Cours à géométrie variable, D.E.A., automne 1996.

Bas Edixhoven, Université de Rennes I.

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This course was given in Rennes at the DEA level, simultaneously with a course on homological and commutative algebra (referred to as Berthelot's course in the text that follows). The two courses, of 30 hours each, were meant to form an introduction to some basic notions and results in algebraic geometry, real algebraic geometry, analytic geometry and differential geometry, preparing for more specialized courses in the second semester. It should be possible to understand these course notes without having followed the other course, by working through Section 1 of Chapter II and Sections 1 and 2 of Chapter III of Hartshorne's book "Algebraic Geometry".

Contents

1 Differentiable varieties

We are going to study various kinds of geometrical objects, such as differentiable varieties, analytic varieties and algebraic varieties. These objects occur in many areas of mathematics, ranging from physics (general relativity, conformal field theory, mechanics) to algebra and number theory (algebraic geometry) and analysis (Lie groups, differential equations). Our time is quite limited (30 hours), so many details will not be discusssed in the lectures. The lectures will therefore be very incomplete; the students taking this course are advised to fill in the gaps as much as possible by studying text books on the subject. Some books I can think of at this moment are: Spivak's series of books (differential geometry), the book by Bott and Tu (differential forms in algebraic topology), Lang's book (differentiable manifolds) and Bourbaki's book (variétés différentielles et analytiques: fascicule de résultats).

We begin with differentiable varieties, also called manifolds. These are usually defined in terms of charts. Intuitively, they are objects that "locally" look like $\mathbb{R}^n$ for some n. We begin by making that precise.

1.1 Definition. *Let X be a set. An atlas for X then consists of the following data: a set I, for each i in I a subset X_i of X, an integer $n_i \geq 0$, an open subset U_i of $\mathbb{R}^{n_i}$ and a bijection $\phi_i\colon U_i \to X_i$. These data are required to satisfy the following conditions. Firstly, the X_i cover X, that is, $\cup_i X_i = X$. Secondly, the charts ϕ_i are compatible, in the sense that we will now explain. For i and j in I let $X_{i,j}$ be $X_i \cap X_j$, and let $U_{i,j}$ be $\phi_i^{-1}X_{i,j}$. Then ϕ_i induces a bijection, still denoted ϕ_i, from $U_{i,j}$ to $X_{i,j}$. Saying that ϕ_i and ϕ_j are compatible means that $U_{i,j}$ is open in U_i, $U_{j,i}$ open in U_j, and the bijection $\phi_j^{-1}\circ\phi_i\colon U_{i,j} \to U_{j,i}$ is differentiable.*

Some remarks are in order here. First of all, the differentiability of $\phi_j^{-1}\circ\phi_i$ in the definition can mean various things. When we just say differentiable, we mean in fact infinitely differentiable, that is, the n_j $\mathbb{R}$-valued functions making up $\phi_j^{-1}\circ\phi_i$ are C^∞-functions on $U_{i,j}$. But we could also consider functions of class C^k for some $k \geq 0$; in that case we will say that the atlas is C^k. Note that the terminology "differentiable" is misplaced in the case $k = 0$; in that case we speak of a topological atlas. The second remark concerns the integers n_i.

1.2 Definition. *Let $k \geq 0$ be an integer or ∞. A variety or manifold of class C^k is a set X equipped with a C^k-atlas. Notation: (X, I, n, U, ϕ).*

For X a C^k-variety and x in X, all n_i for i such that X_i contains x are equal; this integer is called the dimension of X at x; we denote it by $\dim_X(x)$, so that we can view $\dim_X$ as a $\mathbb{Z}$-valued function on X. (For $k > 0$ the equality of the n_i is easy to prove (consider derivatives and use linear algebra); for $k = 0$ one needs some algebraic topology.) Most of the time we will just consider the C^∞ case. As usual, defining the objects to study is not too interesting; we should also say what maps between them we want to consider. For example, we want to say what it means that two manifolds are isomorphic.

1.3 Definition. *Let (X, I, n, U, ϕ) and (Y, J, m, V, ψ) be manifolds. Let f be a map from X to Y. Let x be in X. Then f is called differentiable at x if for every (i, j) such that*

$x \in X_i$ and $f(x) \in Y_j$ the subset $\phi_i^{-1}((f^{-1}Y_j) \cap X_i) \subset \mathbb{R}^{n_i}$ is open and the map $\psi_j^{-1} f \phi_i$ from $\phi_i^{-1}((f^{-1}Y_j) \cap X_i) \subset \mathbb{R}^{n_i}$ to $\mathbb{R}^{m_j}$ is differentiable at $\phi_i^{-1}(x)$. The map f is called differentiable, or a morphism of manifolds, if it is differentiable at all x in X.

Note that this definition does not change if we require the openness and differentiability at x only for one pair (i, j). If $f: X \to Y$ and $g: Y \to Z$ are morphisms, then $g \circ f: X \to Z$ is also a morphism. So we have the category of manifolds: we have objects, morphisms, composition of morphisms, the composition is associative and each object X has an identity morphism id_X. A morphism $f: X \to Y$ is called an isomorphism if and only if there exists a morphism $g: Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. Equivalently: a map $f: X \to Y$ is an isomorphism if and only it is bijective and f and f^{-1} are differentiable. Let us look at some examples of manifolds.

Let $n \geq 0$. Then $\mathbb{R}^n$ with the atlas consisting of the chart $\mathrm{id}_{\mathbb{R}^n}$ is a manifold, that we will denote by $\mathbb{R}^n$. In the same way, every open subset of some $\mathbb{R}^n$ becomes a manifold. If X and Y are manifolds (we have already dropped the atlas from the notation), then $X \times Y$ is easily equipped with an atlas (take $K := I \times J$, $W_{i,j} := U_i \times V_j$, etc.). We leave it as an exercise to the reader to verify that the two projections pr_X and pr_Y from $X \times Y$ to X and Y are differentiable, and that $(X \times Y, \mathrm{pr}_X, \mathrm{pr}_Y)$ has the following universal property: for Z a manifold and morphisms $f: Z \to X$ and $g: Z \to Y$ there exists a unique morphism $h: Z \to X \times Y$ such that $f = \mathrm{pr}_X h$ and $g = \mathrm{pr}_Y h$. Let V be a finite dimensional $\mathbb{R}$-vector space, say of dimension n. Then an isomorphism of $\mathbb{R}$-vector spaces $\phi: \mathbb{R}^n \to V$ makes V into a manifold. It is clear that two such isomorphisms $\phi \neq \phi'$ give different atlases but that id_V is an isomorphism between the manifolds.

For $n \geq 0$ the subset $\mathrm{GL}_n(\mathbb{R})$ of $\mathrm{M}_n(\mathbb{R})$ consisting of invertible n by n matrices with coefficients in $\mathbb{R}$ is an open subset (it is $\det^{-1}(\mathbb{R} - \{0\})$). It is easy to check that the maps $m: \mathrm{GL}_n(\mathbb{R}) \times \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ and $i: \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R})$ given by $m(x, y) = xy$ and $i(x) = x^{-1}$ are differentiable (for i, use the formula for x^{-1} in terms of the matrix of cofactors and $\det(x)$). In general, a group G equipped with an atlas such that the multiplication and inversion are differentiable is called a Lie group. We will see more examples soon.

Our next example is in a sense more interesting, because it is not isomorphic to an open subset of $\mathbb{R}^n$ for any n. We consider the circle S^1 in $\mathbb{R}^2$: it is the set of (x, y) such that $x^2 + y^2 = 1$. One way to make an atlas is the following. The projection on the first coordinate gives a bijection from $\{(x, y) \in S^1 \mid y > 0\}$ to the open interval $]-1, 1[$; the same holds for $\{(x, y) \in S^1 \mid y < 0\}$. We also have the projections on the second coordinate from the sets $\{(x, y) \in S^1 \mid x > 0\}$ and $\{(x, y) \in S^1 \mid x < 0\}$. The four inverses of these maps form an atlas. Another atlas is obtained by restricting the map $(\sin, \cos)$ from $\mathbb{R}$ to $\mathbb{R}^2$ to suitable subsets of $\mathbb{R}$. Yet a third atlas is given by projection from points of S^1. For t in $\mathbb{R}$ consider the line through $(t, 0)$ and $(0, 1)$. This line intersects S^1 in $(0, 1)$ and a unique other point: $(2t/(t^2+1), (t^2-1)/(t^2+1))$. This map gives a bijection from $\mathbb{R}$ to $S^1 - \{(0, 1)\}$. Considering lines through $(0, -1)$ gives a second chart. The map id_{S^1} is an isomorphism between all three atlases for S^1 that we have just seen. It is interesting to note what kind of functions we get from the charts and transition maps (i.e., the $\phi_j^{-1}\phi_i$) in these three cases. In the first case the charts and the transition maps are algebraic functions (they are built up from rational functions and square roots). In the second case the charts are given by the transcendental functions sin and cos, but the transition maps are just translations in $\mathbb{R}$. In the third case all functions are rational functions.

