Lie groups and Lie algebras, D.E.A., 2000-2001.

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These are notes for a course given at Rennes at the DEA level, for students in algebra and geometry, or in analysis, or in probability theory. The notes are based on a handwritten version of about 128 pages by Dominique Cerveau, who has taught the same course during the last few years, and who is of course not responsible for the mistakes in this version. The main result of the course seems to be the Peter-Weyl theorem, which is a generalisation of Fourier theory on the circle to compact Lie groups. I will try to follow his notes quite faithfully, but also I would like to put a little bit more emphasis on representations of Lie groups, with of course the explicit examples for the groups $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{SU}_2(\mathbb{R})$. In particular, I would like to find time to discuss the applications of these examples to quantum mechanics, and, if possible, Gell Man's discovery of quarks in terms of representations of $\mathrm{SU}_3(\mathbb{R})$. The course takes place in 30 hours (10 weeks, 3 hours a week). One reason to type these notes is to have them available on the internet, in a convenient format.

Contents

1 Some topology and differential geometry

1.1 Introduction

In order to even define what an abstract Lie group is, we need the terminology of manifolds. So we begin this course by recalling some facts from topology and differential geometry.

1.2 Topology and quotients

We recall that a topological space X is called separated (or Hausdorff) if for all distinct elements x_1 and x_2 of X there are neighborhoods U_1 and U_2 of x_1 and x_2, respectively, such that $U_1 \cap U_2$ is empty. Equivalently, X is Hausdorff if and only if the diagonal in $X \times X$ (with the product topology) is closed. A compact subset of a separated topological space is closed. If $f\colon X \to Y$ is a continuous bijection with X compact and Y separated, then f is a homeomorphism.

We will view an equivalence relation R on a set X as its graph in $X \times X$. If R is an equivalence relation on a topological space X, then we define the quotient topological space X/R as follows: the set X/R is the set of equivalence classes of R, and a subset U of X/R is open if and only if $p^{-1}U$ is open in X, where $p\colon X \to X/R$ is the quotient map. With this definition, $p\colon X \to X/R$ has the usual universal property: if $f\colon X \to Y$ is continuous and satisfies $f(x_1) = f(x_2)$ for all (x_1, x_2) in R, then there is a unique continuous map $\overline{f}\colon X/R \to Y$ such that $f = \overline{f} \circ p$.

1.2.1 Exercise. Let R be an equivalence relation on a topological space X.

1. Suppose that X/R is separated. Show that R is closed in $X \times X$.

2. Suppose that R is closed in $X \times X$ and that $p\colon X \to X/R$ is open (i.e., that $p(U)$ is open in X/R for every open subset U of X). Show that X/R is separated.

3. Suppose that R is the equivalence relation given by an action of a group G on X. Show that $p\colon X \to X/R$ is open.

1.2.2 Example. Projective spaces. For $n \geq 0$ integer, we have the real and complex n-dimensional projective spaces $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$, defined to be the quotients of $\mathbb{R}^{n+1} - \{0\}$ and $\mathbb{C}^{n+1} - \{0\}$ by the equivalence relation "lying on the same (real or complex) line through 0". These quotients are separated. The fibers of p are precisely the lines through 0.

1.2.3 Example. Elliptic curves. One considers the quotients $\mathbb{C}/\Lambda$, where Λ is a lattice (i.e., discrete subgroup of maximal rank) in $\mathbb{C}$. These quotients are all homeomorphic to the product of two circles $\mathrm{S}^1 \times \mathrm{S}^1$ (and hence are separated). In this case the quotient map p is a covering (revêtement in French), meaning that for each x in $\mathbb{C}/\Lambda$ there is an open neighborhood U of x such that $p^{-1}U \to U$ is isomorphic to a disjoint union of copies of U (think of a stack of pancakes). This makes it possible to give $\mathbb{C}/\Lambda$ the structure of a complex analytic variety (even if we have not yet defined what that should be): the notion of holomorphic complex function on an open subset of $\mathbb{C}/\Lambda$ is clear.

Let (λ_1, λ_2) be a $\mathbb{Z}$-basis of Λ. Then multiplication by λ_1^{-1} transforms Λ into the lattice $\mathbb{Z} + \mathbb{Z}\tau$, with $\tau = \lambda_2/\lambda_1$. Hence the quotient $\mathbb{C}/\Lambda$ is isomorphic to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. This last quotient is often performed in two steps: the quotient of $\mathbb{C}$ by the subgroup $\mathbb{Z}$ is given by $e\colon \mathbb{C} \to \mathbb{C}^*$, $z \mapsto \exp(2\pi i z)$ (note that it is surjective and that the kernel is $\mathbb{Z}$). Let $q = e(\tau)$. Then we conclude that $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ is isomorphic to $\mathbb{C}^*/q^{\mathbb{Z}}$.

1.2.4 Example. Hopf manifolds. Let $n \geq 1$, and let $\lambda_1, \ldots, \lambda_n$ be real numbers with $0 < \lambda_i < 1$. We let the group $\mathbb{Z}$ act on $\mathbb{R}^n - \{0\}$ by:

$$(m, (x_1, \ldots, x_n)) \mapsto (\lambda_1^m x_1, \ldots, \lambda_n^m x_n).$$

The quotient $\mathbb{Z}\backslash(\mathbb{R}^n - \{0\})$ is then homeomorphic to $\mathrm{S}^{n-1} \times \mathrm{S}^1$.

1.2.5 Example. Real and complex tori. One considers quotients of finite dimensional real or complex vector spaces by lattices. The quotients are separated, even compact, and in fact products of circles. (Easy exercise.)

1.2.6 Example. Spaces of lattices. Let $n \geq 0$. We consider the quotient of $\mathrm{GL}_n(\mathbb{R})$ by the action given by right translations of the subgroup $\mathrm{GL}_n(\mathbb{Z})$ (the group of invertible elements in the ring $\mathrm{M}_n(\mathbb{Z})$. Note that $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $\mathrm{M}_n(\mathbb{R})$, which induces the topology on $\mathrm{GL}_n(\mathbb{R})$ that we want to use. The topology induced on $\mathrm{GL}_n(\mathbb{Z})$ is discrete, and the quotient map $p\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{Z})$ is a covering. The quotient $\mathrm{GL}_n(\mathbb{R})/\mathrm{GL}_n(\mathbb{Z})$ is therefore a manifold of dimension n^2. What makes this quotient interesting is that it is the set L of lattices in $\mathbb{R}^n$: the group $\mathrm{GL}_n(\mathbb{R})$ acts on L, this action is transitive, and the stabilizer of $\mathbb{Z}^n$ is $\mathrm{GL}_n(\mathbb{Z})$. We should note that in all the examples until now the equivalence relation comes from a group action.

1.2.7 Example. Non separated quotients. Let us end the examples by some non separated ones.

1. The quotient of $\mathbb{R}$ by the subgroup $\mathbb{Q}$ (or by $\mathbb{Z} + \mathbb{Z}x$ with x not in $\mathbb{Q}$).

2. The quotient of $\mathrm{M}_{m,n}(\mathbb{R})$ (with $m \geq 1$ or $n \geq 1$) by the equivalence relation "equal rank". (Exercise: is this equivalence relation given by a group action? Is every equivalence relation given by a group action?)

1.3 Sub manifolds of $\mathbb{R}^n$

1.3.1 Definition. Let $n \geq 0$ and $U \subset \mathbb{R}^n$ open. A subset V of U is called a closed sub manifold of U if it is closed and if for every v in V there is a diffeomorphism f from an open neighborhood W of v in U to an open neighborhood W' of 0 in $\mathbb{R}^n$ such that $f(V) = W' \cap (\mathbb{R}^k \times \{0\})$ for some k; this k is called the dimension of V at v. Informally speaking: V is a closed sub manifold of U if it is closed and locally at every v in V, up to diffeomorphism, a linear subspace of $\mathbb{R}^n$.

1.3.2 Remark. The notion of diffeomorphism that we use is the one that demands the map to be infinitely differentiable (notation: C^∞), unless we say otherwise.

1.3.3 Example. Let $n \geq 0$, and let $Q\colon \mathbb{R}^n \to \mathbb{R}$ be a quadratic form of rank n on $\mathbb{R}^n$. Then the level sets $Q^{-1}t$ with $t \neq 0$ are closed sub manifolds of $\mathbb{R}^n$.

1.3.4 Example. The spiral $S = \{\exp(t)(\cos(t), \sin(t)) \mid t \in \mathbb{R}\}$ in $\mathbb{C} = \mathbb{R}^2$ is a one-dimensional closed sub manifold of $\mathbb{R}^2 - \{0\}$, but not of $\mathbb{R}^2$ (because it is not closed in $\mathbb{R}^2$).

1.4 Submersions, immersions, embeddings, local models

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$ be open subsets. Let $f\colon U \to V$ be a differentiable map (C^∞). Then f is a *submersion* at $u \in U$ if the derivative $(Df)u\colon \mathbb{R}^n \to \mathbb{R}^p$ is surjective. Suppose now that that is so. After a suitable renumbering of the coordinates of $\mathbb{R}^n$, the map $\phi\colon U \to \mathbb{R}^n$ given by $\phi(x) = (f(x), x_1, \ldots, x_{n-p})$ has bijective derivative $(D\phi)u$ at u. We note that $f = \mathrm{pr}_{\leq p} \circ \phi$, with $\mathrm{pr}_{\leq p}\colon \mathbb{R}^n \to \mathbb{R}^p$ the projection on the first p coordinates. The implicit function theorem (théorème d'inversion locale in french) gives that ϕ induces a diffeomorphism from a suitable open neighborhood of u in U to an open subset U' of $\mathbb{R}^n$, and we have the formula:

$$(f \circ \phi^{-1})(x_1, \ldots, x_n) = (x_1, \ldots, x_p),$$

for all $(x_1, \ldots, x_n)$ in U'. In other words, up to a local diffeomorphism, a submersion is a linear projection. We say that f is a submersion if it is a submersion at all u in U.

1.4.1 Proposition. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$ open subsets, and let $f\colon U \to V$ be a differentiable map. Let v be in V, and suppose that f is a submersion at all u in $f^{-1}\{v\}$. Then $f^{-1}\{v\}$ is a closed sub manifold of U, of dimension $n-p$ at every point.*

Proof. Clear, since this is so for the local model. □

Let again $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$ be open subsets, and $f\colon U \to V$ a differentiable map. We say that f is an *immersion* at u in U if $(Df)u$ is injective. Suppose that that is so. After a suitable renumbering of the coordinates of $\mathbb{R}^p$, the map $\phi\colon U \times \mathbb{R}^{p-n} \to \mathbb{R}^n \times \mathbb{R}^{p-n} = \mathbb{R}^p$ given by $\phi(x,y) = f(x) + \mathrm{i}_{>n}(y)$, with $\mathrm{i}_{>n}\colon \mathbb{R}^{p-n} \to \mathbb{R}^p$ given by $\mathrm{i}_{>n}(y) = (0,y)$, has invertible derivative at $(u,0)$. We note that $f = \phi \circ \mathrm{i}_{\leq n}$. The implicit function theorem gives that ϕ is invertible in a neighborhood of $(u,0)$, and we have the formula:

$$(\phi^{-1} \circ f)(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0),$$

for all $(x_1, \ldots, x_n)$ in a neighborhood of u in U. So, locally at u, f is equivalent to a linear inclusion. The map f is called an immersion if it is an immersion at every point.

In order to state an analog of the previous proposition for images of immersions, we introduce the notion of closed embedding. A map $f\colon X \to Y$ between topological spaces is called a *closed embedding* if it is injective, with closed image, and such that the topology on X induced by Y is the topology of X. Equivalently, f is a closed embedding if it is injective, and if for Z any subset of X, V is closed if and only if $f(V)$ is closed in Y. Equivalently, f is a closed embedding if it induces an isomorphism from X to a closed subspace of Y. (Of course, one also has the analogous notion of open embedding.)

1.4.2 Proposition. *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^p$ open subsets, and let $f\colon U \to V$ be a differentiable map. Then f is called a closed immersion if it is an immersion and a closed embedding. If this is so, then $f(U)$ is a closed sub manifold of V.*

1.4.3 Example. We give two examples of immersions that are not closed immersions.

1. Let x be in $\mathbb{R}$ but not in $\mathbb{Q}$. Take $f\colon \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$, $t \mapsto \overline{t(1,x)}$. In this case the image of f is not closed, it is even dense.

2. (This one would be easier with a picture.) Take $f\colon \mathbb{R} \to \mathbb{R}^2$ given by: $t \mapsto (1,t)$ for $t \leq 0$, and $t \mapsto (\cos(g(t), \sin(g(t))$ for $t \geq 0$, with $g\colon [0,\infty[\to \mathbb{R}$ a C^∞ bijection between $\mathbb{R}_{\geq 0}$ and $[0, 2\pi[$ with $g'(0) = 1$ and $g^{(n)}(0) = 0$ for all $n > 1$.

1.5 Sub Lie groups of $\mathrm{GL}_n(\mathbb{R})$

By definition, the sub Lie groups of $\mathrm{GL}_n(\mathbb{R})$ are the subgroups that are closed sub manifolds (recall that $\mathrm{GL}_n(\mathbb{R})$ is an open subset of $\mathrm{M}_n(\mathbb{R}) = \mathbb{R}^{n^2}$). We start with some very simple examples.

1.5.1 Example. The diagonal subgroup: $\{\mathrm{diag}(t_1, \ldots, t_n) \mid t_1, \ldots, t_n \in \mathbb{R}^*\}$. It is given by the equations $x_{i,j} = 0$ for all $i \neq j$, and it is isomorphic to $(\mathbb{R}^*)^n$.

1.5.2 Example. The triangular subgroup $\{x \in \mathrm{GL}_n(\mathbb{R}) \mid x_{i,j} = 0 \text{ if } i > j\}$. This one is not commutative, but still solvable.

1.5.3 Example. The standard maximal unipotent subgroup:

$$\{x \in \mathrm{GL}_n(\mathbb{R}) \mid x_{i,j} = 0 \text{ if } i > j \text{ and } x_{i,i} = 1 \text{ for all } i\}.$$

In order to see that $\mathrm{SL}_n(\mathbb{R})$ is a sub Lie group of $\mathrm{GL}_n(\mathbb{R})$, it is convenient to prove the following lemma (which is a bit more general than we need).

1.5.4 Lemma. *Let $n \geq 0$, and let a be in $\mathrm{M}_n(\mathbb{R})$. Then $\det\colon \mathrm{M}_n(\mathbb{R}) \to \mathbb{R}$ is a submersion at a if and only if $\mathrm{rank}(a) \geq n-1$.*

Proof. Put $r := \mathrm{rank}(a)$. We take elements p and q in $\mathrm{GL}_n(\mathbb{R})$ such that $a = p1_rq$, with 1_r the diagonal matrix with r ones followed by zeros. The map $\mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$, $x \mapsto pxq$ is a linear bijection, hence a diffeomorphism, so that it suffices to prove that the map $x \mapsto \det(pxq) = \det(p)\det(q)\det(x)$ from $\mathrm{M}_n(\mathbb{R})$ to $\mathbb{R}$ is a submersion at 1_r if and only if $r \geq n-1$. This is done by calculating partial derivatives, using either the definition of det as a sum over the group S_n, or column or row expansion. ☐

1.5.5 Corollary. *For $n \geq 0$, the subgroup $\mathrm{SL}_n(\mathbb{R}) = \det^{-1}\{1\}$ of $\mathrm{GL}_n(\mathbb{R})$ is a sub Lie group, of dimension $n^2 - 1$.*

1.5.6 The orthogonal groups

For $n \geq 0$ we let $\mathrm{O}_n(\mathbb{R})$ be the subgroup $\{x \in \mathrm{GL}_n(\mathbb{R}) \mid x^{\mathrm{t}}x = 1\}$, where x^{t} denotes the transpose of x: $x^{\mathrm{t}}_{i,j} = x_{j,i}$. More conceptually, $\mathrm{O}_n(\mathbb{R})$ is the subgroup of elements g of $\mathrm{GL}_n(\mathbb{R})$ that preserve the standard inner product $\langle x, y\rangle = x^{\mathrm{t}}y = \sum_i x_iy_i$ of $\mathbb{R}^n$ in the sense that g is in $\mathrm{O}_n(\mathbb{R})$ if and only if $\langle gx, gy\rangle = \langle x, y\rangle$ for all x and y in $\mathbb{R}^n$. It is clear, from looking at the defining equations $(x^{\mathrm{t}}x)_{i,j} = \delta_{i,j}$, that $\mathrm{O}_n(\mathbb{R})$ is a closed, compact subgroup of $\mathrm{GL}_n(\mathbb{R})$. In order to see that it is a sub manifold, we consider the map:

$$f\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})^+, \quad a \mapsto a^{\mathrm{t}}a,$$

with $\mathrm{M}_n(\mathbb{R})^+$ the subspace of symmetric matrices of $\mathrm{M}_n(\mathbb{R})$. The map f is polynomial, hence C^∞. Let us compute its derivative at 1, for b in $\mathrm{M}_n(\mathbb{R})$ we have:

$$f(1+b) = (1+b)^{\mathrm{t}}(1+b) = 1 + b^{\mathrm{t}} + b + b^{\mathrm{t}}b = f(1) + (b^{\mathrm{t}} + b) + b^{\mathrm{t}}b.$$

Since $b^{\mathrm{t}} + b$ is linear in b and $b^{\mathrm{t}}b$ homogeneous quadratic, we have $((Df)1)b = b^{\mathrm{t}} + b$. Since every symmetric matrix is of the form $b^{\mathrm{t}} + b$ (think of the decomposition of $\mathrm{M}_n(\mathbb{R})$ into the direct sum $\mathrm{M}_n(\mathbb{R})^+ \oplus \mathrm{M}_n(\mathbb{R})^-$), f is a submersion at 1, and, in a neighborhood of 1, $\mathrm{O}_n(\mathbb{R})$ is a closed sub manifold. In order to finish the proof that $\mathrm{O}_n(\mathbb{R})$ is a closed sub manifold of $\mathrm{GL}_n(\mathbb{R})$ we use the following lemma, whose proof, although quite obvious, illustrates an important principle: the group acts transitively on itself by right or left translations, implying that "everything is the same everywhere".

1.5.7 Lemma. *Let G be a closed subgroup of $\mathrm{GL}_n(\mathbb{R})$. If there exists an open neighborhood U of 1 in $\mathrm{GL}_n(\mathbb{R})$ such that $G \cap V$ is a closed sub manifold of U, then G is a sub Lie group of $\mathrm{GL}_n(\mathbb{R})$.*

Proof. Let g be in G. Let $L_g\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ be the left translation by g: $x \mapsto gx$. This L_g is a diffeomorphism, because it extends to a linear automorphism of $\mathrm{M}_n(\mathbb{R})$. Let U be as in the statement of the lemma. Then $gU = L_gU$ is an open neighborhood of g, and $G \cap gU = g(G \cap U)$, hence $G \cap gU$ is a closed sub manifold of gU. □

Later we will prove a theorem of Cartan and von Neumann that says that each closed subgroup of $\mathrm{GL}_n(\mathbb{R})$ is a sub Lie group.

Note that we do not demand our Lie groups to be connected. For example, the zero dimensional sub Lie groups of $\mathrm{GL}_n(\mathbb{R})$ are the discrete subgroups, i.e., the closed subgroups on which the induced topology is discrete. We have already seen the example $\mathrm{GL}_n(\mathbb{Z})$.

1.5.8 Exercise. Let $n \geq 0$. Show that $\mathrm{GL}_n(\mathbb{C})$ is open in $\mathrm{M}_n(\mathbb{C})$. Show that the subset $\mathrm{U}_n = \{x \in \mathrm{GL}_n(\mathbb{C}) \mid \overline{x}^{\mathrm{t}}x = 1\}$ is a sub Lie group of $\mathrm{GL}_n(\mathbb{C})$, and determine its dimension. These groups are called the unitary groups.

1.5.9 Exercise. For each of the sub Lie groups of $\mathrm{GL}_n(\mathbb{R})$ defined above, determine its number of connected components. Indication: try to use normal forms for endomorphisms of $\mathbb{R}$-vector spaces.

1.6 Abstract manifolds

1.6.1 Definition. Let X be a topological space and let $k \geq 0$ be an integer, ∞ or ω. A C^k-atlas for X consists of the following data: a set I, for each i in I an open subset X_i of X, an integer $n_i \geq 0$, an open subset U_i of $\mathbb{R}^{n_i}$ and a homeomorphism $\phi_i \colon U_i \to X_i$. These data are required to satisfy the following conditions. Firstly, the X_i cover X, that is, $\cup_i X_i = X$. Secondly, the charts ϕ_i are compatible, in the sense that we will now explain. For i and j in I let $X_{i,j}$ be $X_i \cap X_j$, and let $U_{i,j}$ be $\phi_i^{-1} X_{i,j}$. Then ϕ_i induces a bijection, still denoted ϕ_i, from $U_{i,j}$ to $X_{i,j}$. Saying that ϕ_i and ϕ_j are compatible means that $U_{i,j}$ is open in U_i, $U_{j,i}$ open in U_j, and the bijection $\phi_j^{-1} \circ \phi_i \colon U_{i,j} \to U_{j,i}$ is C^k. The notation C^ω means: real analytic, i.e., locally given by a convergent power series.

Note that the terminology "differentiable" is misplaced in the case $k = 0$; in that case we speak of a topological atlas. An other remark concerns the integers n_i.

1.6.2 Definition. Let $k \geq 0$ be an integer, ∞, or ω. A variety or manifold of class C^k is a topological space X, separated and with a countable basis for the topology, equipped with a C^k-atlas. Notation: (X, I, n, U, ϕ).

1.6.3 Remark. Both conditions "separated" and "countable basis" are there only for traditional reasons, there is a good theory without them. See for example [Edix].

For X a C^k-variety and x in X, all n_i for i such that X_i contains x are equal; this integer is called the dimension of X at x; we denote it by $\dim_X(x)$, so that we can view $\dim_X$ as a $\mathbb{Z}$-valued function on X. (For $k > 0$ the equality of the n_i is easy to prove (consider derivatives and use linear algebra); for $k = 0$ one needs some algebraic topology.) Most of the time we will just consider the C^∞ case. As usual, defining the objects to study is not too interesting; we should also say what maps between them we want to consider. For example, we want to say what it means that two manifolds are isomorphic.

1.6.4 Definition. Let (X, I, n, U, ϕ) and (Y, J, m, V, ψ) be manifolds. Let f be a continuous map from X to Y. Let x be in X. Then f is called differentiable at x if for every (i, j) such that $x \in X_i$ and $f(x) \in Y_j$ the map $\psi_j^{-1} f \phi_i$ from $\phi_i^{-1}((f^{-1}Y_j) \cap X_i) \subset \mathbb{R}^{n_i}$ to $\mathbb{R}^{m_j}$ is differentiable at $\phi_i^{-1}(x)$. The map f is called differentiable, or a morphism of manifolds, if it is differentiable at all x in X.

Note that this definition does not change if we require differentiability at x only for one pair (i, j). If $f \colon X \to Y$ and $g \colon Y \to Z$ are morphisms, then $g \circ f \colon X \to Z$ is also a morphism. So we have the category of manifolds: we have objects, morphisms, composition of morphisms, the composition is associative and each object X has an identity morphism id_X. A morphism $f\colon X \to Y$ is called an isomorphism if and only if there exists a morphism $g\colon Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. Equivalently: a map $f\colon X \to Y$ is an isomorphism if and only it is bijective and f and f^{-1} are differentiable.

All concepts of differential calculus in $\mathbb{R}^n$ that are invariant under diffeomorphisms can be transported to C^k-manifolds if $k \geq 1$. For example, a morphism $f\colon X \to Y$ is a submersion (resp., immersion) at x in X if for some i in I such that $x \in X_i$ and for some j in J with $f(x) \in Y_j$ the derivative $(D(\psi_j^{-1} f \phi_i))(\phi_i^{-1} x)$ is surjective (resp., injective). A subset Z of a manifold (X, I, n, U, ϕ) is a closed (resp., open) sub manifold if, for each i in I, $\phi_i^{-1} Z$ is a closed (resp., open) sub manifold of U_i. The name "sub manifold" is justified, because the implicit function theorem gives atlases (and one can verify that any two atlases obtained in that way are equivalent in the sense that the identity map induces an isomorphism).

1.6.5 Exercise. For each of the examples in Section 1.2, give a C^ω-atlas such that the quotient map is a C^ω-submersion.

1.7 A definition of manifolds in terms of functions

The definition of manifold in terms of atlases is natural, but somewhat clumsy. For example, different people may use different atlases for what we would like to call the same manifold. A more natural and better definition is in terms of functions. Suppose that we have a C^k-manifold (X, I, n, U, ϕ). Then, for V an open subset of X, we know what it means for a function $f\colon V \to \mathbb{R}$ to be C^k, and we let $C_X^k(V)$ be the $\mathbb{R}$-algebra space of such functions. (Likewise, for each $l \leq k$ we have $C_X^l(V)$.)

1.7.1 Proposition. *Let X and Y be C^k-manifolds, and f a continuous map from X to Y. Then f is C^k if and only if for each open subset U of Y, and for each g in $C_Y^k(U)$, we have $g \circ f \in C_X^k(f^{-1}U)$.*

Proof. Suppose that f is C^k. Then it follows directly from the definitions, and the fact that compositions of differentiable maps are differentiable, for each open U in Y and $g \in C_Y^k(U)$ the function gf is in $C_X^k(f^{-1}U)$.

Suppose now that for each open U in Y and g in $C_Y^k(U)$ the function gf is in $C_X^k(f^{-1}U)$. Then $X_i \cap f^{-1} Y_j$ is open in X_i, hence $\phi_i^{-1}(X_i \cap f^{-1} Y_j)$ is open in U_i, hence in $\mathbb{R}^{n_i}$. We have to show that the map $\psi_j^{-1} f \phi_i$ from $\phi_i^{-1}(X_i \cap f^{-1} Y_j)$ to $V_j \subset \mathbb{R}^{m_j}$ is C^k. It is equivalent to show that the m_j coordinate functions $x_k\psi_j^{-1}f\phi_i$ of this map are C^k. Now $x_k\psi_j^{-1}$ is in $C_Y^k(Y_j)$, hence $x_k\psi_j^{-1}f$ is in $C_X^k(f^{-1}Y_j)$. It follows that $x_k\psi_j^{-1}f\phi_i$ is C^k. □

The idea to give an other definition of manifolds is simply to study what properties the sub $\mathbb{R}$-algebras $C_X^k(U)$ of the $\mathbb{R}$-algebras of all $\mathbb{R}$-valued functions on U should verify. The following properties are clearly satisfied:

1. for $U \subset X$ open, $f\colon U \to \mathbb{R}$, and U_i $(i \in I)$ an open cover of U, f is in $C_X^k(U)$ if and only if for each i its restriction to U_i is in $C_X^k(U_i)$;
2. for each x in X, there is an open subset U of X, and a homeomorphism f from an open subset V of some $\mathbb{R}^n$ to U, such that for g an $\mathbb{R}$-valued function on U one has $g \in C_X^k(U)$ if and only if $g \circ f$ is C^k on V.

1.7.2 Definition. Let k be a in $\mathbb{Z} \cup \{\infty, \omega\}$. A C^k-manifold is a pair (X, C_X^k), with X a topological space, separated and with a countable basis for the topology, and with C_X the datum, for every open subset U of X, of a subset $C_X^k(U)$ of the set of $\mathbb{R}$-valued functions on U, such that the two conditions above are satisfied. A morphism of C^k-manifolds is a continuous map $f\colon X \to Y$ such that for every open $U \subset Y$ and every g in $C_Y^k(U)$ we have $g \circ f \in C_X^k(f^{-1}U)$.

We leave the verification of the equivalence with the first definition to the reader. Let us just say that one can construct an atlas for (X, C_X^k) by using the homeomorphisms of the second condition. In what follows, we will freely use this definition, using only the notion of C^k functions on open subsets of a manifold. For local computations, we will just use that, locally, (X, C_X^k) is isomorphic to some (U, C_U^k) with U open in some $\mathbb{R}^n$. We finish by remarking that this last defition of manifold is the one used for example in [Vara].

1.8 Tangent spaces and the tangent bundle

Let X be a manifold, of class C^k with $k \geq 1$. For x in X we want to define its tangent space $\mathrm{T}_X(x)$. There are several ways to do this (which are of course equivalent). For a detailed discussion of all of those I know, see [Spiv], Volume I, Chapter 3. We will discuss two of them. Intuitively, the tangent space of X at a point x in X is the first order approximation of X at x. We need it in order to speak of the derivatives of morphisms of manifolds. The second of the two constructions that follow is based on the observation that for a and v in $\mathbb{R}^n$, the map:

$$\partial_v\colon C^1(\mathbb{R}^n) \to \mathbb{R}, \quad f \mapsto \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t}$$

determines v. Indeed, we have $v_i = \partial_v x_i$. Hence knowing v or knowing the map ∂_v, which takes the partial derivative in the direction specified by v, amounts to the same. So in order to get an intrinsic definition of tangent vector (i.e., without using coordinates or charts), we are eventually going to define a tangent vector at x to be a map with some properties that ∂_v has.

1.8.1 The first construction: curves

Let X be a C^k-manifold with $k \geq 1$, and let x be in X. A parametrized curve at x is a differentiable map $c\colon U \to X$ with $U \subset \mathbb{R}$ an open interval containing zero and with $c(0) = x$. For c such a curve and $V \subset X$ open and containing x, we have the map:

$$\partial_c\colon C^k_X(V) \to \mathbb{R}, \quad f \mapsto (f \circ c)'0.$$

We define $\mathrm{T}_X(x)$ to be the set of equivalence classes of parametrized curves at x, for the following equivalence relation. Two such curves c_1 and c_2 are equivalent if and only if for each open neighborhood V of x we have $\partial_{c_1} = \partial_{c_2}$.

We claim that $\mathrm{T}_X(x)$ has an $\mathbb{R}$-vector space structure defined by $\lambda\overline{c_1} + \overline{c_2} = \overline{c_3}$ if and only if $\lambda\partial_{c_1} + \partial_{c_2} = \partial_{c_3}$ for every open neighborhood V of x, and that $\dim \mathrm{T}_X(x) = \dim_X(x)$. To see that, we may suppose that X is itself an open subset of $\mathbb{R}^n$, in which case the details are left to the reader (use the observation of the beginning of this section).

