv_1-v_1),\ldots,(0,tv_n-v_n)).
\end{aligned}$$

Hence we see that the sign here is $\operatorname{sign}\det(t-\mathrm{id})$. It follows that the our orientations on $\mathrm{T}_{X\times X}(x,x)$ and $\mathrm{N}_\Gamma(x,x)\oplus\mathrm{N}_\Delta(x,x)$ are compatible up to the factor $\operatorname{sign}\det(\mathrm{id}-t)$. □

10.4 Example. In order to convince the reader that we did not make a mistake in computing the right signs from the orientations, let us do one example where we show that $\operatorname{sign}\det(\mathrm{id}-\mathrm{T}_f(x))$ is right, and $\operatorname{sign}\det(\mathrm{T}_f(x)-\mathrm{id})$ is wrong. Of course, in this example X should have odd dimension (otherwise the two signs are the same). So we take $X=\mathrm{S}^1$, and we take for f the complex conjugation. Then f has two fixed points: 1 and -1, and at each of them we have that $\mathrm{T}_f(x)$ acts as multiplication by -1 on the tangent space. Hence the sum of the $\operatorname{sign}\det(\mathrm{id}-\mathrm{T}_f(x))$ is two. On the other hand, f^* acts as 1 on $\mathrm{H}^0(X)=\mathbb{R}$, and by -1 on $\mathrm{H}^1(X)=\mathbb{R}[d\phi]$. Hence $\sum_i(-1)^i\mathrm{tr}(f^*,\mathrm{H}^i(X))$ equals two.