We could treat the n-sphere S^n in a similar way (it is defined as the subset of $(x_0, \ldots, x_n)$ in $\mathbb{R}^{n+1}$ such that $x_0^2 + \cdots + x_n^2 = 1$). In particular, the first and third methods we used for $n = 1$ are easily adapted (not the second method, as far as I can see; it has to do with the fact that S^n is simply connected for $n \geq 2$ (well, of course, one has the usual spherical coordinates r, θ and ϕ, but it is not as nice)). But it is more useful to develop a systematic way to make subsets of $\mathbb{R}^n$ that are defined by suitable equations into manifolds. In order to do this we need the implicit function theorem. We will state this theorem in a quite general context, so that it will suffice for the whole course.

1.4 Theorem. *Let K be either $\mathbb{R}$ or $\mathbb{C}$. Let X, Y and Z be normed K-vector spaces, with Y complete. Let U be an open subset of $X \times Y$, $k \geq 1$ and $f: U \to Z$ a C^k-map. Let (x, y) be in U such that the derivative $(D_2 f)(x, y): Y \to Z$ of f with respect to the second variable is an isomorphism of topological vector spaces (i.e., it is bijective and its inverse is continuous). Then there exist open neighborhoods V of x in X and W of y in Y such that $V \times W \subset U$ and for every v in V there exists a unique w in W with $f(v, w) = f(x, y)$. The map $g: V \to Y$ thus defined is C^k. Moreover, if X and Y are finite dimensional and f analytic, then g is analytic.*

For a proof the reader is referred to the standard text books, or to course notes from analysis courses. In the complex case, i.e., $K = \mathbb{C}$, we say that $f: U \to Z$ is C^k if it is so when we view X, Y and Z as $\mathbb{R}$-vector spaces. When we want to talk about differentiability in the complex sense, we will allways explicitly say so. The reason for this terminology is that a function $f: U \to \mathbb{C}$, with $U \subset \mathbb{C}^n$ open, is analytic if and only if it is C^1 in the complex sense (see section 2).

Let us now consider the following situation. We have positive integers n and m, we have an open subset U of $\mathbb{R}^n$ and a C^k map f from U to $\mathbb{R}^m$, for some $k \geq 1$. Let X be the set of zeroes of f: $X := \{x \in U \mid f(x) = 0\}$. We want to equip X with an atlas, in some natural way (for example, the charts should be C^k-maps to $\mathbb{R}^n$). It turns out that at least some conditions have to be satisfied for this to be possible. For example, consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$. Then X is the union of the two coordinate axes; consequently, X, with its induced topology, cannot be a C^0-manifold, since no neighborhood of $(0, 0)$ is homeomorphic to an open interval in $\mathbb{R}$. Note that $(0, 0)$ is special, since both partial derivatives of f vanish at that point, i.e., f has derivative zero at $(0, 0)$. So, in the situation above, we assume that for all x in X the derivative $(Df)x$ is surjective (i.e., f is a submersion at all x in X). Let now x be in X. Let V be the kernel of the linear map $(Df)x: \mathbb{R}^n \to \mathbb{R}^m$. Let $V^\perp$ be the orthogonal of V (for the standard inner product on $\mathbb{R}^n$). We view $\mathbb{R}^n$ as the product $V \times V^\perp$, and hence U as an open subset of $V \times V^\perp$. In this situation we can apply Theorem 1.4, since $(D_2 f)x$ is an isomorphism from $V^\perp$ to $\mathbb{R}^m$. We get an open subset $V' \subset V$ and a C^k-map $g: V' \to V^\perp$ such that x is in $V' \times V^\perp$ and for all v in V' we have $f(v, g(v)) = 0$. Hence the map $\phi: V' \to X$ defined by $\phi(v) = (v, g(v))$ is a chart at x. We can obviously cover X with such charts (for example, take one chart for each x). These charts are compatible because ϕ^{-1} is just the orthogonal projection on $V^\perp$. Note that $\dim_X(x) = \dim V^\perp = n - m$.

1.5 Exercise. Suppose we have $U \subset \mathbb{R}^n$ open, and two C^k-maps $f: U \to \mathbb{R}^m$ and $f': U \to \mathbb{R}^{m'}$, defining the same X and both submersions at all x in X. Then show that any two atlases obtained from the construction above are such that id_X is an isomorphism between them. $\square$

We can now easily give some more examples of Lie groups: the classical matrix groups.

1.6 Example. Let $n \geq 1$. The group special linear group $\mathrm{SL}_n(\mathbb{R})$ is defined as the kernel of the morphism of groups $\det: \mathrm{GL}_n(\mathbb{R}) \to \mathbb{R}^*$. We have to show that for all x in $\mathrm{SL}_n(\mathbb{R})$ the derivative $(D\det)x$ is non-zero. So we have to "compute" $\det(x + \varepsilon y)$ for small ε in $\mathbb{R}$ and any y in $\mathrm{M}_n(\mathbb{R})$. We have:

$$\det(x+\varepsilon y) = \det(x(1+\varepsilon x^{-1}y)) = \det(x)\det(1+\varepsilon x^{-1}y) = 1 + \varepsilon \mathrm{tr}(x^{-1}y) + O(\varepsilon^2), \tag{1.6.1}$$

with $\mathrm{tr}(x^{-1}y)$ the trace of $x^{-1}y$. It follows that $((D\det)x)y = \mathrm{tr}(x^{-1}y)$. This cannot be zero for all y, since x^{-1} is invertible. $\square$

1.7 Example. Let again $n \geq 1$. The orthogonal group $\mathrm{O}_n(\mathbb{R})$ is the subgroup of x in $\mathrm{GL}_n(\mathbb{R})$ that preserve the standard scalar product of $\mathbb{R}^n$, i.e., the x such that $x^t x = 1$. The special orthogonal group is the subgroup $\mathrm{SO}_n(\mathbb{R})$ of x in $\mathrm{O}_n(\mathbb{R})$ with $\det(x) = 1$. We consider the map $f: \mathrm{GL}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})^+$ given by $f(x) = x^t x - 1$, where $\mathrm{M}_n(\mathbb{R})^+$ denotes the set of real symmetric n by n matrices. We have to show that this map is submersive at all x in $\mathrm{O}_n(\mathbb{R})$. For x in $\mathrm{O}_n(\mathbb{R})$, y in $\mathrm{M}_n(\mathbb{R})$ and small ε in $\mathbb{R}$ we have (with $z := x^{-1}y$):

$$\begin{aligned} f(x+\varepsilon y) &= f(x(1+\varepsilon z)) = (1+\varepsilon z)^t x^t x (1+\varepsilon z) - 1 = (1+\varepsilon z^t)(1+\varepsilon z) - 1 = \\ &= \varepsilon(z^t + z) + O(\varepsilon^2). \end{aligned} \tag{1.7.1}$$

It follows that $((Df)x)y = z^t + z$,which clearly shows that $(Df)x$ is surjective. So $\mathrm{O}_n(\mathbb{R})$ is now a manifold. Our computation above also shows that it has everywhere the same dimension, namely $(n^2 - n)/2 = \dim_\mathbb{R}(\mathrm{M}_n(\mathbb{R}))^-$, the dimension of the $\mathbb{R}$-vector space of anti-symmetric n by n matrices. For every x in $\mathrm{O}_n(\mathbb{R})$ we have $1 = \det(x^t x) = \det(x)^2$, hence $\det(x) = \pm 1$. There are x in $\mathrm{O}_n(\mathbb{R})$ with $\det(x) = -1$, hence we have a short exact sequence:

$$1 \to \mathrm{SO}_n(\mathbb{R}) \to \mathrm{O}_n(\mathbb{R}) \to \{\pm 1\} \to 1. \tag{1.7.2}$$

This sequence is split: send -1 to the diagonal matrix $\mathrm{diag}(-1, 1, \ldots, 1)$, for example. If n is odd, we even have a splitting with image in the center: send -1 to -1; hence for odd n we have an isomorphism of Lie groups from $\mathrm{O}_n(\mathbb{R})$ to $\mathrm{SO}_n(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}$. For even n there is not such a splitting, and one cannot do better than say that $\mathrm{O}_n(\mathbb{R})$ is isomorphic to the semi-direct product $\mathrm{SO}_n(\mathbb{R}) \times_\alpha \mathbb{Z}/2\mathbb{Z}$, with $\alpha: \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}(\mathrm{SO}_n(\mathbb{R}))$ the morphism of groups that sends -1 to the inner automorphism given by conjugation by $\mathrm{diag}(-1, 1, \ldots, 1)$. $\square$

1.8 Example. Again, $n \geq 1$. The symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ that preserves the "standard" alternating bilinear form on $\mathbb{R}^{2n}$ that is given by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, in which the coefficients are n by n matrices. One computes easily that $\mathrm{Sp}_{2n}(\mathbb{R})$ is the subset of $\mathrm{GL}_{2n}(\mathbb{R})$ of the x that satisfy $x^t\psi x = \psi$. So we consider the map f from $\mathrm{GL}_{2n}(\mathbb{R})$ to $M_{2n}(\mathbb{R})^-$ given by $f(x) = x^t\psi x - \psi$. Let x be in $\mathrm{Sp}_{2n}(\mathbb{R})$, y in $M_{2n}(\mathbb{R})$ and ε in $\mathbb{R}$. Put $z := x^{-1}y$. Then we have:

$$(1.8.1) \qquad f(x+\varepsilon y) = f(x(1+\varepsilon z)) = (1+\varepsilon z^t)\psi(1+\varepsilon z) - \psi = \varepsilon(z^t\psi + \psi z) + O(\varepsilon^2).$$