1.8.2 The second construction: derivations

For x and v in $\mathbb{R}^n$, the partial derivative $\partial_v\colon C^1(\mathbb{R}^n) \to \mathbb{R}$ satisfies the following properties:

1. it is $\mathbb{R}$-linear;
2. $\partial_v(fg) = f(x)\partial_v(g) + g(x)\partial_v(f)$ (Leibniz's or product rule);
3. $\partial_v(gh) = 0$ if $g(x) = 0 = h(x)$, $g \in C^1(\mathbb{R}^n)$ and $h \in C^0(\mathbb{R}^n)$.

Let again X be a C^k-manifold eith $k \geq 1$. Let x be in X. We let $C^k_{X,x}$ be the set of *germs* of C^k-functions at x: $C^k_{X,x}$ is the set of equivalence classes of pairs (U, f), with U an open neighborhood of x and f in $C^k_X(U)$, for the following equivalence relation. Two such pairs (U_1, f_1) and (U_2, f_2) are equivalent if and only if there is an open neighborhood V of x contained in $U_1 \cap U_2$, such that the functions $f_1|_V$ and $f_2|_V$ are equal. The $\mathbb{R}$-algebra structures on the $C^k_X(U)$ induce one on $C^k_{X,x}$. For U an open neighborhood of x and f in $C^k_X(U)$, the class of (U, f) is called the germ of f at x and is denoted f_x.

We let $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x}, \mathbb{R})$ be the set of maps $\partial\colon C^k_{X,x} \to \mathbb{R}$ that satisfy:

1. ∂ is $\mathbb{R}$-linear;

2. $\partial(fg) = f(x)\partial(g) + g(x)\partial(f)$;

3. $\partial(f) = 0$ if $f = gh$ with $g(x) = 0 = h(x)$, $g \in C^1(\mathbb{R}^n)$ and $h \in C^0(\mathbb{R}^n)$.

Such ∂ that satisfy the first two conditions are called ($\mathbb{R}$-linear) derivations. To emphasize the third condition, we use the notation Der$'$ in stead of Der. Since $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x}, \mathbb{R})$ is closed under $\mathbb{R}$-linear combinations, it is an $\mathbb{R}$-vector space.

By definition, we have an injective $\mathbb{R}$-linear map from $\mathrm{T}_X(x)$ to $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x}, \mathbb{R})$, sending $\overline{c}$ to ∂_c. We claim that this map is an isomorphism. To prove that, it suffices to show that every element of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,x}, \mathbb{R})$ is of the form ∂_c. Since everything is defined intrinsically, and the question is local, we may suppose that X is an open subset of $\mathbb{R}^n$ and that $x = 0$. Let ∂ be in $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,0}, \mathbb{R})$. We claim that $\partial = \sum_{i=1}^n \partial(x_i)\partial_i$, with ∂_i be the element of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,0}, \mathbb{R})$ that sends f to its ith partial derivative at 0. The elements ∂_i are linearly independent because of the relations $\partial_i x_j = \delta_{i,j}$ (where x_j is the jth coordinate function, and $\delta_{i,j}$ the Kronecker symbol). To prove the identity, let f be in $C^k_{X,0}$. Rewriting the equality:

$$\int_0^1 \left(\frac{d}{dt} f(tx)\right) dt = f(x) - f(0),$$

with x in some neighborhood of 0, gives:

$$f(x) = f(0) + \sum_{i=1}^n x_i \int_0^1 (D_i f)(tx)dt = f(0) + \sum_{i=1}^n \partial_i(f)x_i + \sum_{i=1}^n x_i g_i(x),$$

with D_i the ith partial derivative. Since the $D_i f$ are C^{k-1}-functions, the g_i are C^{k-1}, and $\partial(x_i g_i) = 0$ for each i. Applying ∂ to the last identities gives the desired result:

$$\partial = \sum_{i=1}^n \partial(x_i)\partial_i.$$

Now we know that the ∂_i form a basis of $\mathrm{Der}'_{\mathbb{R}}(C^k_{X,0}, \mathbb{R})$. It follows that our map from $\mathrm{T}_X(x)$ to $\mathrm{Der}'_{\mathbb{R}}(C_{X,x}, \mathbb{R})$ is an isomorphism.

1.8.3 Derivatives

Let $f\colon X \to Y$ be a morphism of C^k-manifolds with $k \geq 1$. Let x be in X. Then we have a map:

$$\mathrm{T}_f(x)\colon \mathrm{T}_X(x) \to \mathrm{T}_Y(f(x)), \quad \partial \mapsto \partial \circ f^*, \quad (\mathrm{T}_f(x)\partial)g = \partial(g \circ f),$$

for all g in $C^k_{Y,f(x)}$. Indeed, one verifies immediately from the definitions that $\partial \circ f^*$ is an element of $\mathrm{Der}'_{\mathbb{R}}(C^k_{Y,f(x)}, \mathbb{R})$.

1.8.4 Exercise. For f as above, with X and Y open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, verify that $\mathrm{T}_f(x)$ is given, with respect to the bases ∂_i of $\mathrm{T}_X(x)$ and $\mathrm{T}_Y(f(x))$, by the usual matrix whose (i,j)th coefficient is $(\partial f_i/\partial x_j)x$, with f_i the ith coordinate of f.

1.8.5 Exercise. For X a closed sub manifold of $\mathbb{R}^n$ and x in X, the tangent space $\mathrm{T}_X(x)$ is a subspace of $\mathbb{R}^n$. More generally, if $f\colon X \to Y$ is an immersion at x in X, then $\mathrm{T}_Y(f(x))$ is the image $\mathrm{im}(\mathrm{T}_f(x))$ of $\mathrm{T}_X(x)$ under $\mathrm{T}_f(x)$. Also, if $f\colon X \to Y$ is a submersion at x in X, then $Z := f^{-1}\{f(x)\}$ is, locally at x, a submanifold of X, and one has $\mathrm{T}_Z(x) = \ker(\mathrm{T}_f(x))$.

1.8.6 Exercise. Compute (i.e., give equations for) the tangent spaces at the origin of the subgroups of $\mathrm{GL}_n(\mathbb{R})$ given in Section 1.5. These tangent spaces are subspaces of $\mathrm{M}_n(\mathbb{R})$.

1.8.7 The tangent bundle

Let X be a C^k-manifold with $k \geq 1$. We want to make an object T_X, called the tangent bundle of X, that combines all the $\mathrm{T}_X(x)$. As a set, T_X is just the disjoint union of all the $\mathrm{T}_X(x)$, $x \in X$. But in order to have something useful, for example second derivatives of morphisms of manifolds, it can be useful to equip T_X with the structure of a manifold (it will be a C^{k-1}-manifold, in fact), reflecting the fact that it is a disjoint union of vector spaces. The notion of vector bundle is made exactly for doing this, but at this moment we will not go into this abstraction, and we will just describe T_X itself, and not so much the structures that it has.

We give T_X the structure of a C^{k-1}-manifold. Note that we have a canonical map $p\colon \mathrm{T}_X \to X$, such that the fibre $p^{-1}\{x\}$ over x is $\mathrm{T}_X(x)$. We demand that this map be continuous. Since X itself is locally isomorphic to $\mathbb{R}^n$ (n may vary), we may suppose that X is an open subset of $\mathbb{R}^n$ in order to specify the topology and the structure of C^{k-1}-manifold, as long as the map p will be continuous and the structure in question independent of the isomorphism chosen. So let U be an open subset of X, and $\phi\colon U \to V$ an isomorphism, with $V \subset \mathbb{R}^n$ open. Then we have a bijection:

$$\coprod_{u\in U} \mathrm{T}_\phi(u)\colon \coprod_{u\in U} \mathrm{T}_X(u) \longrightarrow \coprod_{v\in V} \mathrm{T}_{\mathbb{R}^n}(v) = \mathbb{R}^n \times V.$$

We simply demand that these bijections are isomorphisms of C^{k-1}-manifolds. This works, since if $\psi\colon U \to W$ is also an isomorphism, with $W \subset \mathbb{R}^n$ open, then the bijection from T_V to T_W induced by $\psi \circ \phi^{-1}$ is given by:

$$(t, v) \mapsto ((D(\psi \circ \phi^{-1})v)t, \psi(\phi^{-1}v)),$$

which is indeed C^{k-1}.

We note that in the last formula the dependence on t is linear, which means that T_X can be seen as a C^{k-1}-family of $\mathbb{R}$-vector spaces, indexed by X. For f a morphism of C^k-manifolds, T_f is a morphism of C^{k-1}-manifolds, respecting the $\mathbb{R}$-vector space structures in the fibers of p.

1.9 Vector fields and derivations

1.9.1 Definition. Let X be a C^k-manifold with $k \geq 1$. A vector field v on X is a section of $p\colon \mathrm{T}_X \to X$, i.e., a map $v\colon X \to \mathrm{T}_X$ such that $p \circ v = \mathrm{id}_X$. Hence a vector field is just the data of a tangent vector at every x in X. A vector field v is called C^l $(l \leq k-1)$ if the map v is C^l.

1.9.2 Remark. On $\mathbb{R}^n$, a vector field v is a map:

$$v\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n, \quad x \mapsto (v_1(x), x).$$

The vector field v is C^l if and only if the n coordinates of v_1 are. In a lot of text books that only deal with manifolds embdedded in some $\mathbb{R}^n$, the second component is never mentioned (because it is given by the identity function anyway), and vector fields are identified with their first component.

1.9.3 Construction. Let v be a vector field on a C^k-manifold X, with $k \in \{\infty, \omega\}$. Then, viewing $\mathrm{T}_X(x)$ as $\mathrm{Der}_{\mathbb{R}}(C^k_{X,x}, \mathbb{R})$, it makes sense to write, for f in $C^k_X(X)$, $(v(x))f$, which is the derivative of f at x in the direction specified by $v(x)$. The function $D_v(f)\colon x \mapsto (v(x))f$ is again in C^k (verify it locally if you want), and, by construction, $D_v\colon C^k_X(X) \to C^k_X(X)$ is a derivation, which means that for all f and g in $C^k_X(X)$ we have:

$$D_v(fg) = D_v(f)g + fD_v(g).$$

1.9.4 Theorem. *Let X be a C^∞-manifold. The map $v \mapsto D_v$ is an isomorphism of $C^\infty_X(X)$-modules from the $C^\infty_X(X)$-module of C^∞ vector fields on X to the $C^\infty_X(X)$-module $\mathrm{Der}_{\mathbb{R}}(C^\infty_X(X))$ of $\mathbb{R}$-linear derivations from $C^\infty_X(X)$ to itself.*

The proof of this theorem is not so hard; contrary to what some people say, one does not need to suppose that X has a countable basis for its topology. I do not know if the condition that X be separated can be dropped. Because of this theorem, we will not distinguish anymore between vector fields and derivations, on C^∞-manifolds; in fact, we will always view them as derivations.

For X open in $\mathbb{R}^n$, $C_X^\infty(X)$-module $\mathrm{Der}_\mathbb{R}(C_X^\infty(X))$ is free with basis the $\partial/\partial x_i$. Equivalently, every vector field D can be written uniquely as $\sum_i a_i\partial/\partial x_i$, with the a_i in $C_X^\infty(X)$.

1.9.5 Construction. The Lie bracket of two vector fields. Let X be a C^∞-manifold, and D_1 and D_2 two elements of $\mathrm{Der}_\mathbb{R}(C_X^\infty(X))$. Then their commutator:

$$[D_1, D_2] = D_1D_2 - D_2D_1$$

in $\mathrm{End}_\mathbb{R}(C_X^\infty(X))$ is again a derivation (exercise). In general, compositions of derivations are no longer derivations; they are higher order differential operators. It is no suprise that the commutator of two derivations is again a derivation; in general, if D_1 and D_2 are differential operators of orders $\leq d_1$ and $\leq d_2$, respectively, then D_1D_2 and D_2D_1 are both of order $\leq d_1 + d_2$ (just count the number of derivatives), but their difference is of order $\leq d_1 + d_2 - 1$ (the non commutation involves lower order operators).

1.9.6 Exercise. For X open in $\mathbb{R}^n$, compute $[\sum_i f_i\partial/\partial x_i, \sum_j g_j\partial/\partial x_j]$, where the f_i and g_j are C^∞-functions on X.

1.10 Flows

Let X be a C^k-manifold, with $k \in \{\infty, \omega\}$. Let D be a C^k vector field on X. For x in X, we denote by D_x the tangent vector at x given by D. We wonder if there are C^k maps $f\colon U \to X$ with $0 \in U \subset \mathbb{R}$ open, such that for all u in U, we have $(\mathrm{T}_f(u))(d/dt)_u = D_{f(u)}$. (Note that $(d/dt)_u$ is just a complicated notation for the tangent vector $1 \in \mathbb{R} = \mathrm{T}_U(u)$.) Such f are called integral curves of D. In order to find out if they exist, we may suppose that X is an open subset of $\mathbb{R}^n$. Then $D = \sum_i a_i\partial/\partial x_i$ with the a_i in $C_X^k(X)$, and $f = (f_1, \ldots, f_n)$. Applying the definitions gives that the equation we want to solve is:

$$a_i(u) = D_{f(u)}(x_i) = ((\mathrm{T}_f(u))(d/dt)_u)x_i = ((d/dt)_u)(x_i \circ f) = ((d/dt)_u)(f_i) = (df_i/dt)(u).$$

So the equation is:

$$df_i/dt = a_i \circ f, \quad 1 \leq i \leq n,$$

or, equivalently:

$$(df/dt)(u) = a(f(u))$$

which is an ordinary first order differential equation, of which we know (Cauchy's theorem) that for every x in X there is a unique solution f with $f(0) = x$, on a sufficiently small interval $U \ni 0$. We denote this solution by $u \mapsto \phi(u, x)$ or by ϕ_u. The *flow* of D is the maximal solution $\phi\colon W \to X$, with W a neighborhood of $\{0\} \times X$ in $\mathbb{R} \times X$.

By construction, we have $\phi(u_1 + u_2, x) = \phi(u_1, \phi(u_2, x))$, i.e., $\phi_{u_1+u_2} = \phi_{u_1} \circ \phi_{u_2}$, wherever these expressions make sense. This means that $u \mapsto \phi_u$ is a local morphism of groups from $\mathbb{R}$ to $\mathrm{Aut}(X)$.

If X is compact, then the flow exists on $W = \mathbb{R} \times X$, simply because W contains at least a subset of the form $]-\varepsilon, \varepsilon[\times X$ (then use $\phi_{u_1+u_2} = \phi_{u_1} \circ \phi_{u_2}$ to extend).

1.11 A theorem of Frobenius

Later on we will have to use a theorem of Frobenius, which generalizes the integral curves of a vector field to "systems of tangent spaces". A system of tangent spaces V on a C^k-manifold X (with $k \geq 1$) is the data, for every x in X, of a subspace V_x of $\mathrm{T}_X(x)$. If all the V_x are of dimension p, we say that V is a rank p system of tangent spaces. Now suppose that $k = \infty$. We say that V is C^∞ if there is a sub $C_X^\infty(X)$-module $\mathcal{D}$ of $\mathrm{Der}_{\mathbb{R}}(C_X^\infty(X))$ such that for all x in X we have $\mathcal{D}_x = V_x$, where:

$$\mathcal{D}_x = \{D_x \mid D \in \mathcal{D}\} \subset \mathrm{T}_X(x).$$

A C^∞ system of tangent spaces V is called *integrable* if for every x in X there exists a diffeomorphism $\phi\colon U \to \mathbb{R}^n$ such that $\mathrm{T}_\phi V_u = \{(v, \phi(u)) \mid v \in \mathbb{R}^p\}$ for some p. If so, there exists, for every x in X a local sub manifold Y_x of X, such that $V_u = \mathrm{T}_{Y_x}(u)$ for all u in Y_x. A *leaf* of V is a maximal injective immersion $i\colon Y \to X$, with Y connected, such that $\mathrm{T}_Y(y) = V_y$ for all y in Y. Intuitively, such a Y can be obtained by glueing the local Y_x's.

1.11.1 Theorem. (Frobenius) *Let V be a C^∞ system of tangent vectors of X, given by $\mathcal{D} \subset \mathrm{Der}_{\mathbb{R}}(C_X^\infty(X))$. Then V is integrable if and only if $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$.*

1.12 Complex manifolds

Since we also want to be able to study complex Lie groups, such as $\mathrm{GL}_n(\mathbb{C})$ and complex tori (quotients by complex vector spaces by lattices), taking into account the complex structures, we also need the notion of complex manifold. One particularity with complex functions is that a function $f\colon U \to \mathbb{C}$ with U open in $\mathbb{C}^n$ is analytic as soon as it is C^1 in the complex sense, i.e., if it is C^1 (in the real sense) and if for every u in U the derivative $(Df)u\colon \mathbb{C}^n \to \mathbb{C}$ (which is by definition $\mathbb{R}$-linear) is $\mathbb{C}$-linear. For $n = 1$ this is a direct consequence of Cauchy's integral formula.

1.12.1 Proposition. (Cauchy's integral formula) *Let $\Delta \subset \mathbb{C}$ be an open disk, with closure $\overline{\Delta}$ and boundary $\partial\Delta$. Let $f\colon \overline{\Delta} \to \mathbb{C}$ be C^1 in the complex sense (i.e., there is an open neighborhood U of $\overline{\Delta}$ and an extension of f to U that is complex differentiable with continuous derivative). Then for all z be in Δ one has:*

$$f(z) = (2\pi i)^{-1} \int_{\partial\Delta} \frac{f(w)dw}{w-z},$$

where the integral is taken in the counterclockwise direction.

The proof of this proposition uses the theorem of Stokes, or a simple version of it based on the fact that we are just dealing with a disk here. We won't give it here (see analysis textbooks).

1.12.2 Proposition. *Let $n \geq 0$, $U \subset \mathbb{C}^n$ open and $f\colon U \to \mathbb{C}$. Then f is analytic if and only if f is C^1 in the complex sense.*

Proof. We just sketch the proof. Of course, if f is analytic, it is C^1 in the complex sense. So suppose now that f is C^1 in the complex sense. Let us first suppose that $n = 1$. Let z_0 be in U. We will show that f is analytic at z_0. After a translation in $\mathbb{C}$ we may assume that $z_0 = 0$. Let Δ be an open disk with center 0, contained in U. Then for all z in Δ we have:

$$f(z) = (2\pi i)^{-1} \int_{\partial\Delta} \frac{f(w)dw}{w-z} = (2\pi i)^{-1} \int_{\partial\Delta} \left(1 + (z/w) + (z/w)^2 + \cdots\right) f(w)\frac{dw}{w}.$$

From this expression it is easily seen that one has:

$$f(z) = \sum_{n\geq 0} a_n z^n, \qquad \text{with} \quad a_n = (2\pi i)^{-1} \int_{\partial\Delta} w^{-n} f(w)\frac{dw}{w},$$

for all z in Δ.

Let us now do the general case: $n \geq 1$. Let z_0 be in U. We want to show that f is analytic at z_0. After a translation we may assume that $z_0 = 0$. Let $r > 0$ be in $\mathbb{R}$ such that the polydisk (i.e., product of disks) $\Delta := \{z \in \mathbb{C}^n \mid |z_i| < r \text{ for } 1 \leq i \leq n\}$ is contained in U. A repeated application of Cauchy's integral formula gives,for all z in Δ:

$$f(z) = (2\pi i)^{-n} \int_{\partial\Delta} \prod_{i=1}^{n} (w_i - z_i)^{-1} \cdot f(w) dw_1 \cdots dw_n.$$

The power series expansion in the z_i/w_i of the product of the $(w_i - z_i)^{-1}$ gives the result. $\square$

This last proposition is the reason that we do not define what complex differentiable manifolds are: they are just the complex analytic manifolds.

1.12.3 Definition. A complex analytic manifold is a pair (X, C_X), with X a topological space and C_X the datum, for every open subset U of X, of a subset $C_X(U)$ of the set of $\mathbb{C}$-valued functions on U, such that:

1. for $U \subset X$ open, $f \colon U \to \mathbb{R}$, and U_i $(i \in I)$ an open cover of U, f is in $C_X(U)$ if and only if for each i its restriction to U_i is in $C_X(U_i)$;

2. for each x in X, there is an open subset U of X, and a homeomorphism f from an open subset V of some $\mathbb{C}^n$ to U, such that for g an $\mathbb{C}$-valued function on U one has $g \in C_X(U)$ if and only if $g \circ f$ is analytic on V.

Morphisms of complex manifolds are continuous maps that transform analytic functions into analytic functions.

2 Lie groups; first definitions

2.1 Definition. A (real) Lie group is a group G equipped with the structure of a C^∞ manifold, such that the maps:

$$G \times G \longrightarrow G, \quad (x,y) \mapsto xy \qquad \text{and} \quad G \longrightarrow G, \quad x \mapsto x^{-1}$$

are morphisms.

A complex Lie group is a group G equipped with the structure of complex manifold, such that the two maps above are morphisms.

A morphism of Lie groups is a morphism of groups that respects the manifold structures. The unit element of a group will usually be denoted by e.

2.2 Exercise. Show that in the above definition one can replace the two maps by the single map: $(x,y) \mapsto xy^{-1}$.

2.3 Examples. The Lie subgroups of $\mathrm{GL}_n(\mathbb{R})$ are Lie groups. Real tori (quotients of finite dimensional $\mathbb{R}$-vector spaces by lattices) are Lie groups. (In fact, we should specify the manifold structure, but we don't; only a fool could think of some manifold structure that is not the one that we want.) The group $\mathrm{GL}_n(\mathbb{C})$ is a complex Lie group, as well as $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{O}_n(\mathbb{C}) = \{x \in \mathrm{GL}_n(\mathbb{C}) \mid x^{\mathrm{t}}x = 1\}$. But U_n is not a complex Lie group (the defining equations are not analytic since they involve complex conjugation; even more convincing: the dimension of U_n need not be even). Complex tori (quotients of finite dimensional $\mathbb{C}$-vector spaces by lattices) are compact complex Lie groups, and we will see that they are the only compact complex Lie groups.

An important tool in the study of Lie groups is the tangent bundle. Our first concern will be to study the implications of having a group structure.

2.4 Various actions of a Lie group on itself

Let G be a Lie group (real or complex). As with any group, we have the following actions of G on itself:

1. the action by left translations: for x in G we have $l_x\colon G \to G$, $y \mapsto xy$;
2. the right action by right translations: for x in G we have $r_x\colon G \to G$, $y \mapsto yx$ (right action means: $r_{x_1x_2}(y) = r_{x_2}(r_{x_1}(y))$);
3. the action by conjugation: for x in G we have $\mathrm{inn}_x\colon G \to G$, $y \mapsto xyx^{-1}$.

All these actions are by automorphisms of G as a manifold (i.e., all l_x, r_x and inn_x respect the manifold structure). The actions by left and right translations are free and transitive: $xy = y$ implies $x = e$, $yx = y$ implies $x = e$, and for y in G we have $y = ye = ey$. The action by conjugation is not free and not transitive as soon as $G \neq \{e\}$ (exercise). However, the action by conjugation respects also the group structure of G: each inn_x is an automorphism of G as a Lie group. Moreover, $\mathrm{inn}\colon G \mapsto \mathrm{Aut}_{\mathrm{Lie}}(G)$, $x \mapsto \mathrm{inn}_x$, is a morphism of groups (and even of Lie groups, if we put a reasonable manifold structure on $\mathrm{Aut}_{\mathrm{Lie}}(G)$).

We let L denote the vector space $\mathrm{T}_G(e)$ (real or complex). For every x in G we have the two isomorphisms (of $\mathbb{R}$ or $\mathbb{C}$-vector spaces) T_{l_x} and T_{r_x} from L to $\mathrm{T}_G(x)$. These two isomorphisms need not be the same. In fact, since $c_x = r_x^{-1} \circ l_x$, we have $\mathrm{T}_{r_x}(e)^{-1}\mathrm{T}_{l_x}(e) = \mathrm{T}_{c_x}(e)$. This fact will give us the structure of Lie algebra on L, as we will see a bit further. By construction, we have $\mathrm{T}_{c_x}(e)\mathrm{T}_{c_y}(e) = \mathrm{T}_{c_{xy}}(e)$, which means that we have a morphism of Lie groups:

$$G \mapsto \mathrm{GL}(L), \quad x \mapsto \mathrm{T}_{c_x}(e).$$

This morphism is called the adjoint representation of G, which is just a complicated way of saying "the action induced by conjugation on its tangent space at e".

2.5 Trivialization of the tangent bundle

We want to identify all $\mathrm{T}_G(x)$ with $\mathrm{T}_G(e)$, and to do that, we choose to use the left translations. (We could also have used right translations. Anyway, the map $x \mapsto x^{-1}$ from G to G transforms left translations into right translations, so every question about right translations can be transformed into one about left translations. This means that our choice is not so important.) The map:

$$L \times G \longrightarrow \mathrm{T}_G, \quad (v, g) \mapsto \mathrm{T}_{l_g}(v) \in \mathrm{T}_G(g)$$

is clearly bijective: its inverse is given by:

$$\mathrm{T}_G \longrightarrow L \times G, \quad t \mapsto (\mathrm{T}_{l_{p(t)}}(e)^{-1}(t), p(t)),$$

with $p\colon \mathrm{T}_G \to G$ the projection. Since both maps are given in terms of formulas that are composed of morphisms of manifolds, they are morphisms of manifolds. A more rigorous argument for this is the following: the second map is a morphism, and moreover its derivative is everywhere bijective since p is a submersion and $\mathrm{T}_{l_{p(t)}}^{-1}$ is an isomorphism from $\mathrm{T}_G(p(t))$ to L.

2.6 Left, right and bi-invariant vector fields

Let us consider the vector space of vector fields (C^∞ or analytic) on G. The group G acts on it via l: for g in G and D a vector field, viewed as a derivation, we have, for every x in G:

$$(g{\cdot}D)_{gx} = (\mathrm{T}_{l_g}(x))(D_x), \quad (D{\cdot}g)_{xg} = (\mathrm{T}_{r_g}(x))(D_x).$$

A vector field D is called left (resp., right or bi) invariant if for all g in G one has $g{\cdot}D = D$ (resp., $D{\cdot}g = D$ or $g{\cdot}D = D = D{\cdot}g$). Clearly, a left (or right) invariant vector field D is determined by its value D_e at e. Moreover, if v is in $L = \mathrm{T}_G(e)$, then there is a unique left (or right) invariant vector field D with $D_e = v$ (use the trivialization of T_G in the previous section). Hence $D \mapsto D_e$ is a bijection from the space of left (or right) invariant vector fields on G to L.

The situation for bi-invariant vector fields is as follows. A vector field D is bi-invariant if and only if for all g in G one has $D_g = (\mathrm{T}_{l_g}(e))D_e$ and $D_g = (\mathrm{T}_{r_g}(e))D_e$, which means: if and only if $D_g = (\mathrm{T}_{l_g}(e))D_e$ and $(\mathrm{T}_{r_g}(e)^{-1} \circ \mathrm{T}_{l_g}(e))D_e = D_e$ for all g in G. Since $\mathrm{T}_{r_g}(e)^{-1} \circ \mathrm{T}_{l_g}(e)) = \mathrm{T}_{c_g}(e)$, D is bi-invariant if and only if it is left invariant and D_e invariant under the adjoint representation of G.

Let us now compute what $g{\cdot}D$ is if we view D as a derivation. So let f be a (C^∞ or analytic) function on an open part U of G, and let u be in U. Then we compute:

$$\begin{aligned}((gD)f)u &= (gD)_u f = ((\mathrm{T}_{l_g}(g^{-1}u))D_{g^{-1}u})f = D_{g^{-1}u}(f \circ l_g) = \\ &= D_{g^{-1}u}(l_g^*(f)) = D(l_g^*(f))(g^{-1}u) = D(l_g^*(f))(l_{g^{-1}}(u)) = \\ &= (l_{g^{-1}}^*(D(l_g^*(f))))u = ((l_{g^{-1}}^* \circ D \circ l_g^*)f)u.\end{aligned}$$

This horrible but instructive computation tells us that:

$$gD = l_{g^{-1}}^* \circ D \circ l_g^*.$$

Of course, there is an easier way to get this formula. Suppose that we have an isomorphism $\alpha\colon X \to Y$ of say C^∞-manifolds, and a vector field D on X. Then we have the following diagram (the only one that makes sense):

$$\begin{array}{ccc} C^\infty(X) & \xleftarrow{\alpha^*} & C^\infty(Y) \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \alpha_* D} \\ C^\infty(X) & \xrightarrow{(\alpha^{-1})^*} & C^\infty(Y). \end{array}$$

Even more generally, for $f\colon X \to Y$ just a morphism of C^∞-manifolds, and for D in $\mathrm{Der}_\mathbb{R}(C^\infty(X))$, we have $f_*D := D \circ f^* = (f^*)^*(D)$ in $\mathrm{Der}_\mathbb{R}(C^\infty(Y), C^\infty(X))$. But we won't use this last one; it leads to the general concept of derivations $\mathrm{Der}_A(B, M)$ with A a ring, B an A-algebra and M a B-module.

We go back to our Lie group G. Suppose that D_1 and D_2 are left-invariant vector fields on G. Then, the formula $gD = l_{g^{-1}}^* \circ D \circ l_g^*$ implies that $D_1 D_2$ and $D_2 D_1$ are both left-invariant (differential operators), and therefore that $[D_1, D_2]$ is left-invariant.

2.7 The Lie algebra of a Lie group

Let G be a Lie group. Let L denote $\mathrm{T}_G(e)$. For v in L, let D_v denote the unique left-invariant vector field on G such that $(D_v)_e = v$. Then we have a map, called the Lie bracket:

$$[\cdot,\cdot]\colon L \times L \longrightarrow L, \quad (v, w) \mapsto [D_v, D_w]_e.$$

In particular, we have: $[D_v, D_w] = D_{[v,w]}$. The *Lie algebra of* G is then $L = \mathrm{T}_G(e)$, equipped with this map $[\cdot,\cdot]$.

In words, all that we have done to construct the Lie bracket on $\mathrm{Lie}(G)$ is to use the isomorphism (of $\mathbb{R}$-vector spaces) with the space of left-invariant vector fields, to transport the commutator on vector fields to $\mathrm{T}_G(e)$. Therefore, it is very simple to verify that:

1. $[\cdot,\cdot]$ is $\mathbb{R}$-bilinear (and also $\mathbb{C}$-bilinear if G is complex);

2. $[\cdot,\cdot]$ is alternating: $[x, x] = 0$ for all x in $\mathrm{Lie}(G)$;

3. $[\cdot,\cdot]$ satisfies the Jacobi identity: $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$ for all x, y and z in $\mathrm{Lie}(G)$.

2.7.1 Definition. Let k be a field. A k-Lie algebra is a k-vector space L equipped with a map $[\cdot,\cdot]$ from $L \times L$ to L that satisfies the three properties above (with $\mathbb{R}$ replaced by k, of course). If L_1 and L_2 are k-Lie algebras, then a morphism from L_1 to L_2 is a k-linear map $f\colon L_1 \to L_2$ such that for all x and y in L_1 one has: $[f(x), f(y)] = f([x, y])$.

This means that $\mathrm{Lie}(G)$ is a Lie algebra (complex if G is). We will show in a moment that for a morphism $f\colon G_1 \to G_2$ the tangent map $\mathrm{T}_f(e)$ from $\mathrm{Lie}(G_1)$ to $\mathrm{Lie}(G_2)$ is a morphism of Lie algebras, i.e., that Lie is a functor from Lie groups to Lie algebras. But first, let us make a remark that gives the Jacobi identity a more conceptual meaning the just some identity.

2.7.2 Remark. Let k be a field and let L be a k-vector space equipped with a composition law $[\cdot,\cdot]$ that satisfies properties 1 and 2 in the definition of a Lie algebra. For x in L, the map $\mathrm{ad}(x)\colon L \to L$, $y \mapsto [x,y]$, is k-linear. Hence we have a map $\mathrm{ad}\colon L \to \mathrm{End}_k(L)$, $x \mapsto \mathrm{ad}(x)$, from L to the k-vector space of endomorphisms of the k-vector space L. Now this is interesting, since $\mathrm{End}_k(L)$ is also equipped with a Lie bracket, namely, the usual commutator: $[u,v] = uv - vu$. Then one verifies just by writing things out that $[\cdot,\cdot]$ satisfies the Jacobi identity if and only if ad is compatible with the Lie brackets on both sides, i.e., if and only if for all x and y in L we have:

$$\mathrm{ad}([x,y]) = [\mathrm{ad}(x), \mathrm{ad}(y)].$$

In particular, for L a Lie algebra, the map ad is a morphism of Lie algebras.

2.7.3 Proposition. *Lie is a functor from Lie groups to Lie algebras, i.e., for $f : G_1 \to G_2$ a morphism of Lie groups, $\mathrm{Lie}(f) := \mathrm{T}_f(e)\colon \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$ is a morphism of Lie algebras.*

Proof. The proposition we want to prove is a simple consequence of the following lemma (just compose two such diagrams).

2.7.4 Lemma. *Let v be in $\mathrm{Lie}(G_1)$, and let D_v be the left-invariant vector field on G_1 given by v. Put $v' := \mathrm{T}_f(e)v$ in $\mathrm{Lie}(G_2)$, and let $D_{v'}$ be the left-invariant vector field on G_2 given by v'. Then v' is the unique element of $\mathrm{Lie}(G_2)$ for which the following diagram commutes:*

$$\begin{array}{ccc} C^\infty(G_1) & \xleftarrow{f^*} & C^\infty(G_2) \\ \downarrow{\scriptstyle D_v} & & \downarrow{\scriptstyle D_{v'}} \\ C^\infty(G_1) & \xleftarrow{f^*} & C^\infty(G_2). \end{array}$$

Proof. We start by recalling a simple formula: for $f\colon X \to Y$ a morphism of varieties, x in X, v in $\mathrm{T}_X(x)$ and F in $C^\infty(Y)$, we have: $((\mathrm{T}_f(x))v)F = v(f^*(F))$.

Now let F be in $C^\infty(G_2)$ and let g be in G_1. Then we simply compute both sides of our diagram, applied to F and evaluated at g. We have:

$$\begin{aligned} ((f^* \circ D_{v'})F)g = (D_{v'}F)(f(g)) = (D_{v'})_{f(g)}F = ((\mathrm{T}_{l_{f(g)}}(e))v')F = v'(l_{f(g)}^*F) = \\ = ((\mathrm{T}_f(e))v)(l_{f(g)}^*F) = v(f^*(l_{f(g)}^*F)). \end{aligned}$$

On the other hand, we have:

$$((D_v \circ f^*)F)g = (D_v(f^*F))g = (D_v)_g(f^*F) = ((\mathrm{T}_{l_g}(e))v)(f^*F) = v(l_g^*(f^*F)).$$

So in order to show that the diagram commutes, it suffices to show that for all g in G_1 we have $f \circ l_g = l_{f(g)} \circ f$, as morphisms of varieties from G_1 to G_2. But, evaluating at x in G_1, the identity becomes: $f(gx) = f(g)f(x)$, which is true since f is a morphism of groups.

The uniqueness in the lemma follows easily from composing the diagram with the evaluation at e map: $C^\infty(G_1) \to \mathbb{R}$, $h \mapsto h(e)$. Indeed, one side of the composition is $v \circ f^* = \mathrm{T}_f(e)v$, and the other side is v'. $\square$

$\square$

2.8 The Lie algebra of $\mathrm{GL}_n(\mathbb{R})$

Let $G := \mathrm{GL}_n(\mathbb{R})$ (some $n \geq 0$). Since G is an open subset of $\mathrm{M}_n(\mathbb{R})$, we identify the $\mathrm{T}_G(x)$ with $\mathrm{M}_n(\mathbb{R})$, for all x in G. Let a be in $L = \mathrm{T}_G(e) = \mathrm{M}_n(\mathbb{R})$. We wish to describe explicitly the left-invariant vector field D_a on G such that $(D_a)_e = a$. For g in G we have $(D_a)_g = ga$. Now we compute $D_a x_{i,j}$, where the $x_{i,j}$ are the coordinate functions on $\mathrm{M}_n(\mathbb{R})$. For g in G, $(D_a x_{i,j})(g)$ is by definition the derivative of $x_{i,j}$ at g in the direction given by $D_a(g)$, i.e., in the direction ga. So we compute:

$$x_{i,j}(g + \varepsilon ga) = (g + \varepsilon ga)_{i,j} = g_{i,j} + \varepsilon (ga)_{i,j} = x_{i,j}(g) + \varepsilon \sum_k g_{i,k} a_{k,j}.$$

It follows that $D_a(x_{i,j}) = \sum_k x_{i,k} a_{k,j}$. Applying this formula twice gives:

$$(D_a D_b) x_{i,j} = D_a(\sum_k x_{i,k} b_{k,j}) = \sum_{k,k'} x_{i,k'} a_{k',k} b_{k,j} = \sum_{k'} x_{i,k'} (ab)_{k',j} = D_{ab} x_{i,j}.$$

From this we get:

$$[D_a, D_b] x_{i,j} = D_{[a,b]} x_{i,j}.$$

Since the functions $x_{i,j}$ are coordinates on the manifold $\mathrm{GL}_n(\mathbb{R})$, two derivations D and D' such that $D(x_{i,j} = D'(x_{i,j})$ for all i and j are necessarily equal. Hence:

$$[D_a, D_b] = D_{[a,b]},$$

So the Lie bracket for $\mathrm{GL}_n(\mathbb{R})$ is just the ordinary commutator of matrices. The reader should check that if we had used right-invariant vector fields to define the Lie bracket, we would have found the opposite result (use that $x \mapsto x^{-1}$ induces multiplication by -1 on L).

2.9 The Lie algebras of some subgroups of $\mathrm{GL}_n(\mathbb{R})$

Let us now reconsider the subgroups of $\mathrm{GL}_n(\mathbb{R})$ that we considered in Section 1.5. From the computations we did there, it follows that $\mathrm{Lie}(\mathrm{SL}_n(\mathbb{R}))$ is the subspace of $\mathrm{M}_n(\mathbb{R})$ consisting of the elements with trace zero, that $\mathrm{Lie}(\mathrm{SO}_n(\mathbb{R})) = \mathrm{M}_n(\mathbb{R})^-$, the space of anti-symmetric matrices, From the functoriality of the Lie algebra it follows that the Lie bracket for any of these subgroups is just the restriction of the one for $\mathrm{M}_n(\mathbb{R})$. Since $\mathrm{U}_n(\mathbb{R})$ is the set of x in $\mathrm{GL}_n(\mathbb{C})$ that satifsy $\overline{x}^{\mathrm{t}}x = 1$, it follows that $\mathrm{Lie}(U_n(\mathbb{R}))$ is the subset of y in $\mathrm{M}_n(\mathbb{C})$ that satisfy $\overline{y}^t + y = 0$.

Another sequence of "classical groups" that we should have introduced already before is that of the *symplectic groups* $\mathrm{Sp}_{2n}(\mathbb{R})$. As in the case of the orthogonal groups, the symplectic groups are stabilizers of bilinear forms, but this time it is antisymmetric forms. We recall that every non-degenerate antisymmetric bilinear form on a real vector space is given, with respect to a suitable basis, by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We define $\mathrm{Sp}_{2n}(\mathbb{R})$ to be the subgroup of x in $\mathrm{GL}_{2n}(\mathbb{R})$ with $x^{\mathrm{t}}\psi x = \psi$. It is clear that $\mathrm{Sp}_{2n}(\mathbb{R})$ is a closed subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$. To show that it is a Lie subgroup, we consider the map:

$$f\colon \mathrm{GL}_{2n}(\mathbb{R}) \longrightarrow \mathrm{M}_{2n}(\mathbb{R})^-, \quad x \mapsto x^{\mathrm{t}}\psi x - \psi,$$

because $f^{-1}\{0\} = \mathrm{Sp}_{2n}(\mathbb{R})$. We have:

$$(Df)(1)\colon \mathrm{M}_{2n}(\mathbb{R}) \longrightarrow \mathrm{M}_{2n}(\mathbb{R})^-, \quad y \mapsto y^{\mathrm{t}}\psi + \psi y.$$

We will now that $(Df)(1)$ is surjective (and hence f a submersion at 1) by computing its kernel (i.e., $\mathrm{Lie}(\mathrm{Sp}_{2n}(\mathbb{R}))$) and comparing dimensions. So write y as a two by two matrix of n by n matrices: $y = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then a short computation gives:

$$y^t\psi + \psi y = 0 \iff (c^t = c \text{ and } b^t = b \text{ and } d = -a^t).$$

It follows that the kernel has dimension $2n^2 + n$. We note that:

$$2n^2 + n = \dim(\mathrm{M}_{2n}(\mathbb{R})^+) = ((2n)^2 - (2n))/2.$$

Linear algebra then implies that our map $z \mapsto z^t\psi + \psi z$ is surjective. So $\mathrm{Sp}_{2n}(\mathbb{R})$ is a Lie subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$, and $\mathrm{Lie}(\mathrm{Sp}_{2n}(\mathbb{R}))$ is the space of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c^t = c$, $b^t = b$ and $d = -a^t$.

3 The exponential map

The exponential map will be our crucial tool in order to show that the difference between Lie groups and Lie algebras is very small (we will make precise statements later).

So let G be a Lie group, let v be in $\mathrm{Lie}(G)$ and let D_v be the left-invariant vector field on G given by v. Let $\phi_v \colon W \to G$ be the flow of D_v (we recall that W is a neighborhood of $\{0\} \times G$ in $\mathbb{R} \times G$. For x in G, we let $\phi_{v,x} \colon]-\varepsilon, \varepsilon[\to G$ denote $\phi_v(\cdot, x)$ (with $\varepsilon > 0$ depending on v and x). We recall that the differential equation defining $\phi_{v,x}$ is:

$$(\mathrm{T}_{\phi_{v,x}}(t))(1) = (D_v)_{\phi_{v,x}(t)}.$$

As D_v is left-invariant, the $l_g \colon G \to G$ are isomorphisms of varieties that send D_v to D_v, for all g in G. Hence we have, for every g in G:

$$\phi_{v,gx}(t) = g\phi_{v,x}(t).$$

3.1 Proposition. *The flow of D_v is defined on all of $\mathbb{R} \times G$.*

Proof. Let ε be a positive real number such that $\phi_{v,e}$ is defined on $]-\varepsilon, \varepsilon[$. Then, for g in G, the map $\phi_{v,g}$ is defined on the same interval since it is given as: $\phi_{v,g} = l_g \circ \phi_{v,e}$. Then one argues as in the case where we discussed the flow on a compact variety. □

3.2 Proposition. *The map $\phi_{v,e} \colon \mathbb{R} \to G$ is a morphism of groups.*

Proof. The proof is given by the equalities:

$$\phi_{v,e}(t_1 + t_2) = \phi_{v,\phi_{v,e}(t_1)}(t_2) = \phi_{v,e}(t_1)\phi_{v,e}(t_2).$$

□

3.3 Definition. We define $\exp \colon \mathrm{Lie}(G) \to G$ to be the map $v \mapsto \phi_{v,e}(1)$. This is a morphism of C^k-manifolds if G is a C^k-manifold. We note that for all v in $\mathrm{Lie}(G)$ and all t in $\mathbb{R}$ we have:

$$\exp(tv) = \phi_{tv,e}(1) = \phi_{v,e}(t),$$

since multiplying a vector field by a real number t makes the flow go t times as fast. By the previous proposition, the map $t \mapsto \exp(tv)$ from $\mathbb{R}$ to G is a morphism of groups:

$$\exp((t_1 + t_2)v) = \exp(t_1 v) \exp(t_2 v), \quad \text{for all } t_1 \text{ and } t_2 \text{ in } \mathbb{R}.$$

3.4 Proposition. *Let $f\colon G_1 \to G_2$ be a morphism of Lie groups. Then the diagram:*

$$\begin{array}{ccc} \mathrm{Lie}(G_1) & \xrightarrow{\mathrm{Lie}(f)} & \mathrm{Lie}(G_2) \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G_1 & \xrightarrow{f} & G_2 \end{array}$$

is commutative. In particular, a morphism of Lie groups is determined, on the connected component of the identity element, by its derivative at the identity element. So, if G_1 is connected, then the map Lie *from* $\mathrm{Hom}(G_1, G_2)$ *to* $\mathrm{Hom}(\mathrm{Lie}(G_1), \mathrm{Lie}(G_2))$ *is injective.*

Proof. Let v_1 be in $\mathrm{Lie}(G_1)$, let $v_2 = \mathrm{Lie}(f)v_1$, and let D_{v_1} and D_{v_2} be the left-invariant vector fields on G_1 and G_2 given by v_1 and v_2. As in Lemma 2.7.4, for every x in G_1 we have: $(\mathrm{T}_f(x))(D_{v_1})_x = (D_{v_2})_{f(x)}$. It follows immediately from the definition of the flow of a vector field, plus that of tangent mappings, that for all x in G, $f \circ \phi_{v_1,x} = \phi_{v_2,f(x)}$ (namely, $f \circ \phi_{v_1,x}$ satisfies the differential equation plus the initial data for $\phi_{v_2,f(x)}$). Hence the diagram is commutative.

The fact that f is determined by $\mathrm{Lie}(f)$ follows from the fact that $\exp(\mathrm{Lie}(G))$ generates an open (and hence closed) subgroup of G. □

3.5 Proposition. *The tangent map* $\mathrm{T}_{\exp}(0)\colon \mathrm{T}_{\mathrm{Lie}(G)}(0) = \mathrm{Lie}(G) \to \mathrm{Lie}(G) = \mathrm{T}_G(e)$ *is the identity.*

Proof. Let v be in $\mathrm{Lie}(G)$. We consider the composition:

$$\phi_{v,e}\colon \mathbb{R} \xrightarrow{\cdot v} \mathrm{Lie}(G) \xrightarrow{\exp} G, \quad t \mapsto tv \mapsto \exp(tv).$$

Then we simply look at what happens on tangent spaces. The map $t \mapsto tv$ is linear, hence equal to its derivative. Hence $(\mathrm{T}_{\cdot v}(0))(1) = v$. Hence we have: $(\mathrm{T}_{\exp}(0))v = (\mathrm{T}_{\phi_{v,e}}0)1 = v$ (the last equality comes from the definition of $\phi_{v,e}$). □

3.6 Corollary. *The exponential map* $\exp\colon \mathrm{Lie}(G) \to G$ *is a local isomorphism of C^k-manifolds (G being a C^k-manifold), with* $\mathrm{T}_{\exp}0 = \mathrm{id}$.

3.7 Remark. Be careful: exp is not always a morphism of Lie groups; we will see later that exp is a morphism of Lie groups if and only if $[x, y] = 0$ for all x and y in $\mathrm{Lie}(g)$. We will also see (hopefully) the Baker-Campbell-Hausdorff formula that expresses $\exp^{-1}(\exp(a)\exp(b))$ in terms of repeated commutators of a and b (for a and b in a sufficiently small neighborhood of 0).

3.8 Proposition. *For* $\mathrm{GL}_n(\mathbb{R})$ *(and its Lie subgroups) the exponential map is the usual exponential map:*

$$\exp\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad \exp(a) = \sum_{i\geq 0} \frac{1}{i!} a^i = e^a.$$

Proof. We have $\exp(ta) = \phi_{a,1}(t)$, and $\phi_{a,1}$ is defined by the differential equation:

$$(\mathrm{T}_{\phi_{a,1}}(t))(1) = (D_a)_{\phi_{a,1}(t)} = \phi_{a,1}(t)a,$$

with the initial condition: $\phi_{a,1}(0) = 1$. Since $t \mapsto e^{ta}$ satisfies this equation with initial condition, and the solution is unique, we have the desired equality. □

3.9 Examples. 1. For $G = \mathbb{R}$ (the additive group), exp is the identity. One can see this abstractly, by computing the invariant vector fields and solving the differential equation, or by viewing $\mathbb{R}$ as the subgroup of $\mathrm{GL}_2(\mathbb{R})$ via the map $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, and noting that for $a = \left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right)$, $\exp(a) = 1 + a$ since $a^2 = 0$.

2. For $G = \mathbb{R}^*$, the multiplicative group, viewed as an open subset of $\mathbb{R}$ in order to view its tangent spaces as $\mathbb{R}$, the map exp sends t to e^t. Either apply the definitions, or view $\mathbb{R}^*$ as $\mathrm{GL}_1(\mathbb{R})$ for example.

3. For $G = \mathrm{SO}_n(\mathbb{R})$, we recover the fact that the matrices e^a with a antisymmetric are orthogonal. Moreover, we know now that close to 0, this is a diffeomorphism between $\mathrm{M}_n(\mathbb{R})^-$ and $\mathrm{SO}_n(\mathbb{R})$. In particular, we recover that the matrices $\exp\left(\begin{smallmatrix} 0 & -t \\ t & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{smallmatrix}\right)$ are rotations in $\mathbb{R}^2$.

4 Categories and functors

Why introduce these notions? Well, I want to express the relation between Lie groups and Lie algebras as follows (as in [Ser1, II, §8, Thm. 1]):

> The functor Lie that associates to a Lie group its Lie algebra induces an equivalence of categories from the category of Lie groups that are connected and simply connected to that of Lie algebras.

So if we want to do that, we need to know what that means. In this section, the definition of category and functor are given, and some examples. The notion of equivalence will be postponed until we need it, and the notion of simply connected will be explained in the next section, together with some facts about coverings and universal coverings.

4.1 Definition. A category C consists of the following data:

1. a collection (or let us just say set) $\mathrm{Ob}(C)$ of which the elements are called the *objects* of C;
2. for all X and Y in $\mathrm{Ob}(C)$: a set $\mathrm{Hom}_C(X,Y)$ of which the elements are called the *morphims* from X to Y;
3. for all X, Y, and Z in $\mathrm{Ob}(C)$: a map $\mathrm{Hom}_C(Y,Z) \times \mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_C(X,Z)$, denoted $(g,f) \mapsto g \circ f$, called composition of morphisms,

that satisfy:

1. composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$ every time that it makes sense;
2. for all X in $\mathrm{Ob}(C)$ there is an element id_X in $\mathrm{Hom}_C(X,X)$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ whenever these make sense.

In most cases, there is no ambiguity of what the morphisms are supposed to be, so in that case one just defines the category by saying what its objects are.

4.2 Examples. 1. The category Set of sets.

2. The category Grp of groups.
3. The category Top of topological spaces.
4. The category of C^k-manifolds.

5. The categories of Lie groups and of Lie algebras.

4.3 Definition. A functor should be thought of as a morphism of categories. More precisely, if C and D are categories, a covariant (resp., contravariant) functor $F\colon C \to D$ consists a the data:

1. a map $F\colon \mathrm{Ob}(C) \to \mathrm{Ob}(D)$, denoted $X \mapsto F(X)$;
2. for all X and Y in $\mathrm{Ob}(C)$, a map F from $\mathrm{Hom}_C(X, Y)$ to $\mathrm{Hom}_D(F(X), F(Y))$ (resp., to $\mathrm{Hom}_D(F(Y), F(X))$) denoted $f \mapsto F(f)$,

such that $F(g \circ f) = F(g) \circ F(f)$ (resp., $F(g \circ f) = F(f) \circ F(g)$) whenever this makes sense.

4.4 Examples. 1. Lie from Lie groups to Lie algebras (covariant).

2. The functor from C^k-manifolds to $\mathbb{R}$-algebras that associates to X the $\mathbb{R}$-algebra $C^k(X)$ (contravariant).
3. The functor that associated to a vector space its dual (contravariant).

4.5 The homotopy category

To end this section let us just give one interesting example of a useful and highly non-trivial construction with categories. We start with the category Top of topological spaces. For X and Y in Top, f and g in $\mathrm{Hom}(X, Y)$ are said to be *homotopic* if one can interpolate between them, i.e., if there exists a continuous map $F\colon X \times [0, 1] \to Y$ such that $f = F(\cdot, 0)$ and $g = F(\cdot, 1)$. One checks easily that this is an equivalence relation on $\mathrm{Hom}(X, Y)$; we denote it "$\sim$". Then one can check that composition in Top is compatible with $\sim$ in the sense that one gets a category Top$/\sim$ with the same objects as Top, but in which one has replaced $\mathrm{Hom}(X, Y)$ by $\mathrm{Hom}(X, Y)/\sim$.

As an exercise one can show that the point $\{0\}$ and the line $\mathbb{R}$ are isomorphic objects in Top$/\sim$ (a morphism $f\colon X \to Y$ is called an isomorphism if there is a morphism $g\colon Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$).

5 Coverings and universal coverings

5.1 Definition. Let $f\colon X \to Y$ be a morphism of topological spaces (or C^k-manifolds, etc.). Then f is called a covering if for all y in Y there exists an open neighborhood U of y in Y such that $f^{-1}U$ is isomorphic, compatibly with the map f, to a disjoint union of copies of U, i.e., if there is a commutative diagram:

$$\begin{array}{ccccc} X & \longleftarrow & f^{-1}U & \overset{\sim}{\longleftarrow} & \coprod_{x\in f^{-1}\{y\}} U \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \mathrm{can}} \\ Y & \longleftarrow & U & = & U. \end{array}$$

A morphism of coverings from $f_1\colon X_1 \to Y_1$ to $f_2\colon X_2 \to Y_2$ is a commutative diagram:

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ Y_1 & \longrightarrow & Y_2. \end{array}$$

5.2 Example. Coverings of Lie groups. If G is a Lie group, and N is a normal discrete subgroup of G, then the quotient morphism $p\colon G \to G/N$ is a covering. We will give some explicit examples in a moment.

5.3 Exercise. Let $f\colon X \to Y$ be a covering, and let f_0 and f_1 be morphisms from $Z \to X$, such that $f \circ f_0$ and $f \circ f_1$ are homotopic. Show that f_0 and f_1 are homotopic. Hint: show in fact that for $F\colon Z \times [0,1] \to Y$ a homotopy between $f \circ f_0$ and $f \circ f_1$ there exists a unique $\widetilde{F}\colon Z \times [0,1] \to X$ such that $F = f \circ \widetilde{F}$ and $\widetilde{F}(\cdot,0) = f_0$ and $\widetilde{F}(\cdot,1) = f_1$. To prove this, reduce to the case where X is a disjoint union of copies of Y.

5.4 Definition. A topological space X is called *connected* if X has exactly two subsets that are open and closed: $\emptyset$ and X (in particular, X is not empty). Equivalently: X is connected if the $\mathbb{R}$-vector space of locally constant $\mathbb{R}$-valued functions has dimension one.

5.5 Definition. A topological space X is called *simply connected* if every loop in it is contractible, i.e., if every continuous $f\colon \mathrm{S}^1 \to X$ is homotopic to a constant map, i.e, if for every such f there exists a continuous $F\colon \mathrm{S}^1 \times [0,1] \to X$ such that $F(\cdot,0) = f$ and $F(\cdot,1)$ is constant.

5.6 Example. Every contractible topological space is simply connected. In particular, all convex subsets of finite dimensional real vector spaces are simply connected. A bit less trivial: for $n \geq 2$ the n-sphere S^n is simply connected.

5.7 Theorem. *For every connected manifold X, there exists a covering $p_X\colon \widetilde{X} \to X$ with $\widetilde{X}$ connected and simply connected. These coverings have the following property: for $f\colon X \to Y$ a morphism of connected manifolds, and for x in X and y in Y with $y = f(x)$, there exists, for every choice of $\tilde{x}$ in $p_X^{-1}\{x\}$ and $\tilde{y}$ in $p_Y^{-1}\{y\}$, a unique morphism $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ {\scriptstyle p_X}\big\downarrow & & \big\downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y. \end{array}$$

The covers $p_X\colon \widetilde{X} \to X$ are called universal covers. The property above implies that universal covers are unique up to isomorphism (the isomorphisms are not unique, but are unique after the choice of base points). The group $\mathrm{Aut}(p_X)$ of automorphisms of the covering p_X, i.e., the group of commutative diagrams:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & \widetilde{Y} \\ {\scriptstyle p_X}\big\downarrow & & \big\downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y, \end{array}$$

can be identified, after the choice of a point x in X, with the fundamental group $\pi_1(X, x)$ of X (recall that $\pi_1(X, x)$ is the set of homotopy classes of loops in X with base point x, i.e., the class of morphisms $f\colon \mathrm{S}^1 \to X$ with $f(1) = x$ under base point preserving homotopies). The group $\mathrm{Aut}(p_X)$ acts freely on $\widetilde{X}$ with quotient p_X.

We will not prove this theorem at all, but just use it, and illustrate it by some examples. (I would like to say that anyway, if one wants to do research, one has to learn to *use* results without knowing all of their proofs. It is simply not always possible to know the proofs of all the results that one uses.) A few words about the construction of the universal cover. Let x be in X. Then, as a set, $\widetilde{X}$ is just the set of homopoty classes of maps of pointed spaces $f\colon [0,1] \to X$ ($f(0) = x$ and homotopies F are required to satisfy $F(0,t) = F(0,0) = x$ and $F(1,t) = F(1,0)$ for all t). Then one takes a reasonable topology on this set. What is clear at least is that $p^{-1}\{x\}$ is the same as $\pi_1(X, x)$.

It is an interesting exercise to find, for every connected Lie group that we encounter, an explicitly given universal cover. In the special case of Lie groups, one has the following result.

5.8 Proposition. *Let G be a connected Lie group. Then, for any choice of $\tilde{e}$ in $p^{-1}\{e\}$, there is a unique group law on $\widetilde{G}$ such that p is a morphism of Lie groups. Let $\tilde{e}$ be given.*

Then $\ker(p)$ *is a normal discrete subgroup of* $\widetilde{G}$*, contained in the center of* $\widetilde{G}$*, in particular,* $\ker(p)$ *is commutative.*

Proof. We just sketch. Let $\tilde{e}$ be in $p^{-1}\{e\}$. Since the product of two connected and simply connected manifolds is connected and simply connected, $\widetilde{G} \times \widetilde{G}$ is simply connected, and it is the universal cover of $G \times G$. Therefore, there is a unique morphism $\tilde{m}\colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ that is compatible with the multiplication morphism $m\colon G \times G \to G$. This $\tilde{m}$ gives $\widetilde{G}$ a group structure. The fact that $\ker(p)$ lies in the center of $\widetilde{G}$ follows from the fact that for z in $\ker(p)$ both thye left and right translations l_z and r_z are in $\mathrm{Aut}(p)$ and both map $\tilde{e}$ to z, hence are equal. □

5.9 Example. A very simple example is given by the circle S^1. Recall that we view S^1 as the unit circle in $\mathbb{C}^*$, with the induced group structure. In this case, a simply connected (even contractible) cover is given by: $\mathbb{R} \to \mathrm{S}^1$, $x \mapsto \exp(2\pi i x)$, with kernel $\mathbb{Z}$.

5.10 Quaternions, $\mathrm{SU}_2(\mathbb{R})$, $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{SO}_4(\mathbb{R})$

The real quaternion algebra $\mathbb{H}$ is defined to be the sub $\mathbb{R}$-algebra of $\mathrm{M}_2(\mathbb{C})$ consisting of the matrices of the form $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$, with a and b in $\mathbb{C}$. It is just a tiny computation to see that this is indeed a sub-$\mathbb{R}$-algebra. Another way to describe $\mathbb{H}$ is to say that it is a $\mathbb{R}$-vector space with basis $(1, i, j, k)$ and an associative multiplication defined by: $i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$. In our realization of $\mathbb{H}$, one can choose $i = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ and $j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The center of $\mathbb{H}$ is just $\mathbb{R}$: every element in $\mathrm{M}_2(\mathbb{C})$ that commutes with i is diagonal, and then the condition to commute with j implies that it is scalar.

Since $\det\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right) = |a|^2 + |b|^2$, $\mathbb{H}$ is clearly a division algebra (i.e., every non-zero element in it is invertible). For x in $\mathbb{H}$, let $x^* := \overline{x}^{\mathrm{t}}$, the complex conjugate of the transpose of x (i.e., the adjoint of x for the standard hermitian inner product on $\mathbb{C}^2$). Then $x \mapsto x^*$ is clearly an involution of $\mathbb{H}$ (note that $(xy)^* = y^*x^*$), with the property that $N(x) := x^*x = xx^* = \det(x)$ and $\mathrm{tr}(x) := x + x^* \in \mathbb{R}$. Note that $N(xy) = N(x)N(y)$ for all x and y in $\mathbb{H}$. We define a real inner product on the $\mathbb{R}$-vector space $\mathbb{H}$ by: $\langle x, y\rangle = (x^*y + y^*x)/2 = \mathrm{tr}(x^*y)/2 = \mathrm{tr}(yx^*)/2 = \langle x^*, y^*\rangle$. Clearly, the quadratic form associated to this inner product is the norm $N\colon \mathbb{H} \to \mathbb{R}$.