This shows that $((Df)x)y = z^t\psi + \psi z$, so we have to show that the map $z \mapsto z^t\psi + \psi z$ from $\mathrm{M}_{2n}(\mathbb{R})$ to $\mathrm{M}_{2n}(\mathbb{R})^-$ is surjective. To do this, we compute its kernel (this is interesting anyway, since this kernel is what is called the Lie algebra of $\mathrm{Sp}_{2n}(\mathbb{R})$). So write z as a two by two matrix of n by n matrices: $z = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then a short computation gives:

$$(1.8.2) \qquad z^t\psi + \psi z = 0 \iff (c^t = c \text{ and } b^t = b \text{ and } d = -a^t).$$

It follows that the kernel has dimension $2n^2 + n$. This is equal to $\mathrm{M}_{2n}(\mathbb{R})^- = ((2n)^2 - (2n))/2$. Linear algebra then implies that our map $z \mapsto z^t\psi + \psi z$ is surjective. □

1.9 Exercise. Let $n \geq 1$. Show that the Lie groups $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$ and $\mathrm{Sp}_{2n}(\mathbb{R})$ are connected (the last one is more difficult to do). Show also that $\mathrm{Sp}_{2n}(\mathbb{R})$ is contained in $\mathrm{SL}_{2n}(\mathbb{R})$. □

1.10 Exercise. Let $n \geq 1$. Show that $\mathrm{GL}_n(\mathbb{C})$ is a manifold (i.e., make it into one, in the right way). The unitary group $\mathrm{U}_n(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of those x that preserve the standard scalar product on $\mathbb{C}^n$ (the one that sends (v, w) to $v_1\overline{w_1} + \cdots + v_n\overline{w_n}$), or, equivalently, the x with $\overline{x}^t x = 1$. Make $\mathrm{U}_n(\mathbb{R})$ into a manifold. Compute its dimension. Show that it is connected and compact. Do the same things for its subgroup $\mathrm{SU}_n(\mathbb{R})$ consisting of those x with $\det(x) = 1$. □

We have already quite a few examples at our disposal, and it seems a good moment to do something about the foundations again. The reader is certainly aware that up to now each time we had various atlases on one set X, they had the property that id_X was an isomorphism between them. There should be a much more natural way to express this. In fact, we should replace the atlases by something else, giving us an equivalent category (this will be made precise). For example, when one studies groups, it is very clumsy to deal only with groups in terms of generators and relations. So we look for an object associated to an atlas for X such that two atlases such that id_X is an isomorphism between them give exactly the same object. We do this by considering a topology on X and the notion of differentiable functions on open subsets of X.

Let X be a set equipped with a C^k-atlas (some $k \geq 0$) (I, n, U, ϕ). Then we can define a topology on X by saying that a subset V of X is open if and only if for all i in I the subset $\phi_i^{-1}(V \cap X_i)$ is open in $\mathbb{R}^{n_i}$. (The verification that this works is left to the reader, and also that of the following assertion.) A subset V of X is open if and only if for all x in V there exists an i such that the subset $\phi_i^{-1}(V \cap X_i)$ of $\mathbb{R}^{n_i}$ is open. Suppose now that $V \subset X$ is open. Then a function $f: V \to \mathbb{R}$ is called of class C^k if and only if for all i the function $f \circ \phi_i$ on $\phi_i^{-1}(V \cap X_i)$ is C^k. The reader will verify that $f: V \to \mathbb{R}$ is C^k if and only if for all x in V there exists i such that $f \circ \phi_i$ on $\phi_i^{-1}(V \cap X_i)$ is C^k. The set of C^k functions on V will be denoted by $C_X^k(V)$; it is clearly an $\mathbb{R}$-algebra, usually of infinite dimension. For an open subset W contained in V we have the restriction map $\mathrm{res}(V,W): C_X^k(V) \to C_X^k(W)$, that sends a function f on V to its restriction $f|_W$ to W. It is clear that for Z an open subset contained in W we have $\mathrm{res}(V,Z) = \mathrm{res}(W,Z)\mathrm{res}(V,W)$. In Berthelot's course it will be explained that such a collection of sets $C_X^k(V)$ and maps $\mathrm{res}(V,W)$ is what one calls a presheaf on the topological space X, denoted C_X^k. A very important property of this presheaf C_X^k is the following direct consequence of the local nature of a function being C^k: if we have an open subset V of X, and a covering of V by open subsets V_j with the j in some set J, and for each j an element f_j of $C_X^k(V_j)$ such that for all j and j' we have $\mathrm{res}(V_j, V_j \cap V_{j'})f_j = \mathrm{res}(V_{j'}, V_j \cap V_{j'})f_{j'}$, then there exists a unique f in $C_X^k(V)$ such that $\mathrm{res}(V, V_j)(f) = f_j$ for all j. (I apologize for the long sentence.) In general, a presheaf that satisfies this glueing condition will be called a sheaf. For the moment, the notions of sheaf and presheaf are just convenient for us for notational matters; we won't do anything complicated with sheaves for some time. The real work concerning sheaves will be done in Berthelot's course.

Given a set X with an atlas, the object we associate to it is the pair (X, C_X^k), consisting of a topological space and a sheaf of rings on it. Such objects are called ringed spaces, and all geometrical objects we will consider in this course will be ringed spaces. Let us now look at what it means for a map to be a morphism in terms of these sheaves.

1.11 Proposition. *Let X and Y be manifolds, and let $f: X \to Y$ be a map (of sets). Then f is a morphism of manifolds if and only if f is continuous and for each open U in Y and $g \in C_Y(U)$ the function gf is in $C_X(f^{-1}U)$.*

Proof. Suppose that f is a morphism of manifolds, i.e., f is differentiable. Then it follows directly from the definitions, and the fact that compositions of differentiable maps are differentiable, that f is continuous and that for each open U in Y and $g \in C_Y(U)$ the function gf is in $C_X(f^{-1}U)$.

Suppose now that f is continuous and that for each open U in Y and g in $C_Y(U)$ the function gf is in $C_X(f^{-1}U)$. Then $X_i \cap f^{-1}Y_j$ is open in X_i, hence $\phi_i^{-1}(X_i \cap f^{-1}Y_j)$ is open in U_i, hence in $\mathbb{R}^{n_i}$. We have to show that the map $\psi_j^{-1} f \phi_i$ from $\phi_i^{-1}(X_i \cap f^{-1}Y_j)$ to $V_j \subset \mathbb{R}^{m_j}$ is differentiable. It is equivalent to show that the m_j coordinate functions $x_k \psi_j^{-1} f \phi_i$ of this map are differentiable. Now $x_k \psi_j^{-1}$ is in $C_Y(Y_j)$, hence $x_k \psi_j^{-1} f$ is in $C_X(f^{-1}Y_j)$. It follows that $x_k \psi_j^{-1} f \phi_i$ is differentiable. □

We are now ready to formulate a new, improved definition of the category of manifolds. Note that a morphism of manifolds $f: X \to Y$ induces, for every open U in Y, a morphism of $\mathbb{R}$-algebras $f^*(U): C_Y(U) \to C_X(f^{-1}U)$. In the language of sheaves, this is just a morphism of sheaves from C_Y to f_*C_X. For $f\colon X \to Y$ a morphism of topological spaces and F a sheaf on X, f_*F is the sheaf defined by $(f_*F)(U) = F(f^{-1}U)$. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair (f, ϕ) with f a continuous map from X to Y and ϕ a morphism of sheaves from $\mathcal{O}_Y$ to $f_*\mathcal{O}_X$. Let (f, ϕ) be a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ and (g, γ) a morphism from $(Y, \mathcal{O}_Y)$ to $(Z, \mathcal{O}_Z)$. Then $k := gf\colon X \to Z$ is continuous, and for every open $U \subset Z$ we have a morphism of rings $\kappa(U)$ from $\mathcal{O}_Z(U)$ to $\mathcal{O}_X(k^{-1}U)$ obtained by composing $\gamma(U)\colon \mathcal{O}_Z(U) \to \mathcal{O}_Y(g^{-1}U)$ and $\phi(g^{-1}U)\colon \mathcal{O}_Y(g^{-1}U) \to \mathcal{O}_X(f^{-1}g^{-1}U)$. In the case where X, Y and Z are obtained from manifolds as above, these maps are just the maps that do pullback of functions. Anyway, one easily verifies that κ is a morphism of sheaves (i.e., the $\kappa(U)$ are compatible with te restriction maps) so that (k, κ) is a morphism of ringed spaces. This composition of morphisms gives us a category: the category of ringed spaces. So now we also have the notion of isomorphisms between ringed spaces.

But in fact all this is not exactly what we need at this moment. Our ringed spaces (X, C_X) are somehow special: the sheaf C_X is a sheaf of $\mathbb{R}$-valued functions. If $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ are topological spaces with sheaves of $\mathbb{R}$-valued functions, we define a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ to be a continuous map $f\colon X \to Y$ such that for all $U \subset Y$ open and all g in $\mathcal{O}_Y(Y)$ the function $f^*g := gf$ is in $\mathcal{O}_X(f^{-1}U)$. One checks immediately that this also gives us a category (let us call it the category of topological spaces with a sheaf of $\mathbb{R}$-valued functions), hence also a notion of isomorphism. For $(X, \mathcal{O}_X)$ a topological space with a sheaf of $\mathbb{R}$-valued functions, and $U \subset X$ open, we have the ringed space $(U, \mathcal{O}_X|_U)$, with $\mathcal{O}_X|_U(V) = \mathcal{O}_X(V)$ for all open $V \subset U$; this topological space with a sheaf of $\mathbb{R}$-valued functions is called the open subspace U of X. We can now state our improved definition of manifolds.