For x in $\mathbb{H}$ let l_x and r_x denote the left and right multiplication by x from $\mathbb{H}$ to itself: $l_x\colon y \mapsto xy$ and $r_x\colon y \mapsto yx$. Let us first see how these relate to the inner product we just defined, or, equivalently, to the quadratic form N. Well, $N(xy) = N(x)N(y)$, and $N(yx) = N(y)N(x) = N(x)N(y)$, so it follows that l_x and r_x preserve the inner product up to a factor $N(x)$. In particular, l_x and r_x are orthogonal if $N(x) = 1$.

Note that $\{x \in \mathbb{H} \mid N(x) = 1\}$ is the subgroup $\mathrm{SU}_2(\mathbb{R})$ of $\mathbb{H}^*$, i.e., the subgroup of $\mathrm{GL}_2(\mathbb{C})$ of elements fixing the standard hermitian inner product on $\mathbb{C}^2$. The adjoints of l_x and r_x are l_{x^*} and r_{x^*}, respectively (check it, it might be harder than you think!). We want to know the characteristic polynomials of l_x and r_x. For x in $\mathbb{R}$, these are both equal to $(X - x)^4$, of course. So let's suppose that x is not in $\mathbb{R}$. Then $x^* \neq x$, and the minimal polynomial of x over $\mathbb{R}$ is $P_x := (X - x)(X - x^*) = X^2 - \mathrm{tr}(x) + N(x)$. This polynomial is irreducible over $\mathbb{R}$ because if it had a real root λ, $x - \lambda$ would be a non-zero zero divisor in $\mathbb{H}$. Note that P_x is also the minimal polynomial of l_x and r_x. The characteristic polynomial of a linear transformation has the same irreducible factors as its minimal polynomial, so if the minimal polynomial is irreducible, then the characteristic polynomial is just a power of the minimal polynomial. In our case, it follows that the characteristic polynomials of l_x and r_x are both equal to P_x^2, since they are of degree four. In particular, $\det(l_x) = \det(r_x) = N(x)^2$; note that these are positive. For x and y in $\mathbb{H}$ let $b_{x,y}\colon \mathbb{H} \to \mathbb{H}$ be given by: $z \mapsto xzy^*$. Hence $b_{x,y}$ is the composition of the commuting $\mathbb{R}$-linear maps l_x and r_{y^*}. Let λ and $\overline{\lambda}$ be the roots in $\mathbb{C}$ of P_x, and let μ and $\overline{\mu}$ be those of P_y. We claim that the characteristic polynomial of $b_{x,y}$ is the one whose roots are $\lambda\mu$, $\lambda\overline{\mu}$, $\overline{\lambda}\mu$ and $\overline{\lambda}\overline{\mu}$. If x or y is in $\mathbb{R}$, this is clear from what we know already, so suppose that x and y are both not in $\mathbb{R}$. Suppose that r_{y^*} has only one eigenvalue on the λ-eigenspace of l_x (in the complexification of the $\mathbb{R}$-vector space $\mathbb{H}$). Interchanging μ and $\overline{\mu}$ if necessary, we may suppose that this eigenvalue is μ. Then r_{y^*} acts on the λ-eigenspace as multiplication by μ, and on the $\overline{\lambda}$-eigenspace as multiplication by $\overline{\mu}$. It follows that r_{y^*} is a polynomial in l_x, with coefficients in $\mathbb{R}$ (take any Q in $\mathbb{R}[X]$ such that $Q(\lambda) = \mu$). But then $r_{y^*} = l_z$, with $z = Q(x)$. Since $z = z{\cdot}1 = 1{\cdot}y^* = y^*$, we have $z = y^*$, hence $l_z = r_z$. But this contradicts that z is not in the center of $\mathbb{H}$.

Let V be the orthogonal complement of $\mathbb{R}$, with respect to our inner product. Then V is called the space of pure quaternions, I believe. It is a three dimensional $\mathbb{R}$-vector space, with a given inner product (the restriction of $\langle\cdot,\cdot\rangle$). For x in $\mathbb{H}^*$, the map $y \mapsto xyx^{-1}$, from $\mathbb{H}$ to $\mathbb{H}$, induces the identity on $\mathbb{R}$, hence induces a map $c_x\colon V \to V$. Since $N(xyx^{-1}) = N(y)$, c_x is an orthogonal transformation of V. Clearly, $x \mapsto c_x$ defines a morphism of groups from $\mathbb{H}^*$ to $\mathrm{O}(V)$, hence, by restriction to $\mathrm{SU}_2(\mathbb{R})$, a morphism $c\colon \mathrm{SU}_2(\mathbb{R}) \to \mathrm{O}(V)$. The kernel of c is the intersection of $\mathrm{SU}_2(\mathbb{R})$ with the center of $\mathbb{H}$, i.e., the intersection with $\mathbb{R}$; hence $\ker(c) = \{1, -1\}$. Let us show that $\mathrm{im}(c)$ is contained in $\mathrm{SO}(V)$. Let x be in $\mathrm{SU}_2(\mathbb{R})$. Then l_x and $r_{x^{-1}}$ have determinant one (see above). Hence c_x, being equal to $l_x \circ r_{x^{-1}}$, has determinant one. Let us now show that $\mathrm{im}(c) = \mathrm{SO}(V)$. The cheapest way to do that is to use that $c\colon \mathrm{SU}_2(\mathbb{R}) \to \mathrm{SO}(V)$ is a morphism of Lie groups. We know that $\mathrm{SO}_(V)$ is connected (a choice of an orthonormal basis of V gives an isomorphism from $\mathrm{SO}_3(\mathbb{R})$ to

SO(V)), hence if im(c) is an open subgroup of SO(V) then it is also closed, hence equal to SO(V). (Exercise: show that if G is a topological group, and H an open subgroup, that H is closed; hint: the complement of H is a union of translates of H.) To show that im(c) is open, it suffices to show that c is a submersion at 1 in $\mathrm{SU}_2(\mathbb{R})$. But since both $\mathrm{SU}_2(\mathbb{R})$ and SO(V) are of dimension three, that amounts to showing that Lie(c) is injective. Let y be in $\mathbb{H}$. Then we compute:

$$N(1+\varepsilon y) = (1+\varepsilon y)(1+\varepsilon y)^* = 1+\varepsilon(y+y^*)+O(\varepsilon^2).$$

Hence $\mathrm{Lie}(\mathrm{SU}_2(\mathbb{R})) = \{y \in \mathbb{H} \mid y+y^* = 0\}$. Now we compute, for v in V and y in $\mathrm{Lie}(\mathrm{SU}_2(\mathbb{R}))$:

$$(c(1+\varepsilon y))v = (1+\varepsilon y)v(1+\varepsilon y)^{-1} = (1+\varepsilon y)v(1-\varepsilon y+O(\varepsilon^2)) = v+\varepsilon(yv-vy)+O(\varepsilon^2).$$

It follows that Lie(c)(y) is the endormorphism $v \mapsto yv - vy$ of V. The fact that the center of $\mathbb{H}$ is $\mathbb{R}$ implies that this endomorphism is zero if and only if y is in $\mathbb{R}$, which is the case only if $y = 0$ (since $y^* = -y$). Hence Lie(c) is injective.

One can also show by direct computation that im(c) = SO(V). It suffices to show that im(c) contains elements with a given trace in $[-1,3]$ and fixing a given non-zero element of V (classification of orthogonal transformations in dimension three). Let t be in $[-1,3]$ and let a be a non-zero element of V. Then the x with $c_x(a) = a$ are exactly those commuting with a, hence those in $\mathbb{R}[a] = \mathbb{R} \oplus \mathbb{R}a$. Now $\mathbb{R}[a]$ is isomorphic, as $\mathbb{R}$-algebra, to $\mathbb{C}$. Let x be an element of $\mathbb{R}[a]$ with $N(x) = 1$, and let λ and $\overline{\lambda}$ be the roots in $\mathbb{C}$ of P_x. We have seen that the characteristic polynomial of the map $y \mapsto xyx^* = xyx^{-1}$ from $\mathbb{H}$ to $\mathbb{H}$ has as roots λ^2, $\overline{\lambda}^2$, 1 and 1. Hence the eigenvalues of $c(x)$ on the complexification of V are 1 (on $\mathbb{C}a$), λ^2 and $\overline{\lambda}^2$. So one just takes x such that $\lambda = e^{\pm i\phi/2}$, with $\cos(\phi) = (t-1)/2$. We conclude that $\mathrm{SU}_2(\mathbb{R})/\{1,-1\}$ is isomorphic to SO_3.

Let's now study O($\mathbb{H}$). (Any choice of othonormal basis of $\mathbb{H}$, for example $(1,i,j,k)$, gives an isomorphism from $\mathrm{O}_4(\mathbb{R})$ to O($\mathbb{H}$)). For x and y in $\mathbb{H}^*$, recall that $b_{x,y}$ is the $\mathbb{R}$-linear map $z \mapsto xzy^*$, from $\mathbb{H}$ to $\mathbb{H}$. The map $(x,y) \mapsto b_{x,y}$ gives a morphism of groups from $\mathbb{H}^* \times \mathbb{H}^*$ to $\mathrm{GL}_{\mathbb{R}}(\mathbb{H})$. Restricting this map to $\mathrm{SU}_2(\mathbb{R}) \times \mathrm{SU}_2(\mathbb{R})$ gives a morphism of groups $b\colon \mathrm{SU}_2(\mathbb{R}) \times \mathrm{SU}_2(\mathbb{R}) \to \mathrm{O}(\mathbb{H})$, since r_x and l_{y^*} are orthogonal if $N(x) = 1$ and $N(y) = 1$. We want to determine kernel and image of this b. Let's start with the kernel. Suppose that $b(x,y)$ is the identity in O($\mathbb{H}$). Then $1 = x1y^* = xy^*$, which shows that $y = x$. Moreover, for all z in $\mathbb{H}$ we have $z = xzx^* = xzx^{-1}$, which means that x is in the center of $\mathbb{H}$, i.e., in $\mathbb{R}$. It follows that $\ker(b) = \{(1,1),(-1,-1)\}$, a cyclic group of order two. Now the image of b. Since all r_x and l_y have positive determinant, im(b) is contained in SO($\mathbb{H}$). We claim that it is equal to SO($\mathbb{H}$).

To show that, it suffices to show that $\mathrm{Lie}(b)$ is injective, since both groups are Lie groups of dimension four. So let (y_1, y_2) be in $\mathrm{Lie}(\mathrm{SU}_2(\mathbb{R})) \times \mathrm{Lie}(\mathrm{SU}_2(\mathbb{R}))$, and suppose that its image in $\mathrm{Lie}(\mathrm{SO}(\mathbb{H}))$ is zero. Then one has, for all z in $\mathbb{H}$:

$$(1+\varepsilon y_1)z(1+\varepsilon y_2)^* = z + O(\varepsilon^2), \quad \text{hence} \quad y_1 z + z y_2^* = 0.$$

Now $y_2^* = -y_2$. Substituting $z := 1$ gives: $y_2 = y_1$, so that we have $y_1 z - z y_1 = 0$ for all z in $\mathbb{H}$. But that means that y_1 is in $\mathbb{R}$. Since $y_1^* = -y_1$, we have $y_1 = 0$. That does it.

As before, one can also compute directly. It suffices to show that, for every two-dimensional subspace W of $\mathbb{H}$, and for every pair (a, b) in $\mathrm{SO}(W) \times \mathrm{SO}(W^\perp)$, $\mathrm{im}(b)$ contains an element fixing W (and hence $W^\perp$), inducing a and b on W and $W^\perp$, respectively. Let W, a and b be such data. Let (w_1, w_2) be an $\mathbb{R}$-basis of W. We define x' in $\mathbb{H}^*$ by: $x' w_1 = w_2$ (i.e., we have $x' = w_2 w_1^{-1}$). Since x' is a root of $P_{x'}$ which is of degree two, we have $x'W = W$. Likewise, we define y' by: $w_1 y' = w_2$. Let x be in $\mathbb{R}[x']$ and y in $\mathbb{R}[y']$. Then we have $xW \subset W$, $xW^\perp \subset W^\perp$, $Wy \subset W$ and $W^\perp y \subset W^\perp$. Let λ and $\overline{\lambda}$ be the roots in $\mathbb{C}$ of P_x, and μ and $\overline{\mu}$ those of P_y. Then the eigenvalues of $b(x, y)$ on the complexification of $\mathbb{H}$ are $\lambda\mu$, $\lambda\overline{\mu}$, $\overline{\lambda}\mu$ and $\overline{\lambda}\overline{\mu}$. After interchanging, if necessary, μ and $\overline{\mu}$, $b(x, y)$ has eigenvalues $\lambda\mu$ and $\overline{\lambda}\overline{\mu}$ on the complexification of W, and $\lambda\overline{\mu}$ and $\overline{\lambda}\mu$ on the complexification of $W^\perp$. It is now clear that one can choose x and y in $\mathrm{SU}_2(\mathbb{R})$ such that $b(x, y)$ is as desired.

Coming back to universal covers, it is clear that $\mathrm{SU}_2(\mathbb{R})$ is the unit sphere S^3 in $\mathbb{H} = \mathbb{R}^4$, hence it is simply connected. So we conclude that the double cover $c\colon \mathrm{SU}_2(\mathbb{R}) \to \mathrm{SO}_2(\mathbb{R})$ is the universal cover, and the same for $b\colon \mathrm{SU}_2(\mathbb{R})^2 \to \mathrm{SO}_4(\mathbb{R})$.

6 Lie induces an equivalence

We have the functor Lie from the category of Lie groups to that of Lie algebras. We would like to construct a kind of inverse to it. This "inverse" has to be a functor from Lie algebras to Lie groups. Since various non-isomorphic Lie groups can have isomorphic Lie algebras, the "inverse" has to make a choice between the various possibilities; a canonical choice is to take connected and simply connected Lie groups. The following result says that this choice is possible, i.e., gives a functor.

6.1 Theorem. *For every Lie algebra L there exists a pair $(\mathrm{G}(L), i_L)$ with $\mathrm{G}(L)$ a connected and simply connected Lie group, and $i_L\colon L \to \mathrm{Lie}(\mathrm{G}(L))$ an isomorphism of Lie algebras. These pairs have the following property: if L_1 is a Lie algebra, G_2 a Lie group, and $f\colon L_1 \to \mathrm{Lie}(G_2)$ is a morphism, then there exists a unique morphism $\widetilde{f}\colon \mathrm{G}(L_1) \to G_2$ such that the diagram:*

$$\begin{array}{ccc} L_1 & \xrightarrow{\ f\ } & \mathrm{Lie}(G_2) \\ {\scriptstyle i_{L_1}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Lie}(\mathrm{G}(L_1)) & \xrightarrow{\mathrm{Lie}(\widetilde{f})} & \mathrm{Lie}(\mathrm{G}(L_2)) \end{array}$$

is commutative. In particular, such pairs are unique up to unique isomorphism (apply the property above to i'_L and i_L where $(\mathrm{G}(L), i_L)$ and $(\mathrm{G}(L)', i'_L)$ are two such pairs). Hence a choice of such a pair, for every L, gives a functor G from Lie algebras to (connected and simply connected) Lie groups.

Proof. We will use, without proof, the following theorem (for a proof, see [Vara, §3.17] or [BLie, 1, §7, 3, Thm. 3]).

6.1.1 Theorem. (Ado, 1936) *Let k be a field of characteristic zero, and let L be a finite dimensional Lie algebra over k. Then there exists an injective morphism of Lie algebras $f\colon L \to \mathrm{M}_n(k)$ for a suitable integer n.*

Let now L be a finite dimensional real Lie algebra (the complex case will just work in the same way). We view L as a sub Lie algebra of some $\mathrm{M}_n(\mathbb{R})$, via a suitable embedding (which exists by Ado's theorem). We view $\mathrm{M}_n(\mathbb{R})$ as $\mathrm{Lie}(G)$, with $G = \mathrm{GL}_n(\mathbb{R})$.

We let H be the subgroup of G that is generated by the elements $\exp(v)$ with v in L. We would like to take for $\mathrm{G}(L)$ the universal cover of H. The problem is that H is not necessarily a closed subgroup of G (think of maps $\mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ with dense image). And even if it were closed, we would still have to prove that it is a sub Lie group. The solution to the problem is to apply the theorem of Frobenius that was stated in Section 1, and to take a topology on H that is not necessarily the one that is induced by the topology of G. With the right topology, H will be a connected Lie group.

For v in L, we have the corresponding left-invariant vector field D_v on G, and for every g in G, we have the subspace $V_g = \{(D_v)_g \mid v \in L\} = \mathrm{T}_{l_g} L$ of $\mathrm{T}_G(g)$. We let $\mathcal{D}$ be the sub $C^\infty(G)$-module of $\mathrm{Der}_{\mathbb{R}}(C^\infty(G))$ that is generated by the D_v with v in L. If $(v_1, \ldots, v_d)$ is an $\mathbb{R}$-basis of L, then $(D_{v_1}, \ldots, D_{v_d})$ is a $C^\infty(G)$-basis of $\mathcal{D}$ (we just say this because it may help to understand what is going on; we don't need to know that $\mathcal{D}$ is a free module). We have to check that $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$. For that, it suffices to show that for v_1 and v_2 in L, and f_1 and f_2 in $C^\infty(G)$, $[f_1 D_{v_1}, f_2 D_{v_2}]$ is in $\mathcal{D}$. One computes that:

$$[f_1 D_{v_1}, f_2 D_{v_2}] = f_1 D_{v_1}(f_2) D_{v_2} - f_2 D_{v_2}(f_1) D_{v_1} + f_1 f_2 D_{[v_1, v_2]}.$$

The theorem of Frobenius now gives, locally at every point x in G, a closed sub manifold X_x of G such that, for all y in X_x one has $\mathrm{T}_{X_x}(y) = V_y$ (these X_x are called local integral manifolds of the system of subspaces V).

For v in L, let $\phi_v \colon \mathbb{R} \times G \to G$ denote the flow of the vector field D_v. Then, for x in G and small t in $\mathbb{R}$, $\phi_v(t, x)$ stays in X_x, since for all y in G we have $(D_v)_y \in \mathrm{T}_{X_y}(y)$ (one can say: D_v induces vector fields on the X_y, hence the flow stays in the X_y). Now consider $\exp(L)$ and X_e, in a small neighborhood of e in G (say the image of a neighborhood of 0 in L on which exp is a diffeomorphism). Both are d-dimensional sub manifolds, and from what we have just said about the flows, it follows that (in the small neighborhood of e) $\exp(L)$ is contained in X_e, hence they are equal. But then it follows that H is the maximal integral sub manifold (for $\mathcal{D}$) that passes through e.

Let us now change the topology. In the new topology on G, a basis for the set of neighborhoods of x is the set of neighborhoods of x in a closed (local) integral sub manifold X_x. (One should make a drawing here of what is going on.) With this new topology, G is a d-dimensional manifold (but it does not have a countable basis for its topology). As we have promised, the subset H is now a manifold, and it is connected. It is a Lie group, with an injective morphism of Lie groups to G, whose image is not necessarily closed. By construction, $\mathrm{Lie}(H) = L$ (as sub Lie algebras of $\mathrm{Lie}(G)$). We put $\mathrm{G}(L) := \widetilde{H}$.

Let now L_1 ne a finite dimensional Lie algebra, G_2 a Lie group, and $f \colon L_1 \to \mathrm{Lie}(G_2)$ be a morphism. Let Γ_f be the graph of f; it is a Lie sub algebra of $L_1 \times \mathrm{Lie}(G_2)$ such that the projection to L_1 is an isomorphism. Let H be the subgroup of $\mathrm{G}(L_1) \times G_2$ that is generated by $\exp(\Gamma_f)$. The same arguments as above show that H is the image of a morphism $\mathrm{G}(\Gamma_f) \to \mathrm{G}(L_1) \times G_2$ that induces the maps we already had on the level of Lie algebras. The induced morphism $\mathrm{G}(\Gamma_f) \to \mathrm{G}(L_1)$ is a covering, and since both groups are connected and simply connected, it is an isomorphism. Hence $\mathrm{G}(\Gamma_f)$ is the graph of a morphism $\mathrm{G}(f)$ from $\mathrm{G}(L_1)$ to G_2. □

It becomes time to explain what an equivalence of categories is. So here is a definition.

6.2 Definition. Let $F\colon C \to D$ be a functor (covariant, say). Then F is an equivalence of categories if there exists a functor $G\colon D \to C$ and isomorphisms $\alpha(X)\colon X \to G(F(X))$ for all X in $\mathrm{Ob}(C)$ and isomorphisms $\beta(X)\colon X \to F(G(X))$ for all X in $\mathrm{Ob}(D)$, such that for all morphisms f in C and g in D the following diagrams commute:

$$\begin{array}{ccc} X & \xrightarrow{\alpha(X)} & G(F(X)) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G(F(f))} \\ Y & \xrightarrow{\alpha(Y)} & G(F(Y)) \end{array} \qquad \begin{array}{ccc} U & \xrightarrow{\beta(U)} & F(G(U)) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle F(G(g))} \\ V & \xrightarrow{\beta(V)} & F(G(V)). \end{array}$$

If F is an equivalence of categories, then for all X and Y in $\mathrm{Ob}(C)$, the map F from $\mathrm{Hom}_C(X, Y)$ to $\mathrm{Hom}_D(F(X), F(Y))$ is a bijection (the functor is called fully faithful; it is called faithful if all those maps are injective). One can show that a functor $F\colon C \to D$ is an equivalence if and only if F is fully faithful and essentially surjective: for all Y in $\mathrm{Ob}(D)$ there exists an X in $\mathrm{Ob}(C)$ such that Y is isomorphic to $F(X)$.

6.3 Remark. One may be surprised that this definition is complicated. If we would have defined the notion of morphism of functors, then it would have been clear that $\alpha\colon \mathrm{id}_C \to G \circ F$ and $\beta\colon \mathrm{id}_D \to F \circ G$ are isomorphisms of functors. So one can say that G is an inverse of F, up to isomorphism of functors. One may ask, why not simply define an equivalence of categories to be an isomorphism, in the sense that one wants *equalities* $\mathrm{id}_C = G \circ F$ and $\mathrm{id}_D = F \circ G$. The answer is that in daily life, not enough equivalences of categories are isomorphisms of categories. Let us just give one example: if D is the functor that sends a finite dimensional vector space to its dual, then $D \circ D$ is isomorphic to the identity functor, but not equal.

We can now state the main theorem of the first (and worst) part of this course.

6.4 Theorem. *The functor* Lie *from Lie groups to Lie algebras induces an equivalence of categories from that of connected and simply connected Lie groups, to that of Lie algebras. Moreover, every connected Lie group is of the form G/N with G connected and simply connected, and N a discrete subgroup contained in the center of G.*

Proof. Let G be as in the previous theorem. So it is a functor from the category of Lie algebras to that of connected and simply connected Lie groups. We have to produce the isomorphisms $\alpha(G)$ and $\beta(L)$, for G a connected and simply connected Lie group and for L a Lie algebra. Let L be a Lie algebra. Then the construction of $\mathrm{G}(L)$ already gives an isomorphism $i_L\colon L \to \mathrm{Lie}(\mathrm{G}(L))$, this will be $\beta(L)$. Let G be a connected and simply connected Lie group. We have to come up with an isomorphism $\alpha(G)\colon G \to \mathrm{G}(\mathrm{Lie}(G))$. Theorem 6.1 gives us a morphism $\mathrm{G}(\mathrm{Lie}(G)) \to G$, which gives the identity on Lie algebras. Hence it is a local diffeomorphism at the identity element. It is surjective because the image is open and G is connected. Hence it is a covering (the kernel is discrete), and hence it is an isomorphism because G and $\mathrm{G}(\mathrm{Lie}(G))$ are both simply connected. □

6.5 Some consequences of the equivalence

Now that we know that we have Theorem 6.4 at our disposal, it is the moment to look at a few consequences of it. First of all, the equivalence implies that the theory of Lie groups is to a large extent an algebraic theory. (There are connected Lie groups that do not admit a faithful finite dimensional representation, for example the universal cover of $\mathrm{SL}_2(\mathbb{R})$; more on this later.) A lot about Lie groups can be said in terms of their Lie algebras, which are algebraic objects (vector spaces with some bilinear composition law that satisfies certain conditions). Conversely, Lie algebras can be studied in terms of Lie groups. Both aspects have their advantages, and the real advantage is that one can choose on which side one wants to work. For example, integration over compact Lie groups (as we will see later) is an important tool that one has on the side of Lie groups.

6.5.1 Theorem. *Let G be a connected Lie group. Then G is commutative if and only if $\mathrm{Lie}(G)$ is commutative in the sense that $[x, y] = 0$ for all x and y in $\mathrm{Lie}(G)$.*

Proof. Suppose that G is commutative. Then the map $\iota\colon G \to G$ with $\iota(x) = x^{-1}$ is a morphism. By the lemma below, $\mathrm{Lie}(\iota)$ is the endomorphism $x \mapsto -x$ of $\mathrm{Lie}(G)$. This implies that for all x and y in $\mathrm{Lie}(G)$ we have:

$$[x, y] = [-x, -y] = -[x, y],$$

which implies that $[x, y] = 0$, hence that $\mathrm{Lie}(G)$ is commutative.

Suppose now that $\mathrm{Lie}(G)$ is commutative, and let us prove that G is commutative. Since G is a quotient of its universal cover $\widetilde{G}$, it suffices to prove that $\widetilde{G}$ is commutative. So we may suppose that G is simply connected. Now, since Lie induces a bijection from $\mathrm{End}(G)$ to $\mathrm{End}(\mathrm{Lie}(G))$, there is a unique endomorphism ι of G such that $\mathrm{Lie}(\iota)$ is the endomorphism -1 of $\mathrm{Lie}(G)$. If we show that $\iota(y) = y^{-1}$ for all y in G than it is clear that G is commutative. Since G is generated by $\exp(L)$, it suffices to show that for every x in $\mathrm{Lie}(G)$ we have $\exp(-x) = \exp(x)^{-1}$, but that follows from the commutative diagram (see § 3):

$$\begin{array}{ccc} \mathrm{Lie}(G) & \xrightarrow{\mathrm{Lie}(\iota)=-1} & \mathrm{Lie}(G) \\ {\scriptstyle\exp}\downarrow & & \downarrow{\scriptstyle\exp} \\ G & \xrightarrow{\iota} & G \end{array}$$

Note that we did cheat a bit in saying that this theorem is a consequence of the equivalence of categories, because in the proof we did use some more ingredients than just that; it should be possible to give a "categorical" proof, using only what is in the theorem. □

6.5.2 Remark. The condition that G be connected in the previous theorem is clearly necessary: for any Lie group G and a finite group H, $G \times H$ is also a Lie group, and $\mathrm{Lie}(G \times H) = \mathrm{Lie}(G)$. Take H any non commutative group.

6.5.3 Lemma. *Let X_1 and X_2 be C^k-manifolds, with $k \geq 1$. Let $x = (x_1, x_2)$ be in $X = X_1 \times X_2$. Let p_1 and p_2 the projections from X to X_1 and X_2, and let i_1 and i_2 be the maps from X_1 and X_2 to X given by $y_1 \mapsto (y_1, x_2)$ and $y_2 \mapsto (x_1, y_2)$, respectively. Then the tangents maps of these satisfy:*

$$\mathrm{T}_{p_1}(x) \circ \mathrm{T}_{i_1}(x_1) = \mathrm{id}, \quad \mathrm{T}_{p_1}(x) \circ \mathrm{T}_{i_2}(x_2) = 0$$
$$\mathrm{T}_{p_2}(x) \circ \mathrm{T}_{i_1}(x_1) = 0, \quad \mathrm{T}_{p_2}(x) \circ \mathrm{T}_{i_2}(x_1) = \mathrm{id}$$
$$\mathrm{T}_{i_1}(x_1) \circ \mathrm{T}_{p_1}(x) + \mathrm{T}_{i_2}(x_2) \circ \mathrm{T}_{p_2}(x) = \mathrm{id}.$$

In particular, the map $\mathrm{T}_{i_1}(x_1) + \mathrm{T}_{i_2}(x_2)$ from $\mathrm{T}_{X_1}(x_1) \times \mathrm{T}_{X_2}(x_2)$ to $\mathrm{T}_X(x)$ is an isomorphism.

Let G be a Lie group. Then the multiplication map $m\colon G \times G \to G$ has derivative $\mathrm{T}_m((e,e))\colon \mathrm{T}_G(e) \times \mathrm{T}_G(e) = \mathrm{T}_{G\times G}((e,e)) \to \mathrm{T}_G(e)$ equal the sum map: $(x, y) \mapsto x + y$. The tangent map et e of $\iota\colon G \to G$, $\iota(x) = x^{-1}$, is given by $\mathrm{T}_\iota(e)\colon z \mapsto -z$.

Proof. The identity $\mathrm{T}_{p_1}(x) \circ \mathrm{T}_{i_1}(x_1) = \mathrm{id}$ is a direct consequence of $p_1 \circ i_1 = \mathrm{id}_{X_1}$. The same works for $\mathrm{T}_{p_2}(x) \circ \mathrm{T}_{i_2}(x_1) = \mathrm{id}$. As $p_1 \circ i_2$ and $p_2 \circ i_1$ are constant maps, their tangent maps are zero; this gives $\mathrm{T}_{p_1}(x) \circ \mathrm{T}_{i_2}(x_2) = 0$ and $\mathrm{T}_{p_2}(x) \circ \mathrm{T}_{i_1}(x_1) = 0$. The last identity is equivalent to $\mathrm{T}_{i_1}(x_1) + \mathrm{T}_{i_2}(x_2)$ being an isomorphism. But this is a local question, and since it is true for $X_i = \mathbb{R}^{n_i}$, it is true. (This last identity cannot be derived from the functorial nature of tangent spaces plus the property that the tangent space of a point is zero; in fact, taking the functor $\mathrm{T}_X(x) \otimes \mathrm{T}_X(x)$ gives a counter example.)

Let now G be a Lie group and m its multiplication map. Let i_1 and i_2 the morphisms given by $i_1(x) = (x, e)$ and $i_2(x) = (e, x)$. Then we have $m \circ i_1 = \mathrm{id}$ and $m \circ i_2 = \mathrm{id}$. Taking tangent maps shows that $\mathrm{T}_m(e, e)$ is the sum map. To get the tangent map for the inverse, consider the tangent map of $f\colon G \to G \times G$, $x \mapsto (x, x^{-1})$, and use that $m \circ f$ is constant. $\square$

6.5.4 Theorem. *Let G be a simply connected connected Lie group, and V a vector space (real if G is real, complex if G is complex). Then, to give a representation of G on V (i.e., a morphism from G to $\mathrm{GL}(V)$) is equivalent to give a representation of $\mathrm{Lie}(G)$ on V (i.e., a morphism from $\mathrm{Lie}(G)$ to $\mathrm{End}(V)$).*

Proof. This is just saying that the map $\mathrm{Lie}\colon \mathrm{Hom}(G, \mathrm{GL}(V)) \to \mathrm{Hom}(\mathrm{Lie}(G), \mathrm{End}(V))$ is bijective. $\square$

7 The Campbell-Hausdorff formula

It would not do to not say anything in this course about the Campbell-Hausdorff formula (also called Baker-Campbell-Hausdorff formula in some texts, for example [Vara]). The formula in question gives a formula for z such that $\exp(x)\exp(y) = \exp(z)$, in terms of repeated commutators of x and y, such as $[x,y]$, $[x,[x,y]]$, $[y,[x,y]]$, $[x,[y,[x,y]]]$, etc. The formula is universal in the sense that it holds for all x and y sufficiently small in the Lie algebra of any Lie group. The proof of this result is always somewhat long, so we will not give one in this course. A proof that I like a lot (using some concepts as universal enveloping algebra and free Lie algebras) is given in [Ser1]. A proof that is less formal and more for people who prefer analysis over algebra is given in [Vara]. We will just show what the formula is up to degree 3. Knowing the formula up to order two is sufficient for many things, and gives us the insight that the Lie bracket is, up to a factor 1/2, the quadratic term.

So let G be a Lie group. We know that $\mathrm{Lie}(G)$ can be seen as a sub Lie algebra of some $\mathrm{M}_n(\mathbb{R})$ (Ado's theorem). Hence a neighborhood of the identity element of G is "isomorphic" (between quotes because it is closed only for multiplication of elements that are sufficiently close to the identity) to a Lie sub group of $\mathrm{GL}_n(\mathbb{R})$. Hence it suffices to do our computation for $\mathrm{GL}_n(\mathbb{R})$. Now $\mathrm{GL}_n(\mathbb{R})$ has the advantage that it is the group of invertible elements of the ring $\mathrm{M}_n(\mathbb{R})$, which is also its Lie algebra (if equipped only with the Lie bracket). More precisely, the advantage is that we have a simple formula for the inverse of exp, i.e., for log, so that we can write $z = \log(\exp(x)\exp(y))$. Recall that the power series expansions for exp and log are:

$$\exp\colon x \mapsto \sum_{n\geq 0} \frac{1}{n!}x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

$$\log\colon 1 + y \mapsto -\sum_{n\geq 1} \frac{(-1)^n}{n}y^n = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \cdots$$

Using these expansions, it is very simple to show that:

$$(7.1) \qquad \log(\exp(x)\exp(y)) = x + y + \frac{1}{2}[x,y] + \frac{1}{12}\left([x,[x,y]] + [y,[y,x]]\right) + \cdots,$$

for all x and y such that $\exp(x)\exp(y)$ is in a domain of convergence of log.

The Campbell-Hausdorff formula can also be taken as a point of departure for the relation between Lie groups and Lie algebras. It has the advantage that it shows directly that Lie groups are analytic, and it works over other complete fields than $\mathbb{R}$ and $\mathbb{C}$, such as the fields $\mathbb{Q}_p$ of p-adic numbers (important for number theory). See [Ser1].

8 Fundamental groups of some Lie groups

The aim of this section is to compute explicitly the fundamental groups of some Lie groups, so that we know how far these Lie groups are from their universal covers. From what we have said about fundamental groups and homotopy, it is clear that for X and Y connected topological spaces, $f\colon X \to Y$ a morphism that is a homotopy equivalence (i.e., such that $\overline{f}$ is an isomorphism in the homotopy category), and x in X, f induces an isomorphism from $\pi_1(X,x)$ to $\pi_1(Y,y)$. Therefore, it is good to know that some Lie groups are homotopy equivalent to each other. To see this, we use some standard decompositions of $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$, known as "décomposition polaire" in French.

8.1 Proposition. *Let $n \geq 1$. Let $\mathrm{M}_n(\mathbb{R})^{+,+}$ be the set of positive definite symmetric elements of $\mathrm{M}_n(\mathbb{R})$. Then the map:*

$$\mathrm{M}_n(\mathbb{R})^{+,+} \times \mathrm{O}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{R})$$

is a diffeomeorphism.

Proof. First note that the dimensions are right. The theorem says that if we let $\mathrm{O}_n(\mathbb{R})$ act (from the right) on $\mathrm{GL}_n(\mathbb{R})$, then every orbit intersects $\mathrm{M}_n(\mathbb{R})^{+,+}$ in exactly one point, and at that point, the orbit and $\mathrm{M}_n(\mathbb{R})^{+,+}$ are transversal.

Note that $\mathrm{M}_n(\mathbb{R})^{+,+}$ is the set of positive definite symmetric bilinear forms on $\mathbb{R}^n$, i.e., the set of inner products; the identity element 1 corresponds to the standard inner product. We let $\mathrm{GL}_n(\mathbb{R})$ act on $\mathrm{M}_n(\mathbb{R})^{+,+}$ by: $(g,b) \mapsto gbg^{\mathrm{t}}$. Since for every inner product on $\mathbb{R}^n$ there is an orthonormal basis (Gram-Schmidt or so), this action is transitive, hence $\mathrm{M}_n(\mathbb{R})^{+,+}$ is the orbit of 1. By definition, the stabilizer of 1 is $\mathrm{O}_n(\mathbb{R})$. Hence the map $\mathrm{GL}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})^{+,+}$ that sends g to gg^{t} induces a bijection:

$$\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})^{+,+}.$$

On the subset $\mathrm{M}_n(\mathbb{R})^{+,+}$ of $\mathrm{GL}_n(\mathbb{R})$, this map sends b to b^2. We claim that for any b in $\mathrm{M}_n(\mathbb{R})^{+,+}$, there is a unique c in $\mathrm{M}_n(\mathbb{R})^{+,+}$ such that $c^2 = b$, and we will denote it by $\sqrt{b}$. To prove the claim, note that if c is as claimed, then c is diagonalizable simultaneously with b (since they commute, as $b = c^2$), and that the diagonal coefficients of c are then the (positive) square roots of those of b; this already proves uniqueness, the existence is follows from the fact that b is diagonalizable with positive diagonal coefficients. Hence we have indeed that every $\mathrm{O}_n(\mathbb{R})$-orbit intersects $\mathrm{M}_n(\mathbb{R})^{+,+}$ in exactly one point. To show that the intersections are transversal, let us check that the derivative of $\mathrm{M}_n(\mathbb{R})^{+,+} \to \mathrm{M}_n(\mathbb{R})^{+,+}$,

$b \mapsto b^2$, is everywhere injective (hence bijective). So compute, for b in $\mathrm{M}_n(\mathbb{R})^{+,+}$ and c in $\mathrm{M}_n(\mathbb{R})^+$:

$$(b + \varepsilon c)^2 = b^2 + \varepsilon(bc + cb).$$

Suppose that $bc + cb = 0$. We want to conclude that $c = 0$. Since b is diagonalizable, it suffices to show that $c(x) = 0$ for each eigenvector x of b. So suppose that $b(x) = \lambda x$. Then $b(c(x)) = -c(b(x)) = -\lambda c(x)$. But since b is positive definite, all its eigenvalues are positive. Hence indeed $c(x) = 0$. □

8.2 Corollary. *Let $n \geq 1$. The Lie groups $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{O}_n(\mathbb{R})$ are homotopy equivalent. The Lie groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SO}_n(\mathbb{R})$ are homotopy equivalent.*

Proof. Note that $\mathrm{M}_n(\mathbb{R})^{+,+}$ is convex, hence contractible. Note that $\mathrm{SO}_n(\mathbb{R})$ is the connected component of 1 of $\mathrm{O}_n(\mathbb{R})$, and that $\mathrm{SL}_n(\mathbb{R}) \times \mathbb{R}^{*,+}$ is the connected component of 1 of $\mathrm{GL}_n(\mathbb{R})$. □

8.3 Proposition. *Let $n \geq 1$. Let $\mathrm{M}_n(\mathbb{C})^{+,+}$ be the set of positive definite hermitian elements of $\mathrm{M}_n(\mathbb{C})$, i.e., the h such that $(x, y) \mapsto x^{\mathrm{t}} h \overline{y}$ is a hermitian inner product. Let $\mathrm{M}_n(\mathbb{C})^{+,+,1}$ be the subset of $\mathrm{M}_n(\mathbb{C})^{+,+}$ consisting of the h with $\det(h) = 1$. Then the maps:*

$$\mathrm{M}_n(\mathbb{C})^{+,+} \times \mathrm{U}_n(\mathbb{R}) \longrightarrow \mathrm{GL}_n(\mathbb{C}), \quad \mathrm{M}_n(\mathbb{C})^{+,+,1} \times \mathrm{SU}_n(\mathbb{R}) \longrightarrow \mathrm{SL}_n(\mathbb{C})$$

are diffeomeorphisms.

Proof. The set $\mathrm{M}_n(\mathbb{C})^{+,+}$ is the set of matrices that represent positive definite hermitian forms on $\mathbb{C}^n$. Hence $\mathrm{GL}_n(\mathbb{C})$ acts on it by $(g, h) \mapsto g h \overline{g}^{\mathrm{t}}$. This action is transitive and the stabilizer of 1 is $\mathrm{U})n(\mathbb{R})$. Hence we have a bijection from $\mathrm{GL}_n(\mathbb{C})/\mathrm{U}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{C})^{+,+}$, induced by the map $g \mapsto g\overline{g}^{\mathrm{t}}$. The restriction of this map to $\mathrm{M}_n(\mathbb{C})^{+,+}$ is the map $h \mapsto h^2$. As in the real case, one proves that the map from $\mathrm{M}_n(\mathbb{C})^{+,+}$ to itself, that sends h to h^2, is a diffeomorphism. Hence the first map of the theorem is a diffeomorphism.

To see that the second map is bijective, use that for h in $\mathrm{M}_n(\mathbb{C})^{+,+}$ one has $\det(h)$ positive and real, and that for u in $\mathrm{U}_n(\mathbb{R})$, one has $\det(u)$ in S^1. To see that the map is a diffeomorphism, one can show again that the tangent maps of $h \mapsto h^2$ are again injective (and hence bijective), or one can use that the short exact sequences

$$\{1\} \longrightarrow \mathrm{SL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^* \longrightarrow \{1\}$$
$$\{1\} \longrightarrow \mathrm{SU}_n(\mathbb{C}) \longrightarrow \mathrm{U}_n(\mathbb{C}) \xrightarrow{\det} \mathrm{S}^1 \longrightarrow \{1\}$$

are splittable, and hence give isomorphisms of varieties:

$$\mathrm{SL}_n(\mathbb{C}) \times \mathbb{C}^* \cong \mathrm{GL}_n(\mathbb{C}), \quad \mathrm{SU}_n(\mathbb{R}) \times \mathrm{S}^1 \cong \mathrm{U}_n(\mathbb{R}).$$

$\square$

8.4 Corollary. *Let $n \geq 1$. The Lie groups $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{U}_n(\mathbb{R})$ are homotopy equivalent. The Lie groups $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{SU}_n(\mathbb{R})$ are homotopy equivalent.*

Proof. Note that $\mathrm{M}_n(\mathbb{C})^{+,+}$ is convex, hence contractible. Moreover, it is diffeomorphic to $\mathrm{M}_n(\mathbb{C})^{+,+,1} \times \mathbb{R}^{*,+}$, hence $\mathrm{M}_n(\mathbb{C})^{+,+,1}$ is also homotopy equivalent to a point. $\square$

8.5 Definition. Let $f\colon E \to B$ be a morphism of manifolds. Then f is called a fiber bundle if for every b in B there is an open neighborhood U, a manifold F and an isomorphism $g\colon F \times U \to f^{-1}U$ such that $f \circ g$ is the projection $p_1\colon F \times U \to U$. A fiber bundle is a submersion, obviously, and one has F isomorphic to $f^{-1}\{b\}$. If one can take the same F for all b in B (for example, when B is connected), then f is called a fiber bundle with fiber F. The manifold B is called the base, and E is called the total space.

8.6 Some fiber bundles involving Lie groups

For G a Lie group and H a closed sub Lie group, the quotient map $f\colon G \to G/H$ is a fiber bundle with fiber H. Let us give some examples where the quotient map has some geometric interpretation.

Let $n \geq 1$. We let $\mathrm{SO}_n(\mathbb{R})$ act on $\mathbb{R}^n$, by matrix multiplication. Since it acts by isometries that preserve the origin, we get an action of $\mathrm{SO}_n(\mathbb{R})$ on the unit sphere S^{n-1} in $\mathbb{R}^n$. This action on S^{n-1} is transitive (for x in S^{n-1}, we have $\mathbb{R}^n = \mathbb{R}{\cdot}x \oplus (\mathbb{R}{\cdot}x)^\perp$, an orthogonal direct sum decomposition, hence there is an orthonormal basis $(y_1, \ldots, y_n)$ of $\mathbb{R}^n$ with $y_1 = x$). The stabilizer of the last standard basis vector e_n is the subgroup $\mathrm{SO}_{n-1}(\mathbb{R})$. It follows that the map $f\colon \mathrm{SO}_n(\mathbb{R}) \to \mathrm{S}^n$, $g \mapsto g(e_n)$, is a quotient for the right action of $\mathrm{SO}_{n-1}(\mathbb{R})$ on $\mathrm{SO}_n(\mathbb{R})$ by right translations. In particular, f is a fiber bundle with fibre $\mathrm{SO}_{n-1}(\mathbb{R})$. Repeating this argument for $\mathrm{SU}_n(\mathbb{R})$ acting on $\mathbb{C}^n$ gives the following result, that will allow us to apply induction on n to compute fundamental groups.

8.7 Proposition. *For $n \geq 1$, $\mathrm{SO}_n(\mathbb{R})$ admits a morphism to S^n that is a fiber bundle with fiber $\mathrm{SO}_{n-1}(\mathbb{R})$, and $\mathrm{SU}_n(\mathbb{R})$ admits a morphism to S^{2n-1} that is a fiber bundle with fiber $\mathrm{SU}_{n-1}(\mathbb{R})$.*

8.8 Fiber bundles and homotopy theory

For some details and references to more details on the subject of this subsection, see any book on algebraic topology, for instance [BoTu, III.17], or [Hatc]. Let X be a manifold. Then $\pi_0(X)$ is defined to be the set of connected components. Let $I = [0,1]$ be the unit interval. For $n \geq 1$, a map $f\colon I^n \to X$ is called differentiable if it extends to a differentiable map from some neighborhood of I^n in $\mathbb{R}^n$ to X. Let ∂I^n denote the boundary of I^n; it is a union of 2^n copies of I^{n-1}. Suppose x is in X. Then for $n \geq 1$ the nth homotopy group $\pi_n(X,x)$ of (X,x) is defined to be the set of classes of morphisms $f\colon I^n \to X$ with $f\partial I^n = \{x\}$, up to homotopy that fixes $f|_{\partial I^n}$. The group law on $\pi_n(X,x)$ comes from writing $I^n = I^{n-1} \times I$, and using the map $I \coprod I \to I$ given by $x \mapsto x/2$ on the first copy of I, and $x \mapsto (x+1)/2$ on the second. This gives indeed an associative composition on $\pi_n(X,x)$, and one checks that for $f\colon I^n \to X$ constant on ∂I^n with value x, the map g defined by $g(t_1,\ldots,t_n) = f(t_1,\ldots,t_{n-1},1-t_n)$ gives an inverse of the class of f in $\pi_n(X,x)$. For $n \geq 2$, $\pi_n(X,x)$ is commutative.

The basic result on fiber bundles and homotopy groups that we want to use is the following.

8.8.1 Theorem. *Let $f\colon E \to B$ be a fiber bundle with fiber F. Suppose that B and F are connected, that B is simply connected and that $\pi_2(B,b) = 0$, with b in B. Then E is connected, and the morphism $\pi_1(f^{-1}\{b\},e) \to \pi_1(E,e)$ (with e in $f^{-1}\{b\}$) induced by the inclusion is an isomorphism. In particular, if F is simply connected, then E is.*

8.8.2 Remark. More generally, one has the long exact sequence of homotopy groups:

$$\cdots \longrightarrow \pi_n(F,e) \longrightarrow \pi_n(E,e) \longrightarrow \pi_n(B,b) \stackrel{\partial}{\longrightarrow} \pi_{n-1}(F,e) \longrightarrow \cdots .$$

The morphism $\pi_n(B,b) \to \pi_{n-1}(F,f)$ in this sequence is defined as follows. Let $n \geq 2$, and let $g\colon I^n \to B$ induce an element $\overline{g}$ of $\pi_n(B,b)$. Then view I^n as $I^{n-1} \times I$ and g as a homotopy from $g|_{I^{n-1}\times\{0\}}$ to $g|_{I^{n-1}\times\{1\}}$. Since $f\colon E \to B$ is a fiber bundle, there exists a homotopy $G\colon I^{n-1} \times I \to E$ such that $f \circ G = g$. Then $\partial(\overline{g}) = \overline{G|_{I^{n-1}\times\{1\}}}$.

8.9 Corollary. *We have:*

$$\pi_1(\mathrm{SL}_n(\mathbb{C})) = \pi_1(\mathrm{SU}_n(\mathbb{R})) = 0 \textit{ for } n \geq 2,$$
$$\pi_1(\mathrm{SL}_n(\mathbb{R}) = \pi_1(\mathrm{SO}_n(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z} \textit{ for } n \geq 3, \textit{ and}$$
$$\pi_1(\mathrm{SL}_2(\mathbb{R}) = \pi_1(\mathrm{SO}_2(\mathbb{R}) = \mathbb{Z}.$$

9 Closed subgroups are Lie subgroups

We will prove the following theorem, which is due to von Neumann (in the case of subgroups of $\mathrm{GL}_n(\mathbb{R})$) and E. Cartan in general. The proof is a nice application of the Campbell-Hausdorff theorem up to order two. In order to understand the proof given below, it is instructive to first consider the case of subgroups of $\mathbb{R}$ and then of $\mathbb{R}^n$.

9.1 Theorem. *Let G be a real Lie group, and H is closed subgroup. Then H is a closed Lie subgroup of G.*

Proof. The idea of the proof is simply to look with a microscope at what happens close to the element e of G, via the exponential map and the Campbell-Hausdorff formula up to order two. We choose some inner product $\langle\cdot,\cdot\rangle$ on $\mathrm{Lie}(G)$, and we let U be a small ball of positive radius around 0 in $\mathrm{Lie}(G)$. We take U small enough so that we have (Campbell-Hausdorff formula up to order two):

$$\log(\exp(x)\exp(y)) = x + y + R(x,y),$$
$$\log(\exp(x)\exp(y)\exp(-x)\exp(-y)) = [x,y] + S(x,y),$$

with $\|R(x,y)\| \leq c\max(\|x\|,\|y\|)^2$ and $\|S(x,y)\| \leq c\max(\|x\|,\|y\|)^3$ for all x and y in U, with c some real number. Let $L = \{v \in U \mid \exp(v) \in H\}$ in $\mathrm{Lie}(G)$. We have to show that, for U small enough, L is a sub Lie algebra of $\mathrm{Lie}(G)$. What we know about L is that it is closed in U and that it is closed under the operation $(x,y) \mapsto \log(\exp(x)\exp(y))$, whenever the result is in U. In particular, for x in L and n in $\mathbb{Z}$ such that nx is in U, we have $nx \in L$.

We define a subset T of $\mathrm{Lie}(G)$ that is our candidate for $\mathrm{Lie}(H)$. We let T be the union of $\{0\}$ and the set of non-zero v in $\mathrm{Lie}(G)$ for which there exists a sequence x in $L - \{0\}$ that converges to 0 and such that the sequence $n \mapsto \|x_n\|^{-1}x_n$ on the unit sphere of $\mathrm{Lie}(G)$ converges to $\|v\|^{-1}v$. Note that T does not depend on the choice of the inner product, and also not on the choice of U, and that it is a cone: if t is in T and λ is in $\mathbb{R}$, then λt is in T.

9.2 Lemma. *Supppose that t is in T. Then $\exp(t)$ is in H. In particular, $T \cap U$ is in L.*

Proof. Let $t \neq 0$ be in T. Let $\varepsilon > 0$ be small. By the definition of T, we can take $y \neq 0$ in L such that $\|y\| < \varepsilon$ and $\|\|y\|^{-1}y - \|t\|^{-1}t\| < \varepsilon$. Let n be an integer that is closest to $\|t\|/\|y\|$. Then:

$$\|t - ny\| = \|t\|\cdot \big\|\|t\|^{-1}t - \|t\|^{-1}ny\big\| = \|t\|\cdot \big\|\|t\|^{-1}t - \|y\|^{-1}y + \|y\|^{-1}y - \|t\|^{-1}ny\big\|$$
$$\leq \|t\|\cdot \big\|\|t\|^{-1}t - \|y\|^{-1}y\big\| + \|y\|\cdot \big|\|t\|\cdot\|y\|^{-1} - n\big| \leq (\|t\|+1)\varepsilon.$$

Hence $\exp(t)$ is arbitrarily close to elements of H, and since H is closed in G, we have $\exp(t) \in H$. □

9.3 Lemma. *The cone T is a sub Lie algebra of* $\mathrm{Lie}(G)$.

Proof. Let x and y be in T. We have to show that $x+y$ and $[x,y]$ are in T. If $x+y=0$ or $[x,y]=0$ this is true, so we may assume that such is not the case. Since T is a cone, we may suppose that x and y are small, hence in $T \cap U$ and hence in L. But then, for $\lambda \leq 1$ in $\mathbb{R}$ we have:

$$L \ni \log(\exp(\lambda x)\exp(\lambda y)) = \lambda(x+y) + R(\lambda x, \lambda y),$$

which shows that $x+y$ is in T (since $R(\lambda x, \lambda y) = O(\lambda^2)$, the unit vectors associated to $x+y$ and $\log(\exp(\lambda x)\exp(\lambda y))$ get arbitrarily close as λ tends to zero). A similar argument shows that $[x,y]$ is in T. □

9.4 Lemma. *The subset $T \cap U$ of L is open and closed.*

Proof. Since T is a subspace of $\mathrm{Lie}(G)$, the subset $T \cap U$ of U is closed. It remains to prove that $T \cap U$ is open in L. Suppose not. Then there is a sequence x in $L - T$ that converges to some element t in $T \cap U$. After translating the whole sequence $\exp(x)$ by $\exp(-t)$ (which is in H) and taking log of that, we may assume that x converges to zero. We write $\mathrm{Lie}(G) = T \oplus T^\perp$. For every u in U sufficiently small, there are unique t in T and p in $T^\perp$ such that $u = \log(\exp(t)\exp(p))$. For n sufficiently large, let $t(n)$ and $p(n)$ be defined by $x(n) = \log(\exp(t(n))\exp(p(n)))$. Then the sequence p in $T^\perp$ is in L. Since the unit sphere in $T^\perp$ is compact, we can replace p by a subsequence such that $n \mapsto \|p(n)\|^{-1}p(n)$ converges to some element, say v. Then v is in T, but also in $T^\perp$, and is non-zero. This contradiction shows that $T \cap U$ is open in L. □

We have now shown that $H \cap \exp(U)$ is a real analytic submanifold of $\exp(U)$ (by applying exp to $L = \exp^{-1} H \cap \exp(U)$ in U). But then H is a real analytic submanifold of G at every h in H: for every h in H there exists an open neighborhood U of h in G such that $H \cap U$ is an analytic closed sub manifold of U. Since H is closed in G, it is a closed analytic sub manifold of G. □

9.5 Corollary. *Let $f\colon G_1 \to G_2$ be a continuous morphism between two real Lie groups. Then f is real analytic.*

Proof. Apply the preceding Theorem to the graph of f. □

In some other texts (such as [Vara]) the proof that one gives for the Theorem above uses the following proposition, that we state because it might be of some interest.

9.6 Proposition. *Let G be a Lie group, and let x and y be convergent sequences in* $\mathrm{Lie}(G)$*. Then the sequences:*

$$n \mapsto (\exp(x_n/n)\exp(y_n/n))^n \quad \textit{and}$$
$$n \mapsto (\exp(x_n/n)\exp(y_n/n)\exp(-x_n/n)\exp(-y_n/n))^{n^2}$$

in G converge and one has:

$$\lim_{n\to\infty}(\exp(x_n/n)\exp(y_n/n))^n = \exp\left(\lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n)\right),$$
$$\lim_{n\to\infty}(\exp(x_n/n)\exp(y_n/n)\exp(-x_n/n)\exp(-y_n/n))^{n^2} = \exp\left([\lim_{n\to\infty} x_n, \lim_{n\to\infty} y]\right).$$

Proof. Let U be as in the beginning of the proof of the Theorem. Let x and y be sequences as in the Lemma. Then, for n large enough, x_n/n and y_n/n are in U, and we have:

$$n\log(\exp(x_n/n)\exp(y_n/n)) = x_n + y_n + nR(n^{-1}x_n, n^{-1}y_n).$$

Now note that $nR(n^{-1}x_n, n^{-1}y_n) = O(n^{-1})$. This implies the statements in the Lemma concerning the first sequence. For the second sequence, one uses a similar argument. □

10 Representations of SL_2, SU_2 and SO_3

10.1 Representations

For G a group, k a field and V a k-vector space, a (k-linear) action of G on V is an action $G \times V \to V$ of G on the set V such that for each g in G the map $g\cdot\colon V \to V$, $v \mapsto g{\cdot}v$ is k-linear. Equivalently, an action of G on V is a morphism of groups $\alpha\colon G \mapsto \mathrm{GL}(V)$ (to pass from one to the other, define $\alpha(g)$ to be $v \mapsto g{\cdot}v$ and define $G \times V \to V$ by $(g, v) \mapsto (\alpha(g))v$. A *representation* of G over k is defined to be a k-vector space with an action by G. A *morphism* from a representation V to a representation V' of the same group G is a k-linear map $f\colon V \to V'$ that commutes with the actions of G on V and V': for all v in V and g in G one has $f(gv) = g(fv)$. In old fashioned language, such maps f are called intertwining operators. For a group G, one has the category of its representations. In particular, we say that two representations V and V' are isomorphic if they are so in the categorical sense, or, equivalently, if there is an isomorphism $f\colon V \to V'$ of k-vector spaces that commutes with the G-action. Yet an other way to say that two representations V and V' of G are isomorphic is to say that there are bases of V and V' such that for each g in G the matrices of $V \to V\colon\ v \mapsto gv$ and $V' \to V'\colon\ v' \mapsto gv'$ are equal.

An example of a representation of a group G is obtained as follows. Let X be a set with a G-action $G \times X \to X$, let k be a field, and let V be the k-vector space of all k-valued functions on X. Then G acts on V via $(g \cdot f)x = f(g^{-1}x)$. Indeed, this defines a k-linear action (exercise for the reader). Even better: V is a k-algebra, and G acts via k-algebra automorphisms. For $X = G$, with action the left translations, this representation is called the regular representation. Instead of taking all functions on V, one can also take suitable subspaces, such as differentiable functions, or functions with compact support, if X has the necessary structure to define these notions.

For G be a real Lie group and V be a finite dimensional $\mathbb{R}$-vector space, we say that a representation of G on V is differentiable if the map $G \times V \to V$ is, or, equivalently, if the morphism of groups $G \to \mathrm{GL}(V)$ is a morphism of Lie groups. When G is a Lie group, we will always assume representations to be differentiable. The reason that we limit ourselves to finite dimensional representations of Lie groups is not that infinite dimensional ones are not interesting; on the contrary. But for compact groups, the finite dimensional ones are sufficient for all purposes that I know.

10.2 The representations $k[x,y]_d$ of $\mathrm{SL}_2(k)$

Let k be an infinite field, and let $k[x,y]$ be the polynomial algebra in two variables over k. In order to define an action by $\mathrm{SL}_2(k)$ on $k[x,y]$ we view $k[x,y]$ as the set of polynomial functions on k^2. (Indeed, as k is infinite, the notions of polynomial and polynomial function are the same, i.e., two polynomials define the same function if and only if they are equal.) We let $\mathrm{SL}_2(k)$ act on k^2 in the standard way: $(g,a) \mapsto ga$, multiplication of matrix times column vector. Then we have an action by $\mathrm{SL}_2(k)$ on the k-vector space F of all k-valued functions on k^2 (a huge space) given as in the previous section: $(gf)a = f(g^{-1}a)$.

We write out explicitly what happens here. So let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, and let $a = (u,v)$. Then $g^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$, hence $g^{-1}a = (du-bv, -cu+av)$, hence $(gf)(u,v) = f(du-bv, -cu+av)$. The last formula shows that if f is polynomial, then gf is so, too. Hence we have a representation of $\mathrm{SL}_2(k)$ on the sub k-algebra $k[x,y]$ of F, and even better, this action preserves the total degree, i.e., the direct sum decomposition:

$$k[x,y] = \bigoplus_{d \geq 0} k[x,y]_d. \tag{10.2.1}$$

Let us describe explicitly what $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ does with x and y (this is useful, as x and y generate $k[x,y]$). So we compute:

$$\begin{aligned}(gx)(u,v) &= x(du-bv, -cu+av) = du-bv = (dx-by)(u,v)\\ (gy)(u,v) &= y(du-bv, -cu+av) = -cu+av = (-cx+ay)(u,v),\end{aligned}$$

which means:

$$gx = dx - by, \quad gy = -cx + ay.$$

Alas, this is not the formula that I was hoping for. I would have liked to have: $gx = ax+cy$ and $gy = bx+dy$. The reason that I wanted these formulas is that this would have given me the standard two dimensional representation on $k[x,y]_1$ with respect to the basis (x,y).

Now we can do two things. The first one is to change the action of $\mathrm{SL}_2(k)$ on $k[x,y]$, and the other one is to simply choose an other set of generators of $k[x,y]$. We will explain both ways. Actually, let us start by noting that we cannot get the formulas $gx = ax+by$ and $gy = cx+dy$ for the very good reason that those do not give a left action on $k[x,y]$ but a right action (exercise for the reader; I should add, many people get confused by this).