1.12 Definition. *Let $k \geq 0$. A C^k-manifold is a topological space with a sheaf of $\mathbb{R}$-valued functions (X, C_X), that is locally isomorphic (in the category of topological spaces with sheaves of $\mathbb{R}$-valued functions) to some (U, C^k_U), with U an open subset of some $\mathbb{R}^n$ and C^k_U the sheaf of C^k-functions on U. (Note that n may vary.) A morphism from a C^k manifold (X, C_X) to a C^k-manifold (Y, C_Y) is a morphism in the category of topological spaces with a sheaf of $\mathbb{R}$-valued functions.*

Let us now see what it means that this definition is equivalent to the older one. So let, for a moment, **Man** and **Man′** denote the categories of C^k-manifolds in the old and the new sense, respectively. Then we have a functor F from **Man** to **Man′** that sends a set X with a C^k-atlas to the space (X, C^k_X) as explained above, and sends a morphism $f\colon X \to Y$ in **Man** to the morphism (f, f^*) in **Man′**. By saying that our two definitions are equivalent, we mean that F is induces an equivalence of categories from **Man** to **Man′**. By definition, this means that there is a functor G from **Man′** to **Man** such that GF and FG are isomorphic to $\mathrm{id}_{\mathbf{Man}'}$ and $\mathrm{id}_{\mathbf{Man}}$, respectively. General category nonsense says that such a G exists if and only if F is full and faithful (i.e, for all X and Y in **Man** F gives a bijection from $\mathrm{Hom}_{\mathbf{Man}}(X, Y)$ to $\mathrm{Hom}_{\mathbf{Man}'}(F(X), F(Y))$) and essentially surjective (i.e., every $(X, \mathcal{O}_X)$ in **Man′** is isomorphic to the image under F of some object of **Man**). Proposition 1.11 means that F is full and faithful. It remains to show that F is essentially surjective. So let $(X, \mathcal{O}_X)$ be an object of **Man′**. Then we can cover X by open sets X_i such that the $(X_i, \mathcal{O}_X|_{X_i})$ are isomorphic to some (U, C^k_U) with U open in some $\mathbb{R}^n$. These isomorphisms form an atlas on X for which one easily verifies that it induces the sheaf $\mathcal{O}_X$.

We can give another interpretation of the manifold structure on sets of zeroes of equations that we defined just after Theorem 1.4. The situation is the following: we have an open subset U of some $\mathbb{R}^n$, a C^k-map (some $k \geq 1$) $f: U \to \mathbb{R}^m$ such that $(Df)x$ is surjective for all x in $X := f^{-1}0$. Note that X is a closed subset of U. Let Z be any closed subset of U. Then we get a sheaf C^k_Z of $\mathbb{R}$-algebras on Z by:

$$C^k_Z(V) := \{g: V \to \mathbb{R} \,|\, \exists h \in C^k_U(V') \text{ with } V' \subset U \text{ open, } V = V' \cap Z \text{ and } g = h|_V\}.$$

It is left to the reader to show that for X as above, this space (X, C^k_X) is in fact the manifold obtained by the construction mentioned above. (I realize that in this last part I really skipped a lot of details.)

We finish this section with some more examples: projective spaces and Grassmannians.

1.13 Example. Let k be a field, and V a k-vector space. Then we define the projective $\mathbb{P}(V)$ space associated to V to be the set of 1-dimensional subspaces of V (i.e., the set of lines through the origin). Clearly, we have a bijection $(V - \{0\})/k^* \to \mathbb{P}(V)$ induced by the map that sends $v \neq 0$ to the subspace kv. For $n \geq 0$ we define $\mathbb{P}^n(k)$ to be $\mathbb{P}(k^{n+1})$. An element in $\mathbb{P}^n(k)$ will be denoted as $(a_0 : a_1 : \cdots : a_n)$ with $a_i \in k$ not all zero; in this notation we have $(a_0 : \cdots : a_n) = (b_0 : \cdots : b_n)$ if and only if there exists $\lambda \in k^*$ with $b_i = \lambda a_i$ for all i. Clearly, $\mathbb{P}^0(k)$ is just the one point (1). We can describe $\mathbb{P}^1(k)$ as follows: it is the disjoint union of the set $\{(a:1) \,|\, a \in k\}$, that we can identify with k, and the point $(1:0)$ "at infinity". In the same way, $\mathbb{P}^n(k)$ is easily seen to be the disjoint union of k^0, $k^1, \ldots, k^n$. If we want to equip $\mathbb{P}^n(k)$ with an atlas (say for $k = \mathbb{R}$ or $\mathbb{C}$), then these disjoint unions are not so useful: we need "open subsets". So we consider the covering of $\mathbb{P}^n(k)$ by the sets $\mathbb{P}^n(k)_i := \{a_0 : \cdots : a_n \,|\, a_i \neq 0\}$, for $0 \leq i \leq n$. For each i we have a bijection ϕ_i from k^n to $\mathbb{P}^n(k)_i$, sending $(x_1, \ldots, x_n)$ to the point $(a_0 : \cdots : a_n)$ such that $a_i = 1$, $a_j = x_j$ for $j > i$ and $a_j = x_{j+1}$ for $j < i$. The inverse of ϕ_i sends $(a_0 : \cdots : a_n)$ to the $n$tuple of a_j/a_i with $j \neq i$. So, in the usual notation for charts, we have $U_i = k^n$ for all i. For $j > i$ the subset $U_{i,j}$ of $U_i = k^n$ is the set $\{x \in k^n \,|\, x_j \neq 0\}$, which is an open subset if $k = \mathbb{R}$ or $k = \mathbb{C}$. The map $\phi_j^{-1}\phi_i$ from $U_{i,j}$ to k^n sends $(x_1, \ldots, x_n)$ to the $n$tuple consisting of the x_l/x_j with $l < i$, $1/x_j$, and the x_l/x_j with $j \neq l \geq i$. Clearly, these maps are differentiable if $k = \mathbb{R}$ or $\mathbb{C}$. □

1.14 Exercise. Show that the $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$ are connected and compact. Show that $\mathbb{P}^1(\mathbb{R})$ is isomorphic to S^1 and that $\mathbb{P}^1(\mathbb{C})$ is isomorphic to S^2. □

1.15 Example. For k a field, V a k-vector space and $d \geq 0$, let $\mathrm{Gr}_d(V)$ be the set of d-dimensional subspaces of V. Define $\mathrm{Gr}_{d,n}(k) := \mathrm{Gr}_d(k^n)$. Note that $\mathrm{Gr}_1(V) = \mathbb{P}(V)$ and $\mathrm{Gr}_{1,n}(k) = \mathbb{P}^{n-1}(k)$. For $k = \mathbb{R}$ or $\mathbb{C}$, we want to make the $\mathrm{Gr}_{d,n}(k)$ into manifolds. We will do this in two ways: first by making charts, then by considering $\mathrm{Gr}_{d,n}(k)$ as a quotient of $\mathrm{GL}_n(k)$

by a certain subgroup (in this case we can define charts, or a topological space with a sheaf of functions).

Let $n \geq 0$, $d \geq 0$ and a field k be given. Let V be a k-vector space of dimension n. Let x be in $\mathrm{Gr}_d(V)$, i.e., x is a d-dimensional subspace of V. Choose a subspace y of V such that $V = x \oplus y$. Let $\mathrm{Gr}_d(V)_x$ be the subset $\{z \mid z \cap y = \{0\}\}$ of $\mathrm{Gr}_d(V)$. For each z in $\mathrm{Gr}_d(V)_x$ the projection from z to x along y is an isomorphism. It follows that we have a bijection ϕ_x from $\mathrm{Hom}_k(x, y)$ to $\mathrm{Gr}_d(V)_x$ sending f to $\mathrm{im}(\mathrm{id}_x + f)$. Note that $x = \phi_x(0)$. We view the set $\mathrm{Hom}_k(x, y)$ as a k-vector space in the usual way. We leave it to the reader to study the maps ϕ_x and to verify that, for $k = \mathbb{R}$ or $\mathbb{C}$, they form an atlas. It might be a good idea to look first at what happens when one takes various y's for one x, and then to take one y for various x's. Anyway, it is not necessary to do this, since the next method we use shows that the charts are compatible.