The first option: change the action. Instead of defining $(gf)a = f(g^{-1}a)$ we define $(gf)a = f(g^{\mathrm{t}}a)$, with g^{t} the transpose of g. It is left to the reader that this leads indeed to $gx = ax+cy$ and $gy = bx+dy$.

The second option: define $u = y$, $v = -x$. Then we have $k[x,y] = k[u,v]$, and: $gu = au + cv$ and $gv = bu + dv$.

The fact that both ways give us an action that is defined by the same formulas means that the two actions $(gf)a = f(g^{-1}a)$ and $(gf)a = f(g^{\mathrm{t}}a)$ give isomorphic representations. What is behind this is that for every g in $\mathrm{SL}_2(k)$ we have:

$$(g^{\mathrm{t}})^{-1} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) g \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)^{-1}.$$

From now on, we will just write $k[x,y]_d$ for the representation of $\mathrm{SL}_2(k)$ on $k[x,y]_d$ given by the formulas we wanted: $gx = ax + cy$ and $gy = bx + dy$. To finish this section, I want to remark that for $n \geq 3$ there is no longer an element c in $\mathrm{SL}_n(k)$ such that $(g^{\mathrm{t}})^{-1} = cgc^{-1}$ for all g in $\mathrm{SL}_n(k)$. This is easy to see from the characteristic polynomials: cgc^{-1} has the same one as g, but $(g^{\mathrm{t}})^{-1}$ usually does not, for example, for diagonal matrices.

10.3 The $k[x,y]_d$ are irreducible

A representation on a k-vector space V of a group G is called *irreducible* or *simple* if it has exactly two subspaces that are invariant under the action of G ($\{0\}$ and V; in particular, V is not the zero space). In terms of matrices, V being reducible means that, with respect to a suitable basis, all matrices given by the action of elements of G are of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$, with blocks of positive size.

10.3.1 Theorem. *Let k be a field of characteristic zero. Then the representations $k[x,y]_d$ of $\mathrm{SL}_2(k)$ are irreducible for all $d \geq 0$.*

Proof. We will give the proof for the case $k = \mathbb{C}$, as we want to use some Lie algebra arguments (we will do some infinitesimal computations). But as the action of $\mathrm{SL}_2(k)$ on each $k[x,y]_d$ is given by a polynomial map from $\mathrm{SL}_2(k) \times k[x,y]_d$ to $k[x,y]_d$, such infinitesimal computations can be done over any field.

Let $d \geq 0$, and let V be a non-zero subspace of $\mathbb{C}[x,y]_d$ that is invariant for the action of $\mathrm{SL}_2(\mathbb{C})$. We have to show that V is equal to $\mathbb{C}[x,y]_d$. In order to do this, we will use that V is stable for the action of the Lie algebra $L := \mathrm{Lie}(\mathrm{SL}_2(\mathbb{C}))$. So first we compute how L acts on $\mathbb{C}[x,y]$. We recall that L is the subset of $\mathrm{M}_2(\mathbb{C})$ consisting of those elements whose trace equals zero. Hence we have the following $\mathbb{C}$-basis for L:

$$L = \mathrm{Lie}(\mathrm{SL}_2(\mathbb{C})) = \mathbb{C}h \oplus \mathbb{C}a_+ \oplus \mathbb{C}a_-, \tag{10.3.2}$$

where:

$$h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad a_+ = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad \text{and} \quad a_- = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right). \tag{10.3.3}$$

The Lie bracket is given by:

$$[h, a_+] = 2a_+, \quad [h, a_-] = -2a_-, \quad [a_+, a_-] = h. \tag{10.3.4}$$

The action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}[x, y]$ gives a representation of L on $\mathbb{C}[x, y]$ such that for all a in L and f in $\mathbb{C}[x, y]$ we have (modulo ε^2):

$$(1 + \varepsilon a)f = f + \varepsilon(af). \tag{10.3.5}$$

We compute the endomorphisms of the $\mathbb{C}$-vector space $\mathbb{C}[x, y]$ given by h, a_+ and a_-. Applying the definitions gives that for all (i, j) in $\mathbb{N}^2$ we have:

$$\begin{aligned}
(1 + \varepsilon h)x^i y^j &= ((1 + \varepsilon)x)^i((1 - \varepsilon)y)^j = x^i y^j + \varepsilon(i - j)x^i y^j \\
(1 + \varepsilon a_+)x^i y^j &= x^i(\varepsilon x + y)^j = x^i y^j + \varepsilon j x^{i+1} y^{j-1} \\
(1 + \varepsilon a_-)x^i y^j &= (x + \varepsilon y)^i y^j = x^i y^j + \varepsilon i x^{i-1} y^{j+1}.
\end{aligned}$$

Note that the terms $jx^{i+1}y^{j-1}$ and $ix^{i-1}y^{j+1}$ are zero if some exponent is negative. By looking at the terms with a factor ε, we get:

$$h\, x^i y^j = (i - j)x^i y^j, \quad a_+\, x^i y^j = jx^{i+1}y^{j-1}, \quad a_-\, x^i y^j = ix^{i-1}y^{j+1}. \tag{10.3.6}$$

With these formulas at our disposal, we can now easily prove that the $\mathbb{C}[x, y]_d$ are irreducible. Recall that V is a non-zero subspace of $\mathbb{C}[x, y]_d$ that is stable for the action of $\mathrm{SL}_2(\mathbb{C})$. The formula for the action of L on elements f makes it clear that for v in V and a in L we have $av \in V$. So now let $v = \sum v_{i,j}x^i y^j$ be a non-zero element in $\mathbb{C}[x, y]_d$. Take i minimal with the property that $v_{i,j} \neq 0$. Then $a_+^{d-i}v$ is a non-zero element of $\mathbb{C}x^d$ (here that we use that the field over which we work is of characteristic zero). Hence V contains x^d. But then V contains all the $a_-^j x^d$, hence all the $x^{d-j}y^j$ (here we use again that the field is of characteristic zero). □

10.3.7 Remark. The hypothesis that k is of characteristic zero is necessary, as we see by the following example. Let k be a field of characteristic p (for example, $\mathbb{F}_p$). Then the subspace generated by x^p and y^p of $k[x, y]_p$ is $\mathrm{SL}_2(k)$-invariant.

10.3.8 Remark. It is interesting to remark that h, a_+ and a_- act on $\mathbb{C}[x, y]$ as derivations, since for all f in $\mathbb{C}[x, y]$ we have:

$$hf = (x\partial/\partial x - y\partial/\partial y)f, \quad a_+ f = (x\partial/\partial y)f, \quad a_- f = (y\partial/\partial x)f. \tag{10.3.9}$$

We can see this by considering only f of the form $x^i y^j$, but of course these formulas can also be derived directly from the interpretation of h, a_+ and a_- acting by some kind of infinitesimal transformations of $\mathbb{C}^2$. The fact that a Lie group acting on an algebra leads to the Lie algebra acting via derivations is true in general. Derivations from an algebra to itself should be interpreted as infinitesimal automorphisms. Indeed, a vector field on a manifold is the first order approximation of its flow.

In fact, the proof of the irreducibility of the $\mathbb{C}[x,y]_d$ that we have given contained the following intermediate result, which is very useful by itself, for example because it works in the same way for all fields k of characteristic zero.

10.3.10 Theorem. *Let k be a field of characteristic zero. Then all $k[x,y]_d$ are irreducible as representations of the Lie algebra L consisting of matrices of trace zero in $\mathrm{M}_2(k)$.*

The fact that the $\mathbb{R}[x,y]_d$ are irreducible representations of $\mathrm{SL}_2(\mathbb{R})$ follows directly from this theorem, using that a $\mathrm{SL}_2(\mathbb{R})$ invariant subspace is a subrepresentation for $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{R}))$.

10.4 The representations of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$.

10.4.1 Theorem. *Every (finite dimensional) representation (of Lie groups) of $\mathrm{SL}_2(\mathbb{R})$ is isomorphic to a direct sum of copies of the $\mathbb{R}[x,y]_d$. Every complex (finite dimensional) representation of $\mathrm{SL}_2(\mathbb{C})$ (i.e., a finite dimensional $\mathbb{C}$-vector space V with a given morphism of complex Lie groups $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}(V)$) is isomorphic to a direct sum of copies of the $\mathbb{C}[x,y]_d$. In other words, the $\mathbb{R}[x,y]_d$ and $\mathbb{C}[x,y]_d$ give all the irreducible representations, and each representation is semi-simple (also called completely reducible) in the sense that it is a direct sum of irreducible representions.*

The analogous statements hold for finite dimensional representations of the Lie algebra of trace zero elements in $\mathrm{M}_2(k)$, if k is a field of characteristic zero.

Proof. We begin with a little bit of terminology. Let k be a field of characteristic zero, let L denote the Lie algebra of trace zero elements in $\mathrm{M}_2(k)$. Recall that L has a basis (h, a_+, a_-) as above. For V be a finite dimensional representation of L and for λ in k we say that an element v of V is of weight λ if $hv = \lambda v$, and we let $V(\lambda)$ denote the subspace of V of elements of weight λ. Then we have the following result.

10.4.2 Lemma. *Let v be of weight λ. Then a_+v is of weight $\lambda+2$, and a_-v is of weight $\lambda-2$.*

Proof. We have already seen that $[h, a_+] = 2a_+$. Hence:

$$h(a_+v) = a_+(hv) + 2a_+v = (\lambda+2)a_+v.$$

The argument for a_- is similar. $\square$

The essential step in the proof of the theorem is the next proposition.

10.4.3 Proposition. *Let V be a finite dimensional representation of L, let v be a non-zero element of V of some weight λ, such that $a_+v = 0$. Then λ is an integer $d \geq 0$, and the subrepresentation of V generated by v is isomorphic to $k[x,y]_d$.*

Proof. For $i \geq 0$, let $v_i := a_-^i v$. Then we have $v_i \in V(\lambda - 2i)$ by the previous Lemma. We will now prove, by induction on $i \geq 0$, what we can guess from applying the derivations a_+ and a_- formally to x^λ, namely that:

$$a_+v_i = i(\lambda - i + 1)v_{i-1}.$$

This is true by the hypotheses of the proposition for $i = 0$. Since $h = a_+a_- - a_-a_+$ we have, for $i \geq 1$:

$$\begin{aligned} a_+v_i = a_+a_-v_{i-1} = (h + a_-a_+)v_{i-1} = (\lambda + 2 - 2i)v_{i-1} + a_-(i-1)(\lambda - i + 2)v_{i-2} = \\ = (\lambda + 2 - 2i + (i-1)(\lambda - i + 2))v_{i-1} = i(\lambda - i + 1)v_{i-1}. \end{aligned}$$

Note that the for $i = 1$ the occurrence of v_{i-2} is no problem because of the factor $i - 1$.

The v_i are in different eigenspaces for h. Since V is finite dimensional, v_i must be zero for all but finitely many i. Let d be the smallest integer ≥ 0 such that $v_{d+1} = 0$. Then v_d is not zero, and hence $\lambda - d = 0$. It follows that $(v_0, \ldots, v_d)$ is a basis for the representation generated by v. The proof is now finished by comparing what happens in $k[x,y]_d$ and $v := x^d$. □

10.4.4 Proposition. *Let V be an irreducible finite dimensional representation of L. Then V is isomorphic to some $k[x,y]_d$. In particular, V is absolutely irreducible in the sense that for $k \to \overline{k}$ an algebraic closure, $\overline{k} \otimes_k V$ is irreducible as representation of $\overline{k} \otimes_k L$, and hence (Schur's Lemma) $\mathrm{End}_L(V) = k$.*

Proof. Let V be such a representation. Applying the previous proposition to the representation $\overline{k} \otimes_k V$ of $\overline{k} \otimes_k L$, with $k \to \overline{k}$ an algebraic closure, shows that the roots of the characteristic polynomial of h acting on V are all integers. (Namely: for such a root λ, take a non-zero element v in $\ker(h - \lambda)$, and consider the $a_+^i v$.)

Let d be the largest among the roots of the characteristic polynomial of h acting on V. Then d is called the *highest weight* of V. Let v be a non-zero element of $V(d)$. Then we have $a_+v = 0$ because a_+v is in $V(\lambda + 2) = 0$. Hence the proposition gives us an injective morphism of representations $f\colon k[x,y]_d \to V$, sending x^d to v. As V is irreducible, f is an isomorphism.

Theorem 10.3.10 says that for each field k of characteristic zero and for each $d \geq 0$, $k[x,y]_d$ is irreducible as L-module. Hence V is absolutely irreducible. The subspace $\mathrm{End}_L(V) \subset \mathrm{End}_k(V)$ of L-module endomorphisms of V is defined by linear equations, hence any basis of it is also a basis of the subspace $\mathrm{End}_{\overline{k}\otimes_k L}(\overline{k}\otimes_k V)$ of $\mathrm{End}_{\overline{k}}(\overline{k}\otimes_k V)$. But each element of $\mathrm{End}_{\overline{k}\otimes_k L}(\overline{k}\otimes_k V)$ is scalar, because its eigenspaces are subrepresentations. Hence $\mathrm{End}_L(V) = k$. □

Before we go on with the proof of Theorem 10.4.1, we need some definitions.

10.4.5 Definition. Let k be a field, V a k-vector space, and $V^* = \mathrm{Hom}_k(V,k)$ its dual. If a group G acts on V, then it also acts from the right on V^* by $(l,g) \mapsto lg = v(\mapsto l(gv))$. Such a right action on V^* can be transformed in a left action by defining:

$$gl := lg^{-1} = (v \mapsto l(g^{-1}v).$$

With this action, V^* is called the *dual* or *contragredient* representation of V. If V is an A-module for a k-Lie algebra A, then V^* is a right A-module via: $(l,a) \mapsto la = (v \mapsto l(av))$. In order to transform this right A-module structure into a left A-module, one defines:

$$al := -la = (v \mapsto -l(av)).$$

The same terminology applies here: with this A-module structure, V^* is called the *dual* or *contragredient* representation of V.

More generally, if V and W are representations of a group G, then the k-vector space $\mathrm{Hom}_k(V,W)$ of k-linear maps from V to W is a representation of G if we define:

$$(g{\cdot}f)v = g(f(g^{-1}v).$$

Similarly, if V and W are A-modules, for a Lie algebra A, then $\mathrm{Hom}_k(V,W)$ gets an A-module structure if we define:

$$(a{\cdot}f)(v) = a(f(v)) - f(av).$$

The proof of the last part of Theorem 10.4.1 is completed by the next result.

10.4.6 Proposition. *Each finite dimensional representation V of L is isomorphic to a direct sum of irreducible ones.*

Proof. Induction on $\dim(V)$. (In fact, for those who know some homological algebra, the usual short exact sequences reduce the problem to showing that $\mathrm{Ext}^1_L(k,U) = \{0\}$

for irreducible U, which we will do "by hand".) The result is true for $V = \{0\}$, so we assume that $V \neq \{0\}$. Let W be a non-zero submodule of minimal dimension. Then W is irreducible. Applying $\mathrm{Hom}_k(\cdot, W)$ to the short exact sequence:

$$0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0 \tag{10.4.7}$$

gives a short exact sequence of L-modules:

$$0 \longrightarrow \mathrm{Hom}_k(V/W, W) \longrightarrow \mathrm{Hom}_k(V, W) \longrightarrow \mathrm{Hom}_k(W, W) \longrightarrow 0. \tag{10.4.8}$$

Replacing $\mathrm{Hom}_k(W, W) = \mathrm{End}_k(W)$ by its submodule $\mathrm{End}_L(W) = k$ and $\mathrm{Hom}_k(V, W)$ by the inverse image of k in it gives a short exact sequence of L-modules:

$$0 \longrightarrow \mathrm{Hom}_k(V/W, W) \longrightarrow H \longrightarrow k \longrightarrow 0. \tag{10.4.9}$$

By construction, a splitting $s\colon k \to H$ is this sequence is such that $s(1)$ is an L-module morphism from V to W that splits (10.4.7). The existence of such a splitting therefore finishes the proof of the proposition. We will show by induction on $\dim(U)$ that any short exact sequence of L-modules of the form:

$$0 \longrightarrow U \longrightarrow E \longrightarrow k \longrightarrow 0 \tag{10.4.10}$$

is split. If $U = \{0\}$ then the sequence is split. Suppose now that U is irreducible, isomorphic to $k[x, y]_d$, say. If $d = 0$, then $E = k^2$ and $L = [L, L]$ has image in the commutator algebra of $\{\left(\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right)\}$, hence acts as zero on E, so the sequence is split. Suppose that $d > 0$. Dualising gives a short exact sequence of L-modules:

$$0 \longrightarrow k \longrightarrow E^* \longrightarrow k[x, y]_d^* \longrightarrow 0. \tag{10.4.11}$$

Applying Proposition 10.4.3 to the element $(y^d)^*$ of $k[x, y]_d^*$ (with $((x^d)^*, \ldots, (y^d)^*)$) the basis dual to $(x^d, \ldots, y^d)$) gives an isomorphism from $k[x, y]_d$ to $k[x, y]_d^*$. So now we have $k[x, y]_d$ as a quotient of E^*. The eigenvalues of h on E^* are those on $k[x, y]_d^*$ and on k, hence d is the highest weight of E^*, $E^*(d)$ is of dimension one and has trivial intersection with k so surjects to $k[x, y]_d^*(d)$. Now pick l in $E^*(d)$ whose image in $k[x, y]_d^*$ is $(y^d)^*$. Then we have $a_+ l = 0$ because $a_+ l \in E^*(d + 2) = \{0\}$, so Proposition 10.4.3 says that the subrepresentation W of E^* generated by l is isomorphic to $k[x, y]_d$. Since W and $k[x, y]_d^*$ are irreducible, the map from W to $k[x, y]_d^*$ is an isomorphism, and (10.4.11) and (10.4.10)are split.

Suppose now that U is non-zero and reducible. Let U' be a non-zero proper submodule. Then we have a commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & U' & = & U' & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & U & \longrightarrow & E & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \overline{U} & \longrightarrow & \overline{E} & \longrightarrow & k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{10.4.12}
$$

By induction, the bottom row splits, and replacing $\overline{E}$ by the image of such a splitting and E by the inverse image E' of k via $E \to \overline{E}$ gives an exact sequence

$$0 \longrightarrow U' \longrightarrow E' \longrightarrow k \longrightarrow 0$$

that is again split by induction. So we do obtain a splitting from k to E. $\square$

To prove the first part of the theorem, it suffices to say that the sub representations of V for the action of $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ are precisely the sub L-modules. $\square$

10.5 The representations of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$

As a short intermezzo, we note that the results of the previous section show that each representation of the universal cover $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of $\mathrm{SL}_2(\mathbb{R})$ factors through its quotient $\mathrm{SL}_2(\mathbb{R})$. This provides us with an example of a real Lie group that has no faithful (finite dimensional) representation. Of course, for any Lie group G, the space $C^\infty(G)$ is a faithful representation of G.

10.6 The representations of $\mathrm{SU}_2(\mathbb{R})$

As $\mathrm{SU}_2(\mathbb{R})$ is a subgroup of $\mathrm{SL}_2(\mathbb{C})$, every representation of $\mathrm{SL}_2(\mathbb{C})$ gives by restriction a representation of $\mathrm{SU}_2(\mathbb{R})$. In particular, we have the (complex) representations $\mathbb{C}[x,y]_d$, $d \geq 0$, of $\mathrm{SU}_2(\mathbb{R})$. These are the only ones.

10.6.1 Theorem. *Let V be a finite dimensional complex vector space with an action of* $\mathrm{SU}_2(\mathbb{R})$*. Then V is isomorphic to a direct sum of copies of the $\mathbb{C}[x,y]_d$. The same is true for every finite dimensional complex representation of* $\mathrm{Lie}(\mathrm{SU}_2(\mathbb{R}))$*. In fact, restriction from* $\mathrm{SL}_2(\mathbb{C})$ *to* $\mathrm{SU}_2(\mathbb{R})$ *induces an isomorphism from the category of finite dimensional complex representations of* $\mathrm{SL}_2(\mathbb{C})$ *to that of* $\mathrm{SU}_2(\mathbb{R})$*, and the same is true on the level of Lie algebras.*

Proof. As $\mathrm{SU}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ are connected and simply connected, it suffices to prove the very last statement. So let us study the real Lie algebra $L' := \mathrm{Lie}(\mathrm{SU}_2(\mathbb{R}))$, just as we have studied the Lie algebra of $\mathrm{SL}_2(k)$ for any field k. As $\mathrm{SU}_2(\mathbb{R})$ is the subset of matrices of the form $\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ of $\mathrm{M}_2(\mathbb{C})$ with $|a|^2+|b|^2=1$, we have:

(10.6.2)
$$L' = \{\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right) \mid a \in \mathbb{R}i,\ b \in \mathbb{C}\} = \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k,$$

where:

(10.6.3)
$$i = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right), \quad j = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \quad k = \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right).$$

From our computations with quaternions (from which we have taken the notation i, j and k), we have (without new matrix computations):

(10.6.4)
$$[i,j] = 2k, \quad [j,k] = 2i, \quad [k,i] = 2j.$$

These formulas give us a complete description of L'. We are now in for a pleasant surprise.

10.6.5 Proposition. *The inclusion of L' in $L_\mathbb{C} = \mathrm{Lie}(\mathrm{SL}_2(\mathbb{C}))$ induces an isomorphism from the complexification $L'_\mathbb{C} = \mathbb{C} \otimes_\mathbb{R} L'$ of L' to $L_\mathbb{C}$.*

Proof. By definition, the complexification $L'_\mathbb{C}$ is the $\mathbb{C}$-vector space with basis (i,j,k) (or $(1\otimes i, 1\otimes j, 1\otimes k)$ if one wants to avoid confusion), equipped with the Lie bracket given by the formulas (10.6.4) above. On the other hand, it is clear that (i,j,k) is a $\mathbb{C}$-basis of $L_\mathbb{C}$. Since the inclusion of L' into $L_\mathbb{C}$ is a morphism of real Lie algebras, the statement of the proposition is clear. □

Let us now finish the proof of Theorem 10.6.1. In general, if A is a real Lie algebra, then a complex representation V of A is a complex representation of $\mathbb{C} \otimes_\mathbb{R} A$, by the usual properties of tensor products (or just of complexification, if one does not like tensor products). Proposition 10.6.5 implies that the complex representations of L' are the same as those of $L_\mathbb{C}$, and those were described in Theorem 10.4.1. □

10.7 The representations of $\mathrm{SO}_3(\mathbb{R})$

As we have seen in Section 5.10, the Lie group $\mathrm{SO}_3(\mathbb{R})$ is isomorphic to the quotient of $\mathrm{SU}_2(\mathbb{R})$ by its subgroup $\{1, -1\}$. As a consequence, to give a representation of $\mathrm{SO}_3(\mathbb{R})$ is to give a representation of $\mathrm{SU}_2(\mathbb{R})$ that is trivial on $\{1, -1\}$. For $d \geq 0$, the element -1 of $\mathrm{SU}_2(\mathbb{R})$ acts on $\mathbb{C}[x, y]_d$ as $(-1)^d$. That proves the following result.

10.7.1 Theorem. *Viewing* $\mathrm{SO}_3(\mathbb{R})$ *as the quotient of* $\mathrm{SU}_2(\mathbb{R})$ *by* $\{1, -1\}$*, the irreducible complex representations of* $\mathrm{SO}_3(\mathbb{R})$ *are the* $\mathbb{C}[x, y]_d$ *with* $d \geq 0$ *even. All finite dimensional complex representations of* $\mathrm{SO}_3(\mathbb{R})$ *are semi-simple, i.e., direct sums of irreducible ones.*

10.8 Conclusions, references

We have treated the simplest cases of the theory of representations of some simple Lie groups by hand. A good reference for the general theory, of which we have in fact seen all the ingredients, is [Hum]. In the next Section, we will prove by averaging arguments, that all finite dimensional representations of compact Lie groups are semi-simple, just as one does for finite groups. (So this also reproves (in a more simple way) that all representations of $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$ are semi-simple.) An excellent reference for the theory of representations of finite groups is [Ser2]. An interesting topic that we did not discuss in this section is which representations of $\mathrm{SU}_2(\mathbb{R})$ can be realised over the reals, i.e., have all their matrices real, with respect to a suitable basis.

11 Integration on Lie groups

We want to integrate continuous functions with compact support on a Lie group G. What one needs to do that is called a volume form on G. Before choosing a particular kind of volume forms on Lie groups (say left or bi-invariant), we fist discuss what volume forms are on manifolds, and how one integrates them. And, before that, volume forms on finite dimensional real vector spaces. What is behind this is simply that to say what the volume is of a compact subset of a manifold, we first say what it is for a product of intervals in a vector space.

11.1 Volume forms on vector spaces

Let V be a finite dimensional $\mathbb{R}$-vector space, of dimension d, say. A function $\mathrm{Vol}\colon V^d \to \mathbb{R}$ is called a *volume form on* V if it has the following properties:

1. for all λ in $\mathbb{R}^d$ and all v in V^d one has $\mathrm{Vol}(\lambda_1 v_1, \ldots, \lambda_d v_d) = |\lambda_1 \cdots \lambda_d| \mathrm{Vol}(v_1, \ldots, v_d)$;
2. for every σ in S_d and v in V^d one has $\mathrm{Vol}(v_{\sigma(1)}, \ldots, v_{\sigma_d}) = \mathrm{Vol}(v_1, \ldots, v_d)$;
3. for all v in V^d one has (if $d \geq 2$): $\mathrm{Vol}(v_1 + v_2, v_2, \ldots, v_d) = \mathrm{Vol}(v_1, \ldots, v_d)$.

Note that these properties are those for alternating multilinear forms, except that there are no signs. Hence the same arguments as for determinants show that in order to specify a volume form on V is equivalent to give its value on some basis. In particular, if ω is a non-zero alternating multilinear form on V^d, then the volume forms on V are the functions $\mathrm{Vol}_{\omega,\lambda}\colon V^d \to \mathbb{R}$ of the form $v \mapsto \lambda|\omega(v)|$, with λ in $\mathbb{R}$.

11.2 Volume forms on manifolds

Let X be a C^k manifold, with $k \geq 1$. A *volume form on* X then consists of the datum of a volume form Vol_x on $\mathrm{T}_X(x)$ for every x in X. If X is an open subset of $\mathbb{R}^d$, then every volume form Vol on X is of the form $\mathrm{Vol} = f|dx_1 \cdots dx_d|$, where f is a function $X \to \mathbb{R}$ (uniquely determined by Vol) and where $|dx_1 \cdots dx_d|$ has value one on the standard basis $(e_1, \ldots, e_d)$ of $\mathbb{R}^d$. A volume form Vol on X is called C^k if locally it is of the form as above, with f a C^k-function.

The (only) importance of volume forms is that they can be integrated: for Vol a C^0 volume form on a manifold X and for f in $C_c^0(X)$ (continuous functions with compact support) one has $\int_X f\cdot\mathrm{Vol}$ in $\mathbb{R}$ (or in $\mathbb{C}$ if f is complex valued). In order to compute (or to define) this integral, one takes a finite set of coordinate systems X_i on X that cover the support of f, and in each of these coordinate systems one integrates as usual. Then, one does some book keeping to count each contribution exactly once, i.e., one takes the sum of the integrals over the X_i, then subtracts the integrals over the $X_i \cap X_j$, adds the triple intersections, etc. As a matter of notation, we will often denote a volume form by the symbol v (for volume) or by μ (for measure), so that the integrals are denoted by $\int_X fv$ or $\int_X f\mu$. We note that $\int_X fv$ makes sense in fact for any continuous f with compact support on X and with values in a real or complex Banach space (i.e., a complete normed real or complex vector space). (The completeness is necessary so that limits of Riemann sums exist.)

If X is a Riemannian manifold, i.e., a C^∞ manifold equipped with a metric $\langle \cdot, \cdot \rangle$ on its tangent bundle, then we have a natural volume form μ: for x in X say that $\mu(e) = 1$, where e is any orthonormal basis of $\mathrm{T}_X(x)$.

11.3 Invariant volume forms on Lie groups

On an arbitrary manifold, there are no especially nice volume forms, if one has not more structure (as for example a Riemannian structure). The case of Lie groups is nice, as we can use the left translations in order to trivialize the tangent bundle. A volume form v on a Lie group G is called left invariant if it is invariant under left translations, i.e., for every g in G the isomorphism $\mathrm{T}_{l_g} \colon \mathrm{T}_G(e) \to \mathrm{T}_G(g)$ induced by the left translation l_g maps v_e to v_g. The left invariant volume forms (or their effect on $C_c^0(G)$) are called Haar measures. In fact, one can show that every locally compact topological group has such measures, unique up to scalar (and not all zero).

If G is compact, a non-zero left invariant volume form can be uniquely normalized by the condition that $\int_G v = 1$, in which case v is called the *invariant probability measure on G*.

Of course, it is of significant interest whether or not the left invariant volume forms on a Lie group G are also right invariant. This is easily seen to be the case if and only if the representation of G on $\mathrm{Lie}(G)$ induced by conjugation of G on G is such that the composition $G \to \mathrm{GL}(\mathrm{Lie}(G)) \to \mathbb{R}^{*,+}$ obtained by composing with $|\det| \colon \mathrm{GL}(\mathrm{Lie}(G)) \to \mathbb{R}^{*,+}$ is trivial. If this is so, then the volume forms are called bi-invariant. This happens completely automatically in two important cases: when G is compact (since the only compact subgroup of $\mathbb{R}^{*,+}$ is $\{1\}$), and when $[G, G] = G$ (because then every morphism to any commutative group is trivial). The last case occurs for example when G is connected and $[\mathrm{Lie}(G), \mathrm{Lie}(G)] = \mathrm{Lie}(G)$.

Hence the $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$ and so on have bi-invariant volume forms. An example of a group that does not have a non-zero bi-invariant volume form is the subgroup

$\left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix}\right)$ of $\mathrm{GL}_2(\mathbb{R})$, because its action on its Lie algebra composed with det has image $\mathbb{R}^*$.

11.4 Explicit formulas for $\mathrm{GL}_n(\mathbb{R})$

Let $n \geq 0$ and let $G := \mathrm{GL}_n(\mathbb{R})$. Let $L = \mathrm{M}_n(\mathbb{R}))$ denote $\mathrm{Lie}(G))$. We have the coordinate functions $x_{i,j}$ on L and on G ($1 \leq i, j \leq nq$). For g in G, we have $\mathrm{T}_G(g) = L$, simply from the inclusion $G \subset \mathrm{M}_n(\mathbb{R}) = \mathbb{R}^{n^2}$. Note that this identification of $\mathrm{T}_G(g)$ with $L = \mathrm{T}_G(e)$ is not necessarily the same as the one that one gets from left translation by g. In fact, as we have seen when looking at examples of left invariant vector fields, the left invariant vector field D_a corresponding to a in L is given by: $(D_a)_g = ga$ (this follows directly from the equality: $g(1+\varepsilon a) = g + \varepsilon ga$). Hence left translation by g induces the map $g\cdot\colon a \mapsto ga$ from $L = \mathrm{T}_G(e)$ to $\mathrm{T}_G(g) = L$. The determinant of this map is $\det(g)^n$ (note that each column of a behaves as $\mathbb{R}^n$, and that we have n columns). It follows that the volume form given by:

$$\frac{1}{|\det(g)|^n}\left|\prod_{i,j} dx_{i,j}\right| \tag{11.4.1}$$

is left invariant, where $|\prod_{i,j} dx_{i,j}|$ denotes the standard volume form on $L = \mathbb{R}^{n^2}$, i.e., the one that has value one on the standard basis. Note that it is also right invariant because right translation by g also changes the standard volume form $|\prod_{i,j} dx_{i,j}|$ by $|\det(g)|^n$ (use the n rows of a instead of the columns).

11.5 Some computations for $\mathrm{SU}_2(\mathbb{R})$

Let $G := \mathrm{SU}_2(\mathbb{R}) = \{\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right) \mid a, b \text{ in } \mathbb{C} \text{ with } |a|^2 + |b|^2 = 1\}$. By construction, G acts via isometries on $\mathbb{C}^2$ (i.e., preserving the standard hermitian inner product), and hence on the unit sphere S^3 in $\mathbb{C}^2 = \mathbb{R}^4$. But from the matrix description of G we see that the action of G on S^3 is free and transitive, so that $G \to \mathrm{S}^3$, $g \mapsto ge_1$ is a diffeomorphism. It follows that invariant volume forms on G correspond to $\mathrm{SO}_4(\mathbb{R})$-invariant volume forms on S^3. But these are easy to construct using the Riemannian structure on S^3. For example, we have the volume form v on S^3 given by the property that for x in S^3, and (x_1, x_2, x_3) an orthonormal basis of $x^\perp$, $v_x(x_1, x_2, x_3) = 1$.

Let us now compute the volume of G with respect to the volume $\int_G v$ form v, so that the invariant probability measure on G is $\mu := (\int_G v)^{-1} v$. So what we have to compute is the volume of S^3 with respect to v. But that is the same as the derivative at $r = 1$ of the function $B_4\colon r \mapsto \int_{||x||\leq r} |dx_1 \cdots dx_4|$. We compute:

$$B_4(r) = \int_{x=-r}^{r} B_3(\sqrt{r^2-x^2})|dx| = \frac{4}{3}\pi r^4 \int_{-1}^{1} (1-x^2)^{3/2}|dx| = \frac{4}{3}\pi r^4 \int_{-\pi/2}^{\pi/2} \cos(y)^4|dy|$$
$$= \frac{4\pi r^4}{3{\cdot}16} \int_{-pi/2}^{\pi/2} (e^{4ix}+4e^{2ix}+6+4e^{-2ix}+e^{-4ix})|dx| = \frac{1}{2}\pi^2 r^4.$$

It follows that $\int_G v = 2\pi^2$, so that the invariant probability measure on G is $\mu := (1/2\pi^2)v$.

An interesting question that one can ask now is the following: how are the elements of G distributed in the conjugacy classes? In order to make this precise, let us first give an explicit description of the conjugacy classes, and then compute the probability measure that μ induces on it.

11.5.1 Proposition. *Let T be the diagonal subgroup of* $\mathrm{SU}_2(\mathbb{R})$. *Then the map* $\mathrm{S}^1 \to T$ *that sends z to* $\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$ *is an isomorphism. Each conjugacy class intersects T, and the intersection is then of the form* $\{\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right), \left(\begin{smallmatrix} \bar{z} & 0 \\ 0 & z \end{smallmatrix}\right)\}$ *for a unique z in* S^1 *with positive imaginary part. It follows that the map* $\mathrm{tr}/2\colon \mathrm{SU}_2(\mathbb{R}) \to [-1,1]$ *that sends* $(\left(\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right)$ *to* $\Re(a)$ *classifies the conjugacy classes: g_1 and g_2 are in the same class if and only if their images in $[-1,1]$ are equal.*

Proof. This is linear algebra. Let g be in $\mathrm{SU}_2(\mathbb{R})$. Then g has an orthonormal basis of eigenvectors, the two eigenvalues are of absolute value one, and their product is one. □

So the question about distribution into conjugacy classes of elements of $G = \mathrm{SU}_2(\mathbb{R})$ is the question: what is the image of μ on $[-1,1]$ under $\mathrm{tr}/2$? Or, equivalently, what is the probability measure μ' on $[-1,1]$ such that for each continuous function $f\colon [-1,1] \to \mathbb{R}$ we have:

$$\int_{[-1,1]} f\mu' = \int_G (f\circ(\mathrm{tr}/2))\mu = \frac{1}{2\pi^2}\int_{\mathrm{S}^3} (f\circ x_1)v\ ?$$

Taking for f the characteristic functions of the interval $[y, y+\varepsilon]$ (with $-1 \leq y \leq 1$ and $\varepsilon \in \mathbb{R}$ small and positive), we see that $\mu' = g|dt|$, with g given by:

$$\varepsilon g(y) + O(\varepsilon^2) = \int_{t=y}^{y+\varepsilon} g|dt| = \frac{1}{2\pi^2}\int_{\{x\in\mathrm{S}^3 \mid y\leq x_1\leq y+\varepsilon\}} v = \frac{1}{2\pi^2} 4\pi\sin(\phi)^2 \frac{\varepsilon}{\sin(\phi)} + O(\varepsilon^2),$$

where $\cos(\phi) = y$ and $0 \leq \phi \leq \pi$, hence $|dy| = \sin(\phi)|d\phi|$. (Here it is quite useful to draw a picture of the unit circle in $\mathbb{R}^2$.) So we see that:

$$g(y) = \frac{2}{\pi}\sin(\phi) = \frac{2}{\pi}\sqrt{1-y^2},$$

and hence:

$$\mu' = \frac{2}{\pi}\sqrt{1-t^2}|dt|.$$

This means that the trace of an element of $\mathrm{SU}_2(\mathbb{R})$ is unlikely to be close to -2 or to 2, and that the maximum of the distribution of the traces is attained at 0.

It is clear that such computations are not very easy to generalise to higher dimensions. We will use the Peter-Weyl theorem to show how one can compute the analog of μ' without a great effort in more general cases, and to illustrate some interesting phenomenon about the distribution of the traces of the g^n for n fixed and g varying.

11.6 Invariant inner products, and semisimplicity of representations

Let G be a compact Lie group, and let μ be the bi-invariant probability measure on it. Let V be a finite dimensional (real or complex) representation of G. Let $\langle\cdot,\cdot\rangle_0$ be an arbitrary inner product on V (hence hermitian if V is a complex representation). Then we define an inner product $\langle\cdot,\cdot\rangle$ on V by:

$$\langle x, y\rangle = \int_{g\in G} \langle gx, gy\rangle_0 \mu.$$

Indeed, $\langle x, y\rangle$ is an inner product, since it has the right linearity properties in x and y, and it is positive definite since $\langle\cdot,\cdot\rangle_0$ is. Moreover, since μ is invariant, we have for all x and y in V and all g in G that $\langle gx, gy\rangle = \langle x, y\rangle$, i.e., $\langle\cdot,\cdot\rangle$ is G-invariant.

Hence any finite dimensional representation of G has a G-invariant inner product $\langle\cdot,\cdot\rangle$, and with respect to an orthonormal basis of V, the representation takes values in $\mathrm{SO}_n(\mathbb{R})$ or $\mathrm{SU}_n(\mathbb{R})$ (n the dimension of V). This is extremely useful, since it implies that for W a G-invariant subspace, $W^\perp$ (the orthogonal complement with respect to $\langle\cdot,\cdot\rangle$) is G-invariant and one has $V = W \oplus W^\perp$. So this proves quite trivially the following result.

11.6.1 Theorem. *Every finite dimensional representation of a compact Lie group is a direct sum of irreducible representations.*

12 Representations of compact Lie groups and the Peter-Weyl theorem

Let G be a compact Lie group, and let μ be its invariant measure. The theorem of Peter-Weyl gives an orthonormal basis of the complex Hilbert space $L^2(G)$. We recall that by definition, $L^2(G)$ is just the completion of the space $C^0(G)$ of continuous complex valued functions on G with respect to the norm associated to the inner product given by $\langle f_1, f_2\rangle = \int_G f_1\overline{f_2}\mu$. In order to show that a family f_i (i in some set I) is an orthonormal basis, it is enough to show that $\langle f_i, f_j\rangle$ is 1 is $i = j$ and 0 if $i \neq j$, and that the set of f_i is dense. The f_i that are used are "matrix coefficients" of the irreducible representations of G, a notion that we will now explain.

Since we are dealing with hermitian inner products, let us choose them so that they are linear in the first variable, and anti-linear in the second, i.e., $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$, and $\langle y, x\rangle = \overline{\langle x, y\rangle}$. Then, if V is a finite dimensional complex vector space with inner product $\langle\cdot,\cdot\rangle$ and with orthonormal basis $e = (e_1, \ldots, e_d)$, then we have, for every v in V:

$$v = \sum_i \langle v, e_i\rangle e_i.$$

Similarly, if $f\colon V \to W$ is a linear map between finite dimensional complex vector spaces with orthonormal bases v and w, then we have $f(v_i) = \sum_j \langle f(v_i), w_j\rangle w_j$, so that:

$$\mathrm{Mat}(f, v, w)_{i,j} = \langle f(v_i), w_j\rangle,$$

where $\mathrm{Mat}(f, v, w)$ denotes the matrix of f with respect to v and w. In particular, if V is a finite dimensional complex representation of a group G, and if $\langle\cdot,\cdot\rangle$ is an inner product on V and v an orthonormal basis of V, then each g in G gives an element, say $\rho(g)$, of $\mathrm{GL}(V)$, and we have:

$$\mathrm{Mat}(\rho(g), v)_{i,j} = \langle(\rho(g))v_i, v_j\rangle.$$

12.1 Theorem. (Peter-Weyl) *Let G be a compact Lie group, and let μ be its invariant measure. Let I be the set of isomorphism classes of finite dimensional irreducible complex representations of G. For each i in I, let V_i be an irreducible representation in the isomorphism class i, let $\langle\cdot,\cdot\rangle$ be an invariant inner product on V_i and let $v_i = (v_{i,1}, \ldots, v_{i,d_i})$ be an orthonormal basis of V_i. For each (i, j, k) with $1 \leq j, k \leq d_i$, let $f_{i,j,k}$ be the (j, k)th matrix coefficient on G: $f_{i,j,k}(g) = \langle gv_{i,j}, v_{i,k}\rangle$. Then the $\sqrt{d_i}f_{i,j,k}$ form an orthonomal basis of $L^2(G)$. In more intrinsic terms: $L^2(G)$ is isomorphic, viewed as a representation of G via the action of G on itself by right translations, to the Hilbertian direct sum of the $\mathrm{End}_{\mathbb{C}}(V)$, where V runs through the set of isomorphism classes of irreducible representations.*

12.2 Remark. We should note that the theorem of Peter-Weyl as we state it is not the general case: it is valid for compact topological groups. We will prove that the $f_{i,j,k}$ is an orthonormal family (the proof of that works for compact topological groups and just uses the Haar measure. In the case of compact Lie groups embedded in some $\mathrm{GL}_n(\mathbb{C})$, we will prove that the family is a Hilbert basis, i.e., that the subspace that it generates is dense in $L^2(G)$. By classifying the compact Lie groups, one can see that they always admit a faithful finite dimensional representation (we will not do that).

We note that not every compact group admits an embedding into some $\mathrm{GL}_n(\mathbb{C})$. For example, the infinite product $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$, with its product topology, is a compact group that does not admit such an embedding, because if it would, it would be a Lie group by the Theorem of Cartan and von Neumann.

We begin with proving that the family of functions $f_{i,j,k}$ is orthonormal, a statement that is known under the name "Schur's orthogonality relations".

12.3 Proposition. *Let G be a compact Lie group, μ its invariant measure, and V and V' two irreducible representations, with invariant inner products denoted by $\langle\cdot,\cdot\rangle$. Let u and v be in V, and let u' and v' be in V'. Then:*

$$\int_{g\in G}\langle gu,v\rangle\overline{\langle gu',v'\rangle}\mu = 0 \quad \text{if } V \text{ is not isomorphic to } V';$$

$$= \frac{\langle u,u'\rangle\overline{\langle v,v'\rangle}}{\dim(V)} \quad \text{if } V = V'.$$

Proof. Let $f\colon V\to V'$ be any linear map. Then let $F\colon V\to V'$ be the map obtained by averaging f for the action of G on $\mathrm{Hom}_{\mathbb{C}}(V,V')$:

$$F(x) = \int_{g\in G} g(f(g^{-1}x))\mu, \quad \text{for all } x \text{ in } V.$$

In terms of the action of G on $\mathrm{Hom}_{\mathbb{C}}(V,V')$, we simply have $F=\int_{g\in G}(gF)\mu$. Then, by the invariance of μ, we have $gF=F$. Hence F is a morphism of representations. Hence $F=0$ if V and V' are not isomorphic.

Now suppose that V is not isomorphic to V' and take f as follows: $f(x)=\langle x,u\rangle u'$. Then we have $F=0$, so that:

$$0=\langle F(v),v'\rangle = \langle \int_{g\in G} g(f(g^{-1}v))\mu, v'\rangle = \int_{g\in G}\langle g(f(g^{-1}v)),v'\rangle\mu = \int_{g\in G}\langle f(g^{-1}v),g^{-1}v'\rangle\mu$$

$$= \int_{g\in G}\langle\langle g^{-1}v,u\rangle u',g^{-1}v'\rangle\mu = \int_{g\in G}\langle g^{-1}v,u\rangle\langle u',g^{-1}v'\rangle\mu = \int_{g\in G}\overline{\langle gu,v\rangle}\langle gu',v'\rangle\mu.$$

Now suppose that $V = V'$. We take $f\colon V \to V$ as before: $x \mapsto \langle x, u\rangle u'$. Then $F = \lambda \mathrm{id}_V$ for some λ in $\mathbb{C}$. But then we have:

$$\lambda\cdot \dim(V) = \mathrm{tr}(F) = \mathrm{tr}\left(\int_{g\in G}(g\cdot f)\mu\right) = \int g \in G\mathrm{tr}(g\cdot f)\mu = \int_{g\in G}\mathrm{tr}(f)\mu = \mathrm{tr}(f)$$
$$= \langle u', u\rangle.$$

(To see the last equality, use that $V = \mathbb{C}\cdot u' \oplus u'^{\perp}$.) So we see that $\lambda = \langle u', u\rangle / \dim(V)$. Now compute:

$$\begin{aligned}\frac{\overline{\langle u, u'\rangle}\langle v, v'\rangle}{\dim(V)} &= \frac{\langle u', u\rangle}{\dim(V)}\langle v, v'\rangle = \lambda\langle v, v'\rangle = \langle F(v), v'\rangle = \left\langle \int_{g\in G} g(f(g^{-1}v))\mu, v'\right\rangle \\ &= \int_{g\in G}\langle g(f(g^{-1}v)), v'\rangle\mu = \int_{g\in G}\langle f(g^{-1}v), g^{-1}v'\rangle\mu = \int_{g\in G}\langle\langle g^{-1}v, u\rangle u', g^{-1}v'\rangle\mu \\ &= \int_{g\in G}\langle g^{-1}v, u\rangle\langle u', g^{-1}v'\rangle\mu = \int_{g\in G}\langle v, gu\rangle\langle gu', v'\rangle\mu = \int_{g\in G}\overline{\langle gu, v\rangle}\langle gu', v'\rangle\mu.\end{aligned}$$

□

As we remarked, this proposition proves that the family of $f_{i,j,k}$ as in the Peter-Weyl theorem is orthonormal. In order to prove the Peter-Weyl theorem, it therefore suffices to prove that the subspace E of $L^2(G)$ that it generates is dense. An important step in this direction is to show that E is a $\mathbb{C}$-subalgebra, closed under complex conjugation. These two properties are consequences of two very important constructions with representations: tensor products and dualization. We have already seen dualization, and we have already used the tensor product to extend scalars from a field k to an algebraic closure $\overline{k}$; complexification of real vector spaces is usually the first example that one encounters. In order to be able to compute with tensor products of representations, we include a small section on tensor products of vector spaces over a field. For details one may consult any sufficiently advanced book on algebra, for example [Lang].

12.4 Tensor products of vector spaces over a field

Let k be a field. For V and W two k-vector spaces, we have the k-vector space $V \otimes_k W$ (also denoted simply by $V \otimes W$). By construction, $V \otimes W$ is equipped with a map:

$$\otimes\colon V \times W \longrightarrow V \otimes W, \quad (v, w) \mapsto v \otimes w.$$

This map is k-bilinear, and is universal for that in the sense that if $b\colon V \times W \to U$ is a k-bilinear map (with U a k-vector space, of course), then there exists a unique k-linear map $\overline{b}\colon V \otimes W \to U$ such that $b = \overline{b} \circ \otimes$. This universal property shows that the pair $(V \otimes W, \otimes)$ is unique up to unique isomorphism. (The map $\otimes$ is not surjective if $\dim(V) > 1$ and $\dim(W) > 1$, hence not every element of $V \otimes W$ is of the form $v \otimes w$). In order to work with tensor products (of vector spaces at least), it suffices to know the following properties.

12.4.1 Proposition. *The tensor product has the following properties:*

1. *if $v = (v_1, \ldots, v_{\dim(V)})$ and $w = (w_1, \ldots, w_{\dim(W)})$ are bases of V and W, then the $v_i \otimes w_j$ form a basis for $V \otimes W$. In particular $\dim(V \otimes W) = \dim(V)\dim(W)$. We note that even though the bases v and w are ordered, the resulting basis of $V \otimes W$ is not naturally ordered, and in pratice it is often a bad idea to want to put an order on it.*

2. *if $f_1\colon V_1 \to V_2$ and $f_2\colon V_2 \to W_2$ are linear maps between k-vector spaces, then there is a unique linear map*

$$f_1 \otimes f_2\colon V_1 \otimes V_2 \longrightarrow W_1 \otimes W_2,$$

such that for all v_1 in V_1 and v_2 in V_2 we have $(f_1 \otimes f_2)(v_1 \otimes v_2) = f_1(v_1) \otimes f_2(v_2)$.

3. *if V is finite dimensional, then the unique linear map $V^* \otimes W \to \mathrm{Hom}_k(V, W)$ induced by $(l, w) \mapsto (v \mapsto l(v)w)$ is an isomorphism of k-vector spaces. For $W = V$, this isomorphism makes the maps $\mathrm{tr}\colon \mathrm{End}(V) \to k$ and $e\colon V^* \otimes V \to k$ given by $(l, v) \mapsto l(v)$ correspond to each other.*

4. *if b_1 and b_2 are bilinear forms on V_1 and V_2, respectively, then there is a unique bilinear form $b_1 \otimes b_2$ on $V_1 \otimes V_2$ such that $(b_1 \otimes b_2)(v_1 \otimes v_2, v_1' \otimes v_2') = b_1(v_1, v_1')b_2(v_2, v_2')$ for all v_1, v_1' in V_1 and all v_2, v_2' in V_2. If b_1 and b_2 are non degenerate, then so is $b_1 \otimes b_2$.*

12.4.2 Exercise. Let f_1 and f_2 be endomorphisms of finite dimensional k-vector spaces V_1 and V_2. Show that $\mathrm{tr}(f_1 \otimes f_2) = \mathrm{tr}(f_1)\mathrm{tr}(f_2)$.

12.4.3 Definition. For k a field, and G a group acting on two k-vector spaces, we let G act on $V \otimes W$ such that $g \cdot (v \otimes w) = (gv) \otimes (gw)$ for all v in V and w in W. Then we call $V \otimes W$ the tensor product of the representations V and W of G.

12.4.4 Exercise. Check that the isomorphism $V^* \otimes W \to \mathrm{Hom}_k(V, W)$ is an isomorphism of representations, if V and W are representations of a group G.

After these generalities on tensor products of vector spaces we return to the proof of the Peter-Weyl theorem.

12.5 Proposition. *The sub $\mathbb{C}$-vector space E of $L^2(G)$ generated by the $f_{i,j,k}$ is a sub $\mathbb{C}$-algebra closed under complex conjugation.*

Proof. We have to show that products and complex conjugates of the $f_{i,j,k}$ are in E, i.e., are linear combinations of the $f_{i,j,k}$. Let us start with the complex conjugates. Let i be in I, and let $1 \leq j, k \leq d_i$. Let ρ_i denote the morphism $G \to \mathrm{GL}_{\mathbb{C}}(V_i)$ that gives the action of G on V. Since $\langle \cdot, \cdot \rangle$ is G-invariant and the basis v_i is orthonormal, the matrix $\mathrm{Mat}(\rho_i(g), v_i)$ of each $\rho_i(g)$ with respect to v_i is unitary. Hence we have:

$$\overline{\mathrm{Mat}(\rho_i(g), v_i)} = \mathrm{Mat}(\rho_i(g), v_i)^{-1,t}.$$

This means that $\overline{f_{i,j,k}(g)}$ is the (j,k)th coefficient of the matrix of g acting on V_i^*, with respect to the dual basis of v_i. Let i' be the element of I that corresponds to V_i^*, and let $\phi\colon V_{i'} \to V_i^*$ be an isomorphism. The fact that $\phi(v_{i'})$ and v_i^* are bases of the same vector space then implies that $\overline{f_{i,j,k}}$ is a linear combination of the $f_{i',n,m}$, with $1 \leq n, m \leq d_{i'} = d_i$.

For products, we argue basically in the same way. So let i_1 and i_2 be in I. Then the products $f_{i_1,j_1,k_1}(g) f_{i_2,j_2,k_2}(g)$ are the coefficients of the matrix of the action of g on $V_{i_1} \otimes V_{i_2}$, with respect to the basis formed by the $v_{i_1,j_1,k_1} \otimes v_{i_2,j_2,k_2}$. As any finite dimensional representation of G, $V_{i_1} \otimes V_{i_2}$ decomposes into a direct product of irreducible ones, i.e., there are $r \geq 3$, $i_3, \ldots, i_r$ in I, and an isomorphism:

$$\phi\colon V_{i_3} \oplus \cdots \oplus V_{i_r} \xrightarrow{\sim} V_{i_1} \otimes V_{i_2}.$$

It follows that the matrix coefficients $f_{i_1,j_1,k_1} f_{i_2,j_2,k_2}$ are linear combinations of those of the V_{i_j}, $3 \leq j \leq r$. □

12.6 Proof of the Peter-Weyl theorem for subgroups of $\mathrm{GL}_n(\mathbb{C})$.

Let G be a compact subgroup of some $\mathrm{GL}_n(\mathbb{C})$. Let I etc. be as in Theorem 12.1. By Proposition 12.3 the $f_{i,j,k}$ form an orthonormal family. By Proposition 12.5, the sub $\mathbb{C}$-vector space E of $L^2(G)$ that they generate is a sub $\mathbb{C}$-algebra, stable under complex conjugation. It just remains to show that E is dense. But now we use the coordinate functions $x_{i,j}$, $1 \leq i, j \leq n$, of $\mathrm{GL}_n(\mathbb{C})$. These $x_{i,j}$ are linear combinations of the $f_{i,j,k}$, hence are in E. But then we can apply the Stone-Weierstrass theorem, that tells us that any continuous complex valued function f on a compact subset C of $\mathbb{C}^n$ is the limit, for the sup norm, of a sequence of polynomials. It follows that E is dense in $L^2(G)$, because its subspace of continuous functions is dense (for the L^2-norm).

13 Characters, the space of class functions, and the decomposition of some tensor products

In order to compute in a simple way with representations of a compact Lie group, it is very useful to consider its character. To motivate the definition, let us first look at the case of a finite dimensional $\mathbb{C}$-vector space V with a given diagonalisable endormorphism f. Then we have the result that the conjugacy classe of such an f consist precisely of the set of diagonalisable f' that have the same characteristic polynomial as f has. The coefficients of non maximal degree of the characteristic polynomial P_f of f are (up to signs) the elementary symmetric functions of the roots $\lambda_1, \ldots, \lambda_d$ ($d = \dim(V)$)) of P_f. Newton's identities show that these coefficients can be expressed a polynomials with rational coefficients in the $\sum_i \lambda_i^k$, $1 \leq k < d$. But $\sum_i \lambda_i^k$ is just the trace of f^k (clear with respect to a basis with respect to which f is diagonal, or even just upper triangular). So in fact, the conjugacy class of f is determined by the traces of the powers of f.

Suppose now that G is a group, and $\rho\colon G \to \mathrm{GL}_{\mathbb{C}}(V)$ a representation on a finite dimensional $\mathbb{C}$-vector space V. For g in G, the conjugacy class of $\rho(g)$ in $\mathrm{GL}_{\mathbb{C}}(V)$ is determined by the traces of the $\rho(g^k)$. As we will see, the traces of the $\rho(g)$ determine even the isomorphism class of V, if G is a compact Lie group. This is a good motivation for the following definition.

13.1 Definition. Let $\rho\colon G \to \mathrm{GL}_{\mathbb{C}}(V)$ be a representation of a group on a finite dimensional $\mathbb{C}$-vector space. Then we define the *character* of ρ to be the function $\chi = \chi_V \colon G \to \mathbb{C}$, $\chi(g) = \mathrm{tr}(\rho(g))$. We note that if V and V' are isomorphic representations, then $\chi_V = \chi_{V'}$.

13.2 Exercise. Show that the character of the tensor product of two representations is the product of the characters.

The basis results about the relation between representations of compact Lie groups and their characters are given in the following theorem.

13.3 Theorem. *Let G be a compact Lie group, and let μ be its invariant probability measure. Let I be the set of isomorphism classes of irreducible complex representations of G, and for each i in I let V_i be a representation in the class i. Let χ_i denote the character of V_i. Then the χ_i form an orthonormal basis of the closed subspace $L^2(G)^G$ of $L^2(G)$ that consists of elements that are invariant under the action of conjugation by G on itself. (The space $L^2(G)^G$ is called the space of class functions on G.) If V is a complex representation* *of G, then we have:*

$$V \cong \bigoplus_i V_i^{n_i} \quad \textit{and } \chi_V = \sum_i n_i\chi_i, \quad \textit{with } n_i = \langle \chi_V, \chi_i\rangle = \int_{g\in G} \chi_V(g)\overline{\chi_i(g)}\mu.$$

In particular, two finite dimensional complex representations V and W of G are isomorphic if and only if their characters are equal. Moreover, a complex representation V of G is irreducible if and only if $\langle \chi_V, \chi_V\rangle = 1$.

Proof. Let us prove that the χ_i form an orthonormal basis of $L^2(G)^G$. First of all, any character χ_V is in $L^2(G)^G$ because we have, for all x and y in G:

$$\chi_V(xyx^{-1}) = \mathrm{tr}(\rho(xyx^{-1})) = \mathrm{tr}(\rho(x)\rho(y)\rho(x)^{-1}) = \mathrm{tr}(\rho(y)) = \chi_V(y).$$

Let us now show that the χ_i form an orthonormal family. We compute, using Schur's orthogonality relations, for i and j in I:

$$\begin{aligned}\langle \chi_i, \chi_j\rangle &= \int_{g\in G} \chi_i(g)\overline{\chi_j(g)}\mu = \int_{g\in G} \sum_k \rho_i(g)_{k,k}\overline{\sum_l \rho_j(g)_{l,l}}\mu = \sum_{k,l}\int_{g\in G} \rho_i(g)_{k,k}\overline{\rho_j(g)_{l,l}}\mu \\ &= \delta_{i,j}\sum_k \int_{g\in G} |\rho_i(g)_{k,k}|^2\mu = \delta_{i,j}\dim(V_i)\frac{1}{\dim(V_i)} = \delta_{i,j},\end{aligned}$$

where the matrix coefficients are taken with respect to orthonormal bases of V_i and V_j, and where $\delta_{i,j}$ is zero if $i \neq j$ and one if $i = j$.

In order to prove that the χ_i form a Hilbert basis of $L^2(G)^G$, it suffices to prove that, for every i in I, the subspace $\mathrm{Hom}_{\mathbb{C}}(V_i, V_i)^G$ is of dimension one, and hence generated by χ_i. But this is just Schur's Lemma, that says that $\mathrm{Hom}_{\mathbb{C}}(V_i, V_i)^G = \mathbb{C}\cdot\mathrm{id}_{V_i}$.

The proof of the second statement is very simple. Let V be a finite dimensional complex representation of G. Then V is isomorphic to a direct sum of irreducible representations, say to the direct sum of $V_i^{n_i}$, with n_i non-negative integers (almost all zero). Then, by comparing traces of elements of g on V and the direct sum, one gets the result. □

As an application, we will now decompose the tensor products of pairs of irreducible representations of $\mathrm{SU}_2(\mathbb{R})$.

13.4 Example. Proposition 11.5.1 describes the intersections of the conjugacy classes of $\mathrm{SU}_2(\mathbb{R})$ with the diagonal subgroup T, and gives an isomorphism between the diagonal subgroup and S^1. Consequently, we can view a class function on $\mathrm{SU}_2(\mathbb{R})$ as a function f

on S^1 that is invariant under inversion: $f(x^{-1}) = f(x)$ for all x in S^1. One computes that if χ_d denotes the character of $\mathbb{C}[x,y]_d$, then:

$$\chi_d = t^d + t^{d-2} + \cdots + t^{2-d} + t^{-d} = \frac{t^{d+1} - t^{-d-1}}{t - t^{-1}},$$

where t denotes the coordinate function on S^1.