Let us now study $\mathrm{Gr}_{d,n}(k)$ from a different point of view. The group $\mathrm{GL}_n(k)$ acts on the vector space k^n, hence on $\mathrm{Gr}_{d,n}(k)$: an element g sends x to gx, the image of x under g. It is easy to see that $\mathrm{GL}_n(k)$ acts transitively on $\mathrm{Gr}_{d,n}(k)$ (for a given x, choose a basis for x and extend it to a basis of k^n). Let x_0 be the subspace $ke_1 + \cdots + ke_d$ of k^n, where e denotes the standard basis of k^n. The stabilizer $P := \mathrm{GL}_n(k)_x$ of x is the subgroup of $\mathrm{GL}_n(k)$ consisting of those g such that $g_{i,j} = 0$ for all (i, j) with $j \leq d < i$. Hence we get a bijection from $\mathrm{GL}_n(k)/P$ to $\mathrm{Gr}_{d,n}(k)$. Suppose now that $k = \mathbb{R}$ or $\mathbb{C}$. It suffices to equip $\mathrm{GL}_n(k)/P$ with a differentiable structure in order to do so for $\mathrm{Gr}_{d,n}(k)$. As already said above, we can make an atlas for $\mathrm{GL}_n(k)/P$, but we can also make $\mathrm{GL}_n(k)/P$ into a topological space with a sheaf of functions, directly. Since we have already seen numerous atlases, let us first construct the ringed space. For the sake of notation, let $G := \mathrm{GL}_n(k)$, $X := G/P$ and $\pi: G \to X$ be the quotient map. We equip X with the quotient topology: a subset U of X is open if and only if $\pi^{-1}U \subset G$ is open. Then we equip the topological space X with the sheaf of functions that are P-invariant. Note that for $U \subset X$ open, P acts on $C_G(\pi^{-1}U)$ by the formula $(pf)(g) = f(gp)$. For a set S with a P-action, let S^P be the set of elements fixed by P. The we define a sheaf C_X on X by: $C_X(U) := C_G(\pi^{-1}U)^P$. Of course, now we have to verify that the topological space with sheaf of functions (X, C_X) is a manifold. To do this, we have to show that every point has an open neighborhood that it isomorphic to some (U, C_U) with U some open subset of some $\mathbb{R}^n$, and C_U the sheaf of differentiable functions. This is of course almost the same as to make an atlas. Let g be in G. Then the translate gP of P is the orbit of g under P. Let T be the subspace of $\mathrm{M}_n(k)$ consisting of the m with $m_{i,j} = 0$ if $i \leq d$ or $j > d$. Then $\mathrm{M}_n(k)$ is the direct sum of T and the tangent space of P at 1. Let $U_g := T$ and $\phi_g: U_g \to X$ be given by $\phi_g(t) = \pi(g(1 + t))$ (note that $1 + t$ is in G). One verifies that ϕ_g induces an isomorphism between (T, C_T) and $(X_g, C_X|_{X_g})$ with $X_g = \pi(g(1 + T)P)$ open since $(1 + T)P$ is exactly the set of g in G with $\det((g_{i,j})_{1 \leq i,j \leq d}) \neq 0$. What makes this method work is the fact that the map $T \times P \to G$, $(t, p) \to (1 + t)p$, is an isomorphism from $T \times P$ to the open subset $(1 + T)P$ of G. That this is so follows from the simple computation $\left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ ta & tb+d \end{smallmatrix}\right)$.

Let us finish by noting that the charts obtained here and above are in fact the same, so we have now also shown that the charts above are compatible. □

2 Analytic manifolds and varieties

A standard textbook on complex analytic manifolds and varieties is the book "Principles of algebraic geometry", by Griffiths and Harris.

Let $n \geq 0$, $U \subset \mathbb{R}^n$ open and $f: U \to \mathbb{R}$ a function. Then f is said to be (real) analytic at a point u in U if there exists an open neighborhood $u \in V \subset U$ and a formal power series $F = \sum_{i \geq 0} a_i x^i$ in $\mathbb{R}[[x]]$ (multi-index notation: $x = (x_1, \ldots, x_n)$, $i = (i_1, \ldots, i_n)$, $x^i = x_1^{i_1} \cdots x_n^{i_n}$), such that F is convergent on $V - u := \{v - u \mid v \in V\}$ and $f(v) = F(v - u)$ for all v in V. The function f is said to be analytic on U if it is analytic at all u in U. Analytic functions are easily seen to be differentiable, with analytic derivative (use what you know about radii of convergence); hence they are C^∞. The set of analytic functions on U is an $\mathbb{R}$-algebra. Clearly, for $V \subset U$ open, the restriction to V of an analytic function on U is again analytic. Since the property of a function to be analytic is local, it follows that we have a sheaf of $\mathbb{R}$-algebras C_U^ω on U such that, for all open $V \subset U$, $C_U^\omega(V)$ is the $\mathbb{R}$-algebra of analytic functions on V. (The notation C^ω inspired from set theory: ω is the first infinite ordinal.) What makes analytic functions very special compared to C^∞-functions is that on connected open sets they are determined by their power series expansion at one point.

There is a similar definition, of course, of complex analytic functions. For $n \geq 0$ and $U \subset \mathbb{C}^n$ open, a function $f: U \to \mathbb{C}$ is (complex) analytic if it is locally given by a convergent power series. As in the real case, we get the sheaf C_U^ω of analytic functions. If we need to distinguish between the real and the complex case, we will use the notations $C_{\mathbb{R},U}^\omega$ and $C_{\mathbb{C},U}^\omega$.

2.1 Definition. *A real analytic manifold is a topological space with a sheaf of $\mathbb{R}$-valued functions (X, C_X^ω) that is locally of the form (U, C_U^ω), with U an open subset of some $\mathbb{R}^n$ and C_U^ω the sheaf on U of real analytic functions. A complex analytic manifold is a topological space with a sheaf of $\mathbb{C}$-valued functions (X, C_X^ω) that is locally of the form (U, C_U^ω), with U an open subset of some $\mathbb{C}^n$ and C_U^ω the sheaf on U of complex analytic functions. Morphisms are morphisms of topological spaces with a sheaf of $\mathbb{R}$-valued or $\mathbb{C}$-valued functions, respectively.*

2.2 Remark. One can define analytic manifolds in a more general setting, namely, by replacing $\mathbb{R}$ or $\mathbb{C}$ by an arbitrary field equipped with an ultrametric absolute value for which the field is complete and non-discrete (see Bourbaki, "Variétés différentielles et analytiques"). For example, one could work with the fields $\mathbb{Q}_p$ of p-adic numbers (p prime). The fact that groups like $\mathrm{GL}_n(\mathbb{Q}_p)$ can be seen as p-adic manifolds has important consequences for their structure as topological groups. □

It is certainly possible to define (real and complex) analytic manifolds in terms of atlases: one demands that the $\phi_j^{-1}\phi_i$ are analytic, meaning that their n_j coordinates are analytic functions on $U_{i,j}$. As in the previous section, the two definitions are equivalent. In fact, all the atlases of the previous section that were explicitly given are real analytic, hence they give examples of real analytic varieties. More important, the construction given after Theorem 1.4 of a C^k-atlas for sets defined by suitable systems of C^k-equations also works in the analytic case.

2.3 Construction. Let K be $\mathbb{R}$ or $\mathbb{C}$. Suppose that we have positive integers n and m, an open subset U of K^n and an analytic map $f: U \to K^m$. Let $X := f^{-1}0$, and suppose that f is submersive at all x in X: $(Df)x: K^n \to K^m$ is surjective for all x in X. Let x be in X. Let V be the kernel of $(Df)x$, and let $V^\perp$ be its orthogonal (for the standard inner product on K^m (hermitian if $K = \mathbb{C}$)). We view K^n as the product $V \times V^\perp$. Applying Theorem 1.4 gives us an open subset V' of V and an analytic map $g: V' \to V^\perp$ such that x is in $V' \times V^\perp$ and for all v in V' we have $f(v, g(v)) = 0$. Hence the map $\phi: V' \to X$ given by $\phi(v) = (v, g(v))$ is a chart at x. Such charts form an atlas for X for the same reasons as before in section 1. □

It is now easy to give examples of analytic manifolds. Just as in section 1, one sees that $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}_n(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{SU}_n(\mathbb{R})$, are real analytic groups, and that $\mathbb{P}^n(\mathbb{R})$ and $\mathrm{Gr}_{d,n}(\mathbb{R})$ are real analytic manifolds. One also sees that $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, are complex analytic groups, and that $\mathbb{P}^n(\mathbb{C})$ and $\mathrm{Gr}_{d,n}(\mathbb{C})$ are complex analytic manifolds. Note that $\mathrm{SU}_n(\mathbb{R})$ is not a complex analytic manifold, because the defining equations are not analytic (they involve complex conjugation, also: the dimension of $\mathrm{SU}_n(\mathbb{R})$ is not necessarily even).

2.4 Construction. Let (X, C_X^ω) be a real analytic manifold. We will now show how to associate to X a C^∞-manifold. Let $U \subset X$ be open, and $f: U \to \mathbb{R}$. For x in U, we say that f is C^∞ at x, if there is an open neighborhood U' of x in U and an isomorphism between $(U', C_X^\omega|_{U'})$ and (V, C_V^ω) for some open V in some $\mathbb{R}^n$, such that the corresponding function $V' \to \mathbb{R}$ is C^∞. We say that f is C^∞ on U if it is C^∞ at all x in U. Let C_X^∞ be the sheaf on X of C^∞-functions: $C_X^\infty(U)$ is the $\mathbb{R}$-algebra of C^∞-functions on U. One verifies immediately that (X, C_X^∞) is a C^∞-manifold. Another way to see this construction is: choose an atlas for (X, C_X^ω), view this atlas as a C^∞-atlas and take the associated C^∞-manifold.

In the same way, one can of course associate to a C^k-manifold a $C^{k'}$-manifold, whenever $k \geq k'$.

Let (X, C_X^ω) be a complex analytic manifold. Then we can associate a real analytic manifold to it, in more or less the same way. View $\mathbb{C}$ as $\mathbb{R}^2$, say via the $\mathbb{R}$-basis $(1, i)$. Choose an atlas for X. View this atlas as a real analytic atlas (note that the dimension doubles, since $\mathbb{C}^n \cong \mathbb{R}^{2n}$). Let $(X, C_{X,\mathbb{R}}^\omega)$ be the associated real analytic manifold. The elements of $C_{X,\mathbb{R}}^\omega(U)$, for $U \subset X$ open, are those functions $f: U \to \mathbb{R}$ that can locally be expressed in terms of power series in the real and imaginary parts of the coordinates of a chart. □

2.5 Exercise. Show that the real analytic variety associated to $\mathbb{P}^1(\mathbb{C})$ is isomorphic to S^2 (hint: use stereographic projection). □

Before we go on, we state some standard results on analytic functions, in one and more variables. These facts can be found in §0.1 of the book of Griffiths and Harris, and very probably in every book on analytic functions in several variables.

2.6 Proposition. (Cauchy's integral formula) *Let $\Delta \subset \mathbb{C}$ be an open disk, with closure $\overline{\Delta}$ and boundary $\partial\Delta$. Let $f: \overline{\Delta} \to \mathbb{C}$ be C^1 in the complex sense (i.e., there is an open neighborhood*

U of $\overline{\Delta}$ and an extension of f to U that is complex differentiable with continuous derivative). Then for all z be in Δ one has:

$$f(z) = (2\pi i)^{-1} \int_{\partial\Delta} \frac{f(w)dw}{w-z},$$

where the integral is taken in the counterclockwise direction.

The proof of this proposition uses the theorem of Stokes, or a simple version of it based on the fact that we are just dealing with a disk here. We won't give it here (see analysis textbooks).

2.7 Proposition. *Let $n \geq 0$, $U \subset \mathbb{C}^n$ open and $f: U \to \mathbb{C}$. Then f is analytic if and only if f is C^1 in the complex sense.*

Proof. We just sketch the proof. Of course, if f is analytic, it is C^1 in the complex sense. So suppose now that f is C^1 in the complex sense. Let us first suppose that $n=1$. Let z_0 be in U. We will show that f is analytic at z_0. After a translation in $\mathbb{C}$ we may assume that $z_0 = 0$. Let Δ be an open disk with center 0, contained in U. Then for all z in Δ we have:

$$f(z) = (2\pi i)^{-1} \int_{\partial\Delta} \frac{f(w)dw}{w-z} = (2\pi i)^{-1} \int_{\partial\Delta} \left(1 + (z/w) + (z/w)^2 + \cdots\right) f(w)\frac{dw}{w}.$$

From this expression it is easily seen that one has:

$$f(z) = \sum_{n\geq 0} a_n z^n, \qquad \text{with} \quad a_n = (2\pi i)^{-1} \int_{\partial\Delta} w^{-n} f(w) \frac{dw}{w},$$

for all z in Δ.

Let us now do the general case: $n \geq 1$. Let z_0 be in U. We want to show that f is analytic at z_0. After a translation we may assume that $z_0 = 0$. Let $r > 0$ be in $\mathbb{R}$ such that the polydisk (i.e., product of disks) $\Delta := \{z \in \mathbb{C}^n \mid |z_i| < r \text{ for } 1 \leq i \leq n\}$ is contained in U. A repeated application of Cauchy's integral formula gives,for all z in Δ:

$$f(z) = (2\pi i)^{-n} \int_{\partial\Delta} \prod_{i=1}^{n} (w_i - z_i)^{-1} \cdot f(w) dw_1 \cdots dw_n.$$

The power series expansion in the z_i/w_i of the product of the $(w_i - z_i)^{-1}$ gives the result. $\square$

This last proposition is the reason that we did not define what complex differentiable manifolds are: they are just the complex analytic manifolds. So when we speak of the differentiable manifold associated to a complex analytic manifold, we always mean the associated real differentiable manifold.

To indicate that there are surprising differences between the one-variable and more than one variable cases, we state the following theorem.

2.8 Theorem. (Hartog's theorem) *Let $n \geq 2$. For $r > 0$ let $\Delta_n(r) := \{z \in \mathbb{C}^n \mid |z_i| < r\}$. Let $0 < r' < r$. Then any analytic function $f: U := \Delta_n(r) - \overline{\Delta_n(r')} \to \mathbb{C}$ extends to one on $\Delta_n(r)$.*

Proof. Take r'' such that $r' < r'' < r$. Put $V := \Delta_n(r'') - \Delta_n(r')$. For c in $\mathbb{C}^{n-1}$, let V_c be the set of z in V with $(z_2, \ldots, z_n) = c$. For c outside $\Delta_{n-1}(r'')$, V_c is empty. For c in $\Delta_{n-1}(r'') - \Delta_{n-1}(r')$, V_c is $\Delta_1(r'')$, and for c in $\Delta_{n-1}(r')$, V_c is $\Delta_1(r'') - \Delta_1(r')$. We define a function F on V by:

$$F(z) := (2\pi i)^{-1} \int_{|w_1|=r} \frac{f(w_1, z_2, \ldots, z_n) dw_1}{w_1 - z_1}.$$

Then F is complex differentiable, hence analytic. On the V_c with c in $\Delta_{n-1}(r'') - \Delta_{n-1}(r')$, it agrees with f, hence it agrees with f on V. □

Analytic functions are also called holomorphic (I don't know really why). An important property of analytic functions is the following: let U be open and connected in $\mathbb{C}^n$, and $f: U \to \mathbb{C}$ analytic and non-constant; then the real valued function $|f|$ on U has no maximum. This is very easy to prove, so we leave it to the reader. It implies the following result (proof also left to the reader).

2.9 Proposition. *Let (X, C_X^ω) be a compact connected complex analytic manifold. Then $C_X^\omega(X) = \mathbb{C}$.*

2.10 Exercise. Show that the analog of Proposition 2.9 for real analytic functions does not hold. For example: show that the $\mathbb{R}$-vector space of real analytic functions on S^1 is infinite dimensional. □

We end this section with the definition of analytic varieties. These are just ringed spaces that are locally isomorphic to ringed spaces of some kind that we will first describe.

Let K be $\mathbb{R}$ or $\mathbb{C}$. Let $n \geq 0$ and $m \geq 0$. Let $U \subset K^n$ be open, and $f: U \to K^m$ be analytic. Let $X := f^{-1}\{0\}$, with its induced topology. For $V \subset X$ open, $g: V \to K$ and $x \in V$, g is said to be analytic at x if there exists an open neighborhood W of x in U such that g restricted to $W \cap V$ is the restriction to $W \cap V$ of some analytic function on W. Such a function g that is analytic at all x in V is called analytic on V. Let C_X^ω be the sheaf on X of analytic functions. Then we call (X, C_X^ω) the analytic variety (complex if $K = \mathbb{C}$, real if $K = \mathbb{R}$) defined by the system of equations $f = 0$.

2.11 Definition. *A complex (real) analytic variety is a ringed space (X, C_X^ω) that is locally isomorphic to a complex (real) analytic variety defined by a system of equations.*

Note that here it is hard to think of a definition of such objects in terms of an atlas, and that the use of sheaves becomes essential. The study of the local fundamental properties of analytic varieties is postponed until later. For example, we would like to know that allowing arbitrary sets of analytic equations gives the same result. This has to do with certain finiteness properties of sheaves such as $C_{K^n}^\omega$.

3 Algebraic varieties

Some references I can think of for this chapter are: Hartshorne's book "Algebraic geometry", Serre's article "Faisceaux algébriques cohérents", Shafarevich's book "Basic algebraic geometry", a recent book by Eisenbud (in the series "graduate texts in mathematics", Springer), a recent book by Perrin, and the classical texts SGA and EGA (see Harthorne's book for precise references). I recommend the first chapter of Hartshorne's book. Another excellent reference is Mumford's book "The red book of varieties and schemes"

Recall that differentiable manifolds have to do with differentiable functions, and analytic manifolds and varieties with analytic functions. It is clear that functions defined by rational functions are still nicer, and also that they occur at lots of places in mathematics. So it is useful to set up a kind of geometry using this kind of functions. This geometry is called algebraic geometry. It is meant to be a systematic study of the sets of solutions of systems of polynomial equations. Here one would take these polynomials to have coefficients in some fixed ring A (commutative), and study the solutions with values in A-algebras B. In this course we will take A to be an arbitrary algebraically closed field, but it should be noted that the language of schemes (see Hartshorne, Chapter 2) makes it possible to use arbitrary rings. Restricting ourselves to an algebraically closed field k has the advantage that the the set of solutions with values in k itself of a system of polynomial equations almost determines the set of equations (see later, when we discuss Hilbert's Nullstellensatz). Maybe we will have time to say something about non-algebraically closed fields and about real algebraic geometry.