13.5 Theorem. (Clebsch-Gordon) *As representations of* $\mathrm{SU}_2(\mathbb{R})$ *(and hence also as representations of* $\mathrm{SL}_2(\mathbb{C})$ *and of* $\mathrm{Lie}(\mathrm{SL}_2(\mathbb{C}))$*) we have, for all* $n \geq 0$ *and* $m \geq 0$*:*

$$\mathbb{C}[x,y]_n \otimes \mathbb{C}[x,y]_m \cong \bigoplus_{\substack{|n-m| \leq k \leq n+m \\ d \equiv n+m\ (2)}} \mathbb{C}[x,y]_k.$$

Proof. We will show that both sides have the same character. Given what we already know, this means that we have to prove that:

$$\frac{t^{n+1} - t^{-n-1}}{t - t^{-1}} \cdot \frac{t^{m+1} - t^{-m-1}}{t - t^{-1}} = \sum_{\substack{|n-m| \leq k \leq n+m \\ d \equiv n+m\ (2)}} \frac{t^{k+1} - t^{-k-1}}{t - t^{-1}}.$$

But this identity is clear if we suppose that $n \geq m$ and write the second factor as $t^m + t^{m-2} + \cdots + t^{-m}$. □

13.6 The space $L^2(G)$ as a representation of $G \times G$

We start with a generality. If G_1 and G_2 are groups that act on k-vector spaces V_1 and V_2, then we let $G_1 \times G_2$ act on $V_1 \otimes V_2$ by $(g_1, g_2) \cdot (v_1 \otimes v_2) = (g_1 v_1) \otimes (g_2 v_2)$.

13.6.1 Proposition. *Let* G_1 *and* G_2 *be compact Lie groups. If* V_1 *and* V_2 *are irreducible representations of* G_1 *and* G_2*, then* $V := V_1 \otimes V_2$ *is an irreducible representation of* $G_1 \times G_2$*. Every irreducible representation of* $G_1 \times G_2$ *is of that form.*

Proof. Observe first that $\chi_V(g_1, g_2) = \chi_1(g_1)\chi_2(g_2)$. That gives:

$$\begin{aligned}\langle \chi_V, \chi_V \rangle &= \int_{(g_1,g_2) \in G_1 \times G_2} \chi_V(g_1,g_2)\overline{\chi_V(g_1,g_2)}\mu_1\mu_2 = \int_{g_1 \in G_1} \chi_1(g_1)\mu_1 \cdot \int_{g_2 \in G_2} \chi_2(g_2)\mu_2 \\ &= \langle \chi_1, \chi_1 \rangle \cdot \langle \chi_2, \chi_2 \rangle = 1.\end{aligned}$$

Hence V is irreducible. The fact that the matrix coefficients of the $V_i \otimes V_j'$, where the V_i and V_j' run through the irreducible representations of G_1 and G_2, generate a dense subspace of $L^2(G_1 \times G_2)$, implies that all irreducible representations of $G_1 \times G_2$ are of that form (use the Peter-Weyl theorem for all three groups). □

13.6.2 Remark. Is it possible to give a proof of the proposition that does not use characters and that does not use the Peter-Weyl theorem?

Let now G be a compact Lie group. In order to understand why $L^2(G)$, viewed as a representation of G via the action of G on itself via right translations, is isomorphic to the (Hilbert) sum of the $\mathrm{End}_{\mathbb{C}}(V_i)$, we note that actually the group $G \times G$ acts from the right on G via $(x, y) \cdot z = x^{-1}zy$. This induces an action of $G \times G$ on $C^0(G)$ by:

$$((x, y) \cdot f)z = f(x^{-1}zy).$$

This action extends to the completion $L^2(G)$. It is not hard to see that the decomposition $L^2(G) = \widehat{\bigoplus}_i \mathrm{End}_{\mathbb{C}}(V_i)$ is stable for this action, and that $\mathrm{End}_{\mathbb{C}}(V_i)$ is the irreducible representation $V_i^* \otimes V_i$ of $G \times G$, where $G \times G$ acts as $(x, y) \cdot (l \otimes v) = (xl) \otimes (yv)$.

14 Some equidistribution results

The aim of this section is to first compute the measure on the set of conjugacy classes of $\mathrm{SU}_2(\mathbb{R})$ that is induced from the Haar measure, using the Peter-Weyl theorem and the characters of the irreducible representations. Secondly, we will derive that certain powers of elements of $\mathrm{SU}_2(\mathbb{R})$ are equidistributed in the conjugacy classes.

14.1 Second computation of the distribution of $\mathrm{SU}_2(\mathbb{R})$ into conjugacy classes

As we have seen in Example 13.4, to give a continuous class function on $G = \mathrm{SU}_2(\mathbb{R})$ is the same as giving a continuous function on $\mathrm{S}^1 = T$, invariant under $a \mapsto a^{-1}$. As in Section 11.5, but now for S^1 instead of $[-1,1]$, we define a measure μ'' on S^1 by the property that for each continuous class function $f\colon G \to \mathbb{C}$ we have:

$$\int_{\mathrm{S}^1} f_{|\mathrm{S}^1}\mu'' = \int_G f\mu.$$

Of course, the group S^1 also has its Haar measure, and we will write it as $\frac{1}{2\pi}\cdot|dt/t| = \frac{1}{2\pi}|d\phi|$, where $t = e^{\phi i}$ is the coordinate function on S^1. Hence there is a unique function $h\colon \mathrm{S}^1 \to \mathbb{C}$ such that:

$$\mu'' = h\frac{1}{2\pi}\cdot|dt/t|.$$

We will now compute this function h from a few properties of it that we know. First of all, we know that h is real valued (since integrating real valued functions gives real results). Secondly, we know that $h(a) = h(a^{-1})$ for all $a \in \mathrm{S}^1$. Thirdly, we know that for all d_1 and d_2 non negative integers we have:

(14.1.1)
$$\int_{\mathrm{S}^1} \chi_{d_1}\overline{\chi_{d_2}}\cdot h\frac{1}{2\pi}\cdot|dt/t| = \langle \chi_{d_1}, \chi_{d_2}\rangle,$$

where the χ_d are the characters of the irreducible representations of G. We write:

$$h = \sum_{n\in\mathbb{Z}} h_n t^n,$$

with the h_n in $\mathbb{C}$. The fact that $h(z)$ is real for all $z \in \mathrm{S}^1$ gives:

$$\sum_n h_n z^n = h(z) = \overline{h(z)} = \sum_n \overline{h_n}\overline{z}^n = \sum_n \overline{h_n} z^{-n}, \quad \text{so that } \overline{h_n} = h_{-n} \text{ for all } n.$$

The fact that $h(z^{-1}) = h(z)$ for all z in S^1 shows that $h_n = h_{-n}$ for all n, so that we have, for all n:

$$h_n = \overline{h_n}, \quad h_n = h_{-n}.$$

Applying (14.1.1) with $d_1 = d_2 = 0$ gives:

$$1 = \frac{1}{2\pi}\int_{\mathrm{S}^1} h\, d\phi = \frac{1}{2i\pi}\int_{\mathrm{S}^1} \sum_n h_n t^n \frac{dt}{t} = h_0.$$

Applying it with $d_1 = 0$ and $d_2 = 1$, 2 etc. gives $h_1 = h_{-1} = 0$, $h_2 = h_{-2} = -1/2$, and $h_n = 0$ for $|n| \geq 3$. Hence we have:

$$\begin{aligned}\mu'' &= h\frac{1}{2\pi}\cdot|dt/t| = \frac{1}{2}(2 - t^2 - t^{-2})\frac{1}{2\pi}\cdot|dt/t| = -\frac{1}{2}(t - t^{-1})^2\frac{1}{2\pi}\cdot|dt/t| \\ &= -\frac{1}{2}(2i\sin(\phi))^2\frac{1}{2\pi}\cdot|d\phi| = \frac{1}{\pi}\sin(\phi)^2|d\phi|.\end{aligned}$$

It is a simple computation that this is consistent with the measure on $[-1, 1]$ that we computed by hand (it is the projection to the x-axis of the measure that we have here on S^1). (Use that $x = \cos(\phi)$, hence $|dx| = \sin(\phi)|d\phi|$.) What is particularly interesting in the formulas that we have obtained here is that μ'' is $(2 - t^2 - t^{-2})/2$ times the Haar measure on S^1, i.e., a Laurent polynomial times that measure. The following result is implied immediately by this formula, noting that the only exponents that occur are 0, 2 and -2.

14.1.2 Theorem. *Let n be an integer, not in $\{-2, -1, 0, 1, 2\}$. Then the nth powers of the elements of $\mathrm{SU}_2(\mathbb{R})$ are equidistributed in the conjugacy classes of $\mathrm{SU}_2(\mathbb{R})$, in the sense that for every continuous class function f on $\mathrm{SU}_2(\mathbb{R})$ we have:*

$$\int_{g\in G} f(g^n)\mu = \int_{\mathrm{S}^1} f_{|\mathrm{S}^1}|dt/t|.$$

Proof. Let n be an integer, not in $\{0, \pm1, \pm2\}$. It suffices to show that for all continuous functions $f\colon \mathrm{S}^1 \to \mathbb{C}$ we have:

$$\int_{z\in\mathrm{S}^1} f(z^n)\frac{1}{2}(2 - z^2 - z^{-2})\frac{dz}{2i\pi z} = \int_{z\in\mathrm{S}^1} f(z^n)\frac{dz}{2i\pi z}$$

As the f_k, $k \in \mathbb{Z}$, given by $f_k(z) = z^k$ are a basis of $L^2(\mathrm{S}^1)$, it suffices to show the equality for the f_k, in which case it is clear.

Another more probabilistic way of proving the result is to argue as follows. The new measure on S^1 that we get is the image of μ'' under the map $F_n\colon \mathrm{S}^1 \to \mathrm{S}^1$, $z \mapsto z^n$; we denote it by $F_{n,*}\mu''$. As F_n is a morphism of groups, with kernel the group μ_n of nth roots of unity, we have:

$$F_{n,*}\mu'' = \left(\frac{1}{n}\sum_{\zeta\in\mu_n}\delta_\zeta\right) * \mu'',$$

where the multiplication is convolution, and where the δ_ζ are Dirac measures. The fact that $\sum_{\zeta\in\mu_n}\zeta^2 = 0$ implies the result. ☐

15 Quelques exercices; exemple pour l'examen.

Tous les documents seront autorisés. La durée de l'examen sera de 3 heures.

1. Donner une base $e = (e_1, e_2, \dots)$ de l'algèbre de Lie du groupe de Lie réel $\mathrm{SO}_3(\mathbb{R})$, et exprimer les $[e_i, e_j]$ en combinaisons linéaires des e_k.

2. Soit b la forme bilinéaire sur $\mathbb{R}^3$ donnée par:

$$b(x,y) = x^t u y, \quad \text{avec} \quad u = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   Notons G le sous-groupe de $\mathrm{GL}_3(\mathbb{R})$ des g tels qui stabilisent b, c'est à dire, tels que $b(gx, gy) = b(x,y)$ pour tous les x et y dans $\mathbb{R}^3$. C'est un sous-groupe de Lie de $\mathrm{GL}_3(\mathbb{R})$ par le théorème de Cartan et von Neumann.

   (a) Montrer que les algèbres de Lie complexes $\mathbb{C}\otimes_{\mathbb{R}} \mathrm{Lie}(G)$ et $\mathbb{C}\otimes_{\mathbb{R}} \mathrm{Lie}(\mathrm{SO}_3(\mathbb{R}))$ sont isomorphes. (Indication: le plus simple est d'argumenter en termes de formes bilinéaires complexes.) Cette partie est peut-être dure; elle ne sert pas dans le reste, mais vous pouvez en conclure que G est de dimension 3.

   (b) Calculer une base $f = (f_1, \dots)$ de $\mathrm{Lie}(G)$ vue comme sous espace de $\mathrm{M}_3(\mathbb{R})$, et exprimer les $[f_i, f_j]$ en combinaisons linéaires des f_k.

   (c) Calculer $\exp(ta)$ pour tout t dans $\mathbb{R}$, et pour:

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

   (d) Est-ce que G est compact? (Indication: utiliser la partie précédente, ou considérer directement l'intersection de G avec le sous-groupe $\mathrm{GL}_2(\mathbb{R})$ de $\mathrm{GL}_3(\mathbb{R})$ plongé par $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$.)

3. Combien de représentations complexes irréductibles de dimension 12 (à isomorphisme près) ont $\mathrm{SU}_2(\mathbb{R}) \times \mathrm{SU}_2(\mathbb{R})$ et $\mathrm{SO}_4(\mathbb{R})$?

4. En quelles représentations irréductibles et avec quelles multiplicités se décompose la représentation $V\otimes V$ de $\mathrm{SO}_3(\mathbb{R})$, avec $V = \mathbb{C}^3$ la représentation fournie par l'inclusion de $\mathrm{SO}_3(\mathbb{R})$ dans $\mathrm{GL}_3(\mathbb{C})$.

16 Further theory

This part of the notes consists of some indications about the general theory that we did not discuss in the course; it is not part of the material that will be tested in the exam. The material of this section can be found in any sufficiently advanced text on Lie groups, as for example [Vara].

Let L be a Lie algebra (real or complex). Then L has a maximal sub Lie algebra S that is an ideal and that is *solvable*: it has a filtration $S = S^0 \supset S^1 \supset \cdots$ with $S^i = 0$ for i sufficiently large such that each S^{i+1} is an ideal in S^i with S^i/S^{i+1} commutative (see [Vara, Ch. 3, §3.8]). This S is called the *solvable radical* of L, and it is denoted $\mathrm{rad}(L)$. A Lie algebra L is called *semisimple* if $\mathrm{rad}(L) = 0$. Hence in general a Lie algebra L is an *extension* of its maximal semisimple quotient $L/\mathrm{rad}(L)$ by its maximal solvable ideal $\mathrm{rad}(L)$. In fact, L is even a semidirect product of $L/\mathrm{rad}(L)$ by $\mathrm{rad}(L)$ (Levi decomposition, see [Vara, Ch. 3, § 3.14]).

The typical example of a solvable Lie algebra is the one of upper triangular matrices. The finite dimensional complex representations of a solvable Lie algebra are upper triangular, with respect to a suitable basis (this is a theorem of Lie, see [Vara, Ch. 3, Thm. 3.7.3], or [Ser1, Part I, Thm. 5.1]). Equivalently, all irreducible finite dimensional complex representations of a solvable Lie algebra L are one dimensional.

The semisimple complex Lie algebras are classified as follows (see [Vara, Ch. 4]). Such a Lie algebra L is the product of its minimal ideals L_i, which are simple complex Lie algebras. The simple complex Lie algebras have a particularly simple classification: they arise in four series: A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$ and D_n $(n \geq 4)$, and there are six exceptional ones: G_2, F_4, E_6, E_7 and E_8. The four series are also called the classical ones, as they correspond to the groups $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, and $\mathrm{SO}_{2n}(\mathbb{C})$.

The classification of the real semisimple Lie algebras is a bit harder, but well known, and obtained from the complex case

These results on Lie algebras that we have cited here have of course their consequences for Lie groups (recall that $G \mapsto \mathrm{Lie}(G)$ is an equivalence between Lie algebras and simply connected connected Lie groups). The theory of finite dimensional representations of semisimple Lie groups is well known (see [Vara, Ch. 4]). The main point is that each such representation is a direct sum of irreducible ones, and that the irreducible ones are classified by their *highest weight*.

Just to give an example, the equidistribution result that we have seen in Section 14 has a simple analog for arbitrary compact connected semisimple Lie groups; one can deduce it from [Vara, Ch. 4, Cor. 4.13.8] and [Vara, Ch. 4,(4.13.9)].

In order to finish the discussion on the general theory, it is worthwile to note that there are other fields than $\mathbb{R}$ and $\mathbb{C}$, that are important (in particular in number theory for example), such as finite fields and the p-adic fields $\mathbb{Q}_p$. The p-adic fields do have a norm for which they are locally compact, and for them there does exist a theory of Lie groups almost the same as what we have seen; see [Ser1] for example. For finite fields, and in fact for arbitrary fields, one has the theory of algebraic groups, where the differential geometry is replaced by algebraic geometry. For an account of this, we recommend [Spri].

17 Some physics

This section still has to be written. It will contain some results about an orthonormal basis of $L^2(\mathrm{S}^2)$ obtained from the Peter-Weyl theorem, some application of that to the hydrogen atom, and a short description of the representation theory of $\mathrm{SU}_3(\mathbb{R})$ behind Gell-Man's quark model.

Examen "Groupes et algèbres de Lie", DEA Mathématiques, 27/03/2001.

Tous les documents sont autorisés, ainsi que l'utilisation des calculettes. La durée de l'examen est de 3 heures.

1. Soit $n \geq 1$ un entier. Soit G un sous-groupe de Lie de $\mathrm{GL}_n(\mathbb{R})$, et soit L son algèbre de Lie, vue comme sous-algèbre de Lie de $\mathrm{M}_n(\mathbb{R})$. L'action de G sur lui-même par conjugaison fixe l'élément neutre 1 de G, donc induit une action sur L car L est l'espace tangent de G en 1. Ceci donne donc un morphisme $\rho\colon G \to \mathrm{GL}_{\mathbb{R}}(L)$ de groupes de Lie. Comme Lie est un foncteur, ceci nous donne un morphisme d'algèbres de Lie
$$\mathrm{Lie}(\rho)\colon L \longrightarrow \mathrm{End}_{\mathbb{R}}(L).$$
Montrer que pour tout x et y dans L on a: $(\mathrm{Lie}(\rho)x)y = [x, y]$. Indication: calculer ce que fait la conjugaison par $1 + \varepsilon x$ sur y, pour ε dans $\mathbb{R}$ petit.

2. Notons $G := \mathrm{SL}_2(\mathbb{R})$ et $L := \mathrm{Lie}(\mathrm{SL}_2(\mathbb{R}))$. Soit $\rho\colon G \to \mathrm{GL}_{\mathbb{R}}(L)$ la représentation de G sur L donnée par l'action de G sur lui-même comme dans l'exercice précédent. Donner la décomposition en irréductibles de la représentation L, c'est à dire, les multiplicités des $\mathbb{R}[x, y]_d$. Incidation: utiliser le résultat de l'exercice précédent, et appliquer ce que le cours dit sur les représentations de $\mathrm{SL}_2(\mathbb{R})$.

3. Soit h la forme hermitienne sur $\mathbb{C}^2$ donnée par $h(x, y) = x_1\overline{y_1} - x_2\overline{y_2}$. Soit G le sous-groupe de $\mathrm{GL}_2(\mathbb{C})$ des g tels que $h(gx, gy) = h(x, y)$ pour tous x et y dans $\mathbb{C}^2$.
   (a) Montrer que G est un sous-groupe fermé de $\mathrm{GL}_2(\mathbb{C})$. C'est donc un groupe de Lie réel.
   (b) Calculer la dimension de G.
   (c) Est-ce que G est compact?

4. Soit $V = \mathbb{C}^3$ la représentation complexe de $\mathrm{SO}_3(\mathbb{R})$ donnée par l'inclusion de $\mathrm{SO}_3(\mathbb{R})$ dans $\mathrm{GL}_3(\mathbb{C})$. En quelles représentations irréductibles et avec quelles multiplicités se décompose $V \otimes V \otimes V$? Indication: penser à $\mathrm{SU}_2(\mathbb{R})$, et utiliser que $V \otimes V \otimes V$ est la même chose que $(V \otimes V) \otimes V$.

5. (a) Montrer que $\mathrm{SL}_2(\mathbb{C})/\{1,-1\}$ et $\mathrm{SO}_3(\mathbb{C})$ sont isomorphes. Indication: considérer une représentation convenable V de $\mathrm{SL}_2(\mathbb{C})$ et montrer que $\mathrm{SL}_2(\mathbb{C})$ fixe une forme quadratique de rang maximal sur V.

   (b) Est-ce que $\mathrm{SL}_2(\mathbb{R})/\{1,-1\}$ est isomorphe à $\mathrm{SO}_3(\mathbb{R})$?

Exam “Lie groups and Lie algebras”, DEA Mathématiques, 27/03/2001.

All texts can be used during the exam. Calculators too. The exam lasts three hours.

1. Let $n \geq 1$ be an integer. Let G be a sub Lie group of $\mathrm{GL}_n(\mathbb{R})$, and let L be its Lie algebra, viewed as a sub Lie algebra of $\mathrm{M}_n(\mathbb{R})$. The action of G on itself by conjugation fixes the identity element 1 of G, hence induces an action on L because L is the tangent space of G at 1. This gives a morphism $\rho\colon G \to \mathrm{GL}_{\mathbb{R}}(L)$ of Lie groups. As Lie is a functor, this gives us a morphism of Lie algebras

$$\mathrm{Lie}(\rho)\colon L \longrightarrow \mathrm{End}_{\mathbb{R}}(L).$$

Show that for all x and y in L one has: $(\mathrm{Lie}(\rho)x)y = [x, y]$. Hint: compute the conjugation of $1 + \varepsilon x$ on y, for ε in $\mathbb{R}$ small.

2. Let $G := \mathrm{SL}_2(\mathbb{R})$ and $L := \mathrm{Lie}(\mathrm{SL}_2(\mathbb{R}))$. Let $\rho\colon G \to \mathrm{GL}_{\mathbb{R}}(L)$ be the representation of G on L given by the action of G on itself as in the preceding exercise. Give the decomposition in irreducibles of the representation L, i.e., the multiplicities of the $\mathbb{R}[x, y]_d$. Hint: use the result of the preceding exercise, and apply what we have seen in the course on the representations of $\mathrm{SL}_2(\mathbb{R})$.

3. Let h be the Hermitian form on $\mathbb{C}^2$ given by $h(x, y) = x_1\overline{y_1} - x_2\overline{y_2}$. Let G be the subgroup of $\mathrm{GL}_2(\mathbb{C})$ consisting of the g such that $h(gx, gy) = h(x, y)$ for all x and y in $\mathbb{C}^2$.

   (a) Show that G is a closed subgroup of $\mathrm{GL}_2(\mathbb{C})$. Hence it is a real Lie group.

   (b) Compute the dimension of G.

   (c) Is G compact?

4. Let $V = \mathbb{C}^3$ be the complex representation of $\mathrm{SO}_3(\mathbb{R})$ given by the inclusion of $\mathrm{SO}_3(\mathbb{R})$ in $\mathrm{GL}_3(\mathbb{C})$. In which irreducible representations and with which multiplicities decomposes $V \otimes V \otimes V$? Hint: think of $\mathrm{SU}_2(\mathbb{R})$, and use that $V \otimes V \otimes V$ is the same as $(V \otimes V) \otimes V$.

5. (a) Show that $\mathrm{SL}_2(\mathbb{C})/\{1,-1\}$ and $\mathrm{SO}_3(\mathbb{C})$ are isomorphic. Hint: consider a suitable representation V of $\mathrm{SL}_2(\mathbb{C})$ and show that $\mathrm{SL}_2(\mathbb{C})$ fixes a quadratic form of maximal rank on V.

   (b) Is $\mathrm{SL}_2(\mathbb{R})/\{1,-1\}$ isomorphic to $\mathrm{SO}_3(\mathbb{R})$?

Corrigé de l'Examen du 27/03/2001.

1. On note que pour g dans G la conjugaison par g sur $\mathrm{GL}_n(\mathbb{R})$ induit la conjugaison par g sur $\mathrm{M}_n(\mathbb{R})$, donc que l'automorphisme $\rho(g)$ du sous-espace L de $\mathrm{M}_n(\mathbb{R})$ est donné par conjugaison par g. On a $(1+\varepsilon x)^{-1} = 1 - \varepsilon x + O(\varepsilon^2)$, donc:

$$(1+\varepsilon x)y(1+\varepsilon x)^{-1} = y + \varepsilon(xy - yx) + O(\varepsilon^2).$$

   Cela montre que $(\mathrm{Lie}(\rho)x)y = [x,y]$.

2. Dans la section 10 on trouve que L a une base (h, a_+, a_-), avec:

$$[h, a_+] = 2a_+, \quad [h, a_-] = -2a_-, \quad [a_+, a_-] = h.$$

   Il en résulte que le plus grand poids de h agissant sur L est 2, et que a_+ est un "highest weight vector". Donc, par la Proposition 10.4.3, L contient une copie de la représentation $\mathbb{R}[x,y]_2$. Comme L et $\mathbb{R}[x,y]_2$ sont tous les deux de dimension 3, L et $\mathbb{R}[x,y]_2$ sont des représentations isomorphes de G.

3. Notons $u := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. On a alors, pour tout x et y dans $\mathbb{C}^2$:

$$h(x,y) = x^{\mathrm{t}} u \, \overline{y}.$$

   Un élément g de $\mathrm{GL}_2(\mathbb{C})$ est dans G si et seulement si $g^{\mathrm{t}} u g = u$. Ceci montre que G est un sous-groupe fermé de $\mathrm{GL}_2(\mathbb{C})$, et donc, par le théorème de Cartan et Von Neumann, un groupe de Lie réel. Pour déterminer la dimension de G, calculons son algèbre de Lie L. Soit a dans $\mathrm{M}_2(\mathbb{C})$. Pour ε réel et petit on a:

$$(1+\varepsilon a)^{\mathrm{t}} u \overline{(1+\varepsilon a)} = (1+\varepsilon a^{\mathrm{t}}) u (1+\varepsilon \overline{a}) = u + \varepsilon(a^{\mathrm{t}} u + u\overline{a}) + O(\varepsilon^2).$$

   Donc $a \in L$ si et seulement si $a^{\mathrm{t}} u + u\overline{a} = 0$. Un calcul donne que cette condition est équivalente à: $a_{1,1} \in i\mathbb{R}$, $a_{2,2} \in i\mathbb{R}$, et $a_{2,1} = \overline{a_{1,2}}$. On trouve une $\mathbb{R}$-base:

$$\left(\left(\begin{smallmatrix} i & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & i \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)\right).$$

   Donc G est de dimension 4. Comme G contient tous les $\exp(t\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)) = \left(\begin{smallmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{smallmatrix}\right)$ (avec t dans $\mathbb{R}$), G n'est pas compacte car non borné dans $\mathrm{M}_2(\mathbb{C})$.

4. Notons que $\mathrm{SO}_3(\mathbb{R}) = \mathrm{SU}_2(\mathbb{R})/\{1,-1\}$, donc les représentations de $\mathrm{SO}_3(\mathbb{R})$ sont celles de $\mathrm{SU}_2(\mathbb{R})$ où -1 opère trivialement. Montrons d'abord que $V \cong \mathbb{C}[x,y]_2$ en tant que représentation de $\mathrm{SU}_2(\mathbb{R})$. Comme V est de dimension 3, il suffit de montrer que V est irréductible ($\mathbb{C}[x,y]_2$ est la seule représentation irréductible de dimension 3 de $\mathrm{SU}_2(\mathbb{R})$). Comme $\mathrm{SO}_3(\mathbb{R})$ fixe le produit scalaire (hermitien) standard de V, il suffit de montrer que dans V il n'y a pas de droite complexe fixé par $\mathrm{SO}_3(\mathbb{R})$. Pour une droite $\mathbb{C}v$ de $V = \mathbb{C}^3$ il existe toujours un élément w de $\mathbb{R}^3$ tel que $\mathbb{C}w \neq \mathbb{C}v$ et tel que $\langle v, w\rangle \neq 0$; les éléments de $\mathrm{SO}_3(\mathbb{R})$ d'axe $\mathbb{R}w$ ne fixent pas $\mathbb{C}v$.

   Donc $V \cong \mathbb{C}[x,y]_2$. Par le Théorème 13.5 on a:

$$\mathbb{C}[x,y]_2 \otimes \mathbb{C}[x,y]_2 \cong \mathbb{C}[x,y]_4 \oplus \mathbb{C}[x,y]_2 \oplus \mathbb{C}[x,y]_0,$$

   donc, encore par le même Théorème:

$$\begin{aligned}(V\otimes V)\otimes V &\cong (\mathbb{C}[x,y]_4 \otimes \mathbb{C}[x,y]_2) \oplus (\mathbb{C}[x,y]_2 \otimes \mathbb{C}[x,y]_2) \oplus (\mathbb{C}[x,y]_0 \otimes \mathbb{C}[x,y]_2)\\ &\cong \mathbb{C}[x,y]_6 \oplus \mathbb{C}[x,y]_4^2 \oplus \mathbb{C}[x,y]_2^3 \oplus \mathbb{C}[x,y]_0.\end{aligned}$$

5. (a) Nous choisissons pour représentation $V := \mathrm{Lie}(\mathrm{SL}_2(\mathbb{C}))$ de $\mathrm{SL}_2(\mathbb{C})$, où l'action est induite par conjugaison comme dans l'exercice 1. Notons ρ le morphisme $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{\mathbb{C}}(V)$ correspondant. Comme $\mathrm{SL}_2(\mathbb{C})$ est connexe, $\ker(\rho)$ est égal au centre de $\mathrm{SL}_2(\mathbb{C})$, donc à $\{1,-1\}$. Comme l'action sur $V \subset \mathrm{M}_2(\mathbb{C})$ est par conjugaison, le déterminant est fixé. Visiblement, $\left(\begin{smallmatrix} a & c\\ b & -a\end{smallmatrix}\right) \mapsto -a^2 - bc$ est une forme quadratique de rang 3 sur V. Par rapport à une base convenable, la forme bilinéaire symétrique associée à cette forme quadratique est donnée par la matrice identité, ce qui nous donne un morphisme $f\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{O}_3(\mathbb{C})$ avec $\ker(f) = \{1,-1\}$. Pour g dans $\mathrm{O}_3(\mathbb{C})$ on a $\det(g) = \pm 1$, donc, par connexité de $\mathrm{SL}_2(\mathbb{C})$, f a son image dans $\mathrm{SO}_3(\mathbb{C})$. Comme $\mathrm{SO}_3(\mathbb{C})$ est connexe (ceci mérite quelques détails) et de même dimension que $\mathrm{SL}_2(\mathbb{C})$, on a $\mathrm{SL}_2(\mathbb{C})/\{1,-1\} \cong \mathrm{im}(f) = \mathrm{SO}_3(\mathbb{C})$.

   On peut aussi prendre $V := \mathbb{C}[x,y]_2$. Dans ce cas, $\mathrm{SL}_2(\mathbb{C})$ fixe la forme quadratique $ax^2 + bxy + cy^2 \mapsto b^2 - 4ac$, le discriminant.

   (b) Non, car sinon $\mathrm{SL}_2(\mathbb{R})/\{1,-1\}$ serait connexe, donc $\mathrm{SL}_2(\mathbb{R})$ aussi.

Le barême appliqué est: 2+3+5+5+5=20.

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