So let k now be an arbitrary algebraically closed field. As we have already said, our geometric objects will allways be ringed spaces. A ringed space $(X, \mathcal{O}_X)$ will be called a variety over k if it is locally of some type, just as in the cases of differentiable and analytic varieties. So first we want to specify this "type". As a set, this is quite easy. Let $n \geq 0$ and $S \subset A := k[x_1, \ldots, x_n]$ be a set of polynomials. Then we define $\mathrm{V}(S)$ to be the set of common zeroes of all elements of S:

$$\mathrm{V}(S) := \{a \in k^n \mid f(a) = 0 \text{ for all } f \in S\}.$$

These sets $\mathrm{V}(S)$ are the sets we are interested in, we call them algebraic subsets of k^n. Now we have to give them a topology. Clearly, since we want to work with rational functions, polynomial functions should be continuous. Note that saying this already supposes a topology on k. We want the subset $\{0\}$ of k to be closed. Then it follows that the subsets of $\mathrm{V}(S)$ of the form $\mathrm{V}(S) \cap \mathrm{V}(T)$ must be closed. One easily notes the following: $\emptyset$ and k^n are algebraic, $\cap_{i\in I}\mathrm{V}(S_i) = \mathrm{V}(\cup_{i\in I}S_i)$. For S_1 and S_2 subsets of A, let S_1S_2 be the set $\{f_1f_2 \mid f_i \in S_i\}$. Then one has $\mathrm{V}(S_1) \cup \mathrm{V}(S_2) = \mathrm{V}(S_1S_2)$. This leads to the following definition.

3.1 Definition. *The Zariski topology on k^n is the topology for which a subset of k^n is closed if and only if it is algebraic. The Zariski topology on a subset of k^n is the induced topology.*

So now our algebraic subsets of k^n are topological spaces. Before we proceed to describe the sheaf of functions that we want to consider, let us look a bit closer at this topology. For $n = 1$, our topology on k itself has as closed sets only the finite sets and k itself; in particular, the topology is not separated (note that k is infinite). The reader should not be discouraged by this fact, one gets used to it. Another property of the Zariski topology is that the topology on k^2 is strictly finer than the product topology from the two factors k. Namely, the diagonal in k^2 is not closed in the product topology. Now some nice facts about the Zariski topology. For S a subset of A and $I \subset A$ the ideal generated by S we have $\mathrm{V}(S) = \mathrm{V}(I)$. Now A is a noetherian ring, hence every ideal $I \subset A$ is finitely generated. It follows that the algebraic subsets of k^n are of the form $\mathrm{V}(f_1, \ldots, f_r)$, i.e., defined by finitely many equations. (Exercise: show that for a given $n > 0$ there is no bound on r.) For f in A we let $\mathrm{D}(f)$ be the complement in k^n of $\mathrm{V}(f)$; $\mathrm{D}(f)$ is called the fundamental open subset of k^n defined by f.

3.2 Lemma. *The* $\mathrm{D}(f)$ *form a basis for the Zariski topology on* k^n.

Proof. The $\mathrm{D}(f)$ are open by definition. Let a be in k^n and let U be an open subset of k^n containing a. The complement Y of U in k^n is of the form $\mathrm{V}(S)$ for some S. Since a is not in Y, there is an element f in S such that $f(a) \neq 0$. Then $\mathrm{D}(f)$ contains a and is contained in U. □

We can now define our sheaves.

3.3 Definition. *Let* $X \subset k^n$ *be an algebraic set, let* $U \subset X$ *be open and let* $f: U \to k$ *be a function on* U*. Then* f *is called regular at* x *in* U *if there is an open neighborhood* V *of* x *in* U*, and elements* g *and* h *of* A*, with* $V \subset \mathrm{D}(h)$ *and* $f(y) = g(y)/h(y)$ *for all* y *in* V*. The function* f *is called regular if it is regular at all* x *in* U*. We denote by* $\mathcal{O}_X$ *the sheaf on* X *of regular functions.*

3.4 Definition. *An affine algebraic variety over* k *is a topological space with a sheaf of* k*-valued functions that is isomorphic to some* $(X, \mathcal{O}_X)$*, with* X *an algebraic set in some* k^n *and* $\mathcal{O}_X$ *its sheaf of regular functions. An algebraic variety over* k *is a topological space with a sheaf of* k*-valued functions* $(X, \mathcal{O}_X)$ *that is locally an affine algebraic variety over* k*. Morphisms of algebraic varieties over* k *are morphisms of topological spaces with a sheaf of* k*-valued functions.*

3.5 Remark. In Hartshorne's book the definition of (algebraic) variety over k is a lot more restrictive: he requires them to be quasi-projective and irreducible (terms that hopefully will become clear later in the course). At this moment, we just want to be able to talk about algebraic varieties, and to illustrate the usefulness of ringed spaces. We wait a bit before going into the study of the fundamental properties of algebraic varieties. (The local properties, such as dimension and smoothness, are part of Berthelot's course.) □

3.6 Example. Of course, every affine algebraic variety over k is an algebraic variety over k. The matrix groups $\mathrm{GL}_n(k)$, $\mathrm{SL}_n(k)$, $\mathrm{SO}_n(k)$, $\mathrm{O}_n(k)$, $\mathrm{Sp}_{2n}(k)$ are examples of affine algebraic varieties. For example, $\mathrm{GL}_n(k)$ can be seen as the algebraic subset of $\mathrm{M}_n(k) \times \mathrm{M}_n(k)$ defined by the system of equations $xy = 1$; the other groups are subgroups of some $\mathrm{GL}_m(k)$ defined by polynomial equations. So these groups are examples of affine algebraic groups.

For f in $k[x_1, \ldots, x_n]$, the open subset $\mathrm{D}(f)$ of k^n, with its induced topology and sheaf, is in fact affine: it can be identified with the closed subset of $k^{n+1} = k^n \times k$ consisting of those (a, b) such that $f(a)b = 1$. (It is because of this that in the definition of algebraic variety we demand that locally the ringed space is isomorphic to an affine algebraic variety, and not only an open subvariety of such.) Every open subset of an algebraic variety over k, with its induced sheaf of regular functions, is an algebraic variety over k. □

3.7 Construction. If $(X, \mathcal{O}_X)$ is an algebraic variety over k, and $Y \subset X$ a closed subset, we can give Y the structure of an algebraic variety over k as follows. First, give Y the induced topology. Then, for $U \subset Y$ open and $f: U \to k$ a function, one says that f is regular at y in Y if there exists an open neighborhood V of y in X and an element g in $\mathcal{O}_X(V)$ such that $f(v) = g(v)$ for all v in $V \cap U$. This gives us a ringed space $(Y, \mathcal{O}_Y)$. To see that it is an algebraic variety over k, it suffices to prove it for X affine, say $X = \mathrm{V}(S)$ in some k^n. Then Y is $X \cap \mathrm{V}(T) = \mathrm{V}(S \cup T)$ for some T, and one verifies by hand that the sheaf $\mathcal{O}_Y$ coincides with the sheaf of regular functions on $\mathrm{V}(S \cup T)$. □

3.8 Example. The ringed space $(k^n, \mathcal{O}_{k^n})$ is called affine n-space over k, and will be denoted by $\mathbb{A}^n(k)$. The projective spaces $\mathbb{P}^n(k)$ and more generally the Grassmannians $\mathrm{Gr}_{d,n}(k)$ are equipped with the structure of algebraic variety over k by the charts given in §1.13 and §1.15. Indeed, one verifies that the $U_{i,j}$ are Zariski-open, and the $\phi_j^{-1} \circ \phi_i$ given by regular functions. A set S of homogeneous elements of the graded ring $k[x_0, \ldots, x_n]$ (graded by the total degree) defines a closed subset $\mathrm{V}(S)$ of $\mathbb{P}^n(k)$, hence an algebraic variety over k (also denoted $\mathrm{V}(S)$). It can be shown, by the usual process of homogenization and dehomogenization, that all closed subsets of $\mathbb{P}^n(k)$ are of this form. Varieties that can be embedded as closed subvarieties of some $\mathbb{P}^n(k)$ are called projective algebraic varieties. A nice feature of projective algebraic varieties is that they have properties similar to those of compact analytic varieties. More on this later.
□

3.9 Remark. The algebraic varieties $\mathbb{P}^n(k)$ and $\mathrm{Gr}_{d,n}(k)$ can be described by charts involving regular functions. This is something very special for algebraic varieties; it implies for example that they are "rational varieties". Varieties that do not admit any chart are easy to give. For example, the plane algebraic curve given by the equation $y^2 = x^3 - 1$ (suppose that k has characteristic different from 2 and 3) has that property. More precisely: every solution of this equation in the field $k(t)$ is actually a solution in k. I do not know if there is a very simple proof for this. □

3.10 Exercise. Show that $\mathbb{A}^1(k)$ is homeomorphic to $\mathbb{P}^1(k)$. Show that $\mathbb{A}^2(k)$ is homeomorphic to neither $\mathbb{A}^1(k)$ nor $\mathbb{P}^2(k)$. □

3.11 Construction. We will now explain how to associate, to an algebraic variety $(X, \mathcal{O}_X)$ over $\mathbb{C}$, a complex analytic variety $(X^{\mathrm{an}}, C^{\omega}_{X^{\mathrm{an}}})$. Of course, for $X = \mathrm{V}(f_1, \ldots, f_r)$ in some $\mathbb{C}^n$ we want X^{an} to be the analytic variety in $\mathbb{C}^n$ defined by the f_i. It is possible to show that this construction for affines is well-defined and "glues", but we do not really have the necessary techniques for that yet. So let us consider the following construction. Let $(X, \mathcal{O}_X)$ be an algebraic variety over $\mathbb{C}$. As a set, we put $X^{\mathrm{an}} := X$. Now we put a topology on X^{an}. For $U \subset X$ open and $f \in \mathcal{O}_X(U)$, let $\mathrm{V}_{<1}(f)$ be the set $\{x \in U \mid |f(x)| < 1\}$. We take the coarsest topology for which the sets $\mathrm{V}_{<1}(f)$ are open, i.e., the open sets of X^{an} are arbitrary unions of finite intersections of sets of the type $\mathrm{V}_{<1}(f)$. Let $U \subset X^{\mathrm{an}}$ be open and $f: U \to \mathbb{C}$ a function. Then f is said to be analytic at x in U if there exist $n \geq 0$, regular functions $f_1, \ldots, f_n$ on some open neighborhood of x with $|f_i(x)| < 1$ and a formal power series $F \in \mathbb{C}[[z_1, \ldots, z_n]]$ that converges on the polydisc of radius one, such that $f(y) = F(f_1(y), \ldots, f_n(y))$ for all y in some neighborhood of x. This gives us a sheaf $C^{\omega}_{X^{\mathrm{an}}}$.

Let us now check that for $X = \mathrm{V}(f_1, \ldots, f_r) \subset \mathbb{C}^n$, the ringed space $(X^{\mathrm{an}}, C^{\omega}_{X^{\mathrm{an}}})$ is the analytic variety defined by the system of equations $f_1 = 0, \ldots, f_r = 0$. First of all, let us verify that our topology on X^{an} is the topology induced from $\mathbb{C}^n$. Clearly, the open sets of X^{an} are open for the induced topology, since the absolute value of a regular function is a continuous function for that topology. On the other hand, the intersection of X with polydiscs $\cap_{i=1}^{n} \mathrm{V}_{<1}(x_1 - a_i)$ (where $x_1, \ldots, x_n$ are the coordinate functions) are open in our topology on X^{an}. It follows that the two topologies are the same. We leave it to the reader that the two notions of analytic functions are the same.

It is straightforward to check that this construction defines a functor from the category of algebraic varieties over $\mathbb{C}$ to the category of complex analytic varieties. A reference for this construction is Appendix B in Hartshorne's book. □

Because of the course on semi-algebraic geometry in the second semester, let us also give the definition of a real algebraic variety. A subset X of $\mathbb{R}^n$ is called algebraic if it is the set of zeros of a subset of $\mathbb{R}[x_1, \ldots, x_n]$. Just as in the case of an algebraically closed field, these algebraic sets are the closed sets for a topology, called the Zariski topology, on $\mathbb{R}^n$. Note that this topology coincides with the topology induced by the inclusion $\mathbb{R}^n \subset \mathbb{C}^n$.

Let $X \subset \mathbb{R}^n$ be an algebraic set, equipped with the Zariski topology. Let $U \subset X$ be open, and $f: U \to \mathbb{R}$. Then f is called regular at x in U if and only if there exists an open neighborhood V of x in U, and elements g and h of $\mathbb{R}[x_1, \ldots, x_n]$, such that, for all y in V, $h(y) \neq 0$ and $f(y) = g(y)/h(y)$. The function f is called regular if it is regular at all x in U. The set of regular functions on U is an $\mathbb{R}$-algebra; we denote it by $\mathcal{O}_X(U)$. This gives us the sheaf $\mathcal{O}_X$ on X of regular functions.

3.12 Definition. *An affine real algebraic variety is a topological space with a sheaf of $\mathbb{R}$-valued functions that is isomorphic to some $(X, \mathcal{O}_X)$ with X an algebraic set in some $\mathbb{R}^n$ and $\mathcal{O}_X$ its sheaf of regular functions. Morphisms of real algebraic varieties are morphisms of topological spaces with a sheaf of $\mathbb{R}$-valued functions.*

3.13 Exercise. Let k be an algebraically closed field. Determine the global regular functions on $\mathbb{A}^1(k)$ and on $\mathbb{P}^1(k)$. □

4 Vector bundles

Before defining what a vector bundle is, let us study a most important example: the tangent bundle of a manifold. So first we recall what the tangent spaces of a manifold at its points are.

4.1 Tangent spaces

Let (X, C_X) be a manifold, say of class C^1 at least. For x in X we want to define its tangent space. There are several ways to do this (which are of course equivalent). For a detailed discussion of all of those I know, see Spivak's book, Volume I, Chapter 3. We will discuss some of them. Intuitively, the tangent space of X at a point x in X is the first order approximation of X at x. We need it in order to speak of the derivatives of morphisms of manifolds.

Suppose that we have an atlas (X, I, n, U, ϕ) for X. Let x be in X. A tangent vector v at x will then be a compatible system of pairs (i, v_i), with v_i in $\mathbb{R}^{n_i}$, for the i in I such that x is in X_i. The compatibility is defined as follows. Let i and j be in I with $x \in X_i$ and $x \in X_j$. Then the transition isomorphism $\phi_j^{-1}\phi_i$ from $U_{i,j}$ to $U_{j,i}$ has the property that:

(4.1.1) $$(D(\phi_j^{-1}\phi_i))(\phi_i^{-1}x) \text{ sends } v_i \text{ to } v_j.$$

Since for every such pair (i, j) the map $(D(\phi_j^{-1}\phi_i))(\phi_i x)$ is an isomorphism of $\mathbb{R}$-vector spaces from $\mathbb{R}^{n_i}$ to $\mathbb{R}^{n_j}$, it is clear that a compatible system of (i, v_i) is determined by any of its elements, and that such an element can be arbitrary in $\mathbb{R}^{n_i}$. So to give such a compatible system, it is equivalent to give, for one i in I with $X_i \ni x$, an element v_i of $\mathbb{R}^{n_i}$. In particular, the set of such compatible systems, that we call the tangent space of X at x and that we denote $\mathrm{T}_X(x)$, has a natural structure of $\mathbb{R}$-vector space and is, via this construction, isomorphic to the $\mathbb{R}$-vector space $\mathbb{R}^{n_i}$.

Our second description of $\mathrm{T}_X(x)$ uses parametrized curves, and does not require charts. Let x be in X. A parametrized curve at x is a differentiable map $c \colon U \to X$ with $U \subset \mathbb{R}$ an open interval containing zero and with $c(0) = x$. We want to define the tangent space at x as the set of equivalence classes of such curves, where c_1 and c_2 are to be equivalent if and only if they give the same tangent vector. Of course, we do not want to use the previous definition in terms of charts, so we want another way to say that c_1 and c_2 define the same tangent vector. One way to do this is the following. Let $c \colon U \to X$ be a parametrized curve at x, and f in $C_X(V)$ with V an open neighborhood of x. Then, after shrinking U if necessary, $c^*f := fc$ is a differentiable function on U; let $(fc)'(0)$ be its derivative at 0. Then we say that c_1 and c_2 are equivalent if for all open neighborhoods V of x in X and all f in $C_X(V)$ we have $(fc_1)'(0) = (fc_2)'(0)$. Now we have some work to do: we have to show that this relation is an equivalence relation, and that the set of equivalence classes is in some natural way (this will be made precise below) an $\mathbb{R}$-vector space and as such isomorphic to the one defined above. The relation is clearly an equivalence relation. Let c be a parametrized curve at x, and i in I with $X_i \ni x$. Then $\phi_i^{-1}c$ is defined at 0; we get an element $v_i := (D(\phi_i^{-1}c))0$ of $\mathbb{R}^{n_i}$. One checks immediately that this is a compatible system of (i, v_i) in the sense explained above. If c_1 and c_2 are equivalent, then they give the same v_i (view ϕ_i^{-1} as an $n_i$tuple of functions). Suppose now that c_1 and c_2 give the same v_i. We want to show that c_1 and c_2 are equivalent. For doing this, we may suppose that $X = U_i$ and that $\phi_i = \mathrm{id}_X$. Then we know that $(Dc_1)0 = (Dc_2)0$ (consider partial derivatives). But then we have, for f in $C_X(V)$, that

$$(4.1.2) \qquad (fc_1)'0 = ((Df)x)((Dc_1)0) = ((Df)x)((Dc_2)0) = (fc_2)'0.$$

This shows that, as a set, the set of equivalence classes of c is the same as $\mathrm{T}_X(x)$ as above, so we will use the same notation for both. The $\mathbb{R}$-vector space structure on $\mathrm{T}_X(x)$ has the following interpretation (we denote the class of a curve c by $[c]$): $[c_1] + [c_2] = [c_3]$ if and only if for all V and f we have $(fc_1)'(0) + (fc_2)'(0) = (fc_3)'(0)$. Likewise: $a[c_1] = [c_2]$, for a in $\mathbb{R}$, if and only if $a(fc_1)'(0) = (fc_2)'(0)$ for all f. So indeed we have a description of $\mathrm{T}_X(x)$ that does not use charts.

From the previous description it is just a small step to the third and last one. But in order to define it, it is really convenient to use the notions of germ and stalk. So here follows a short intermezzo.

Suppose that X is a topological space, F a sheaf (of sets) on X and x an element of X. The stalk F_x of F at x is then defined as the direct limit $\lim_{\to U \ni x} F(U)$ (the notion of direct limit will be explained in the seminar; one can look it up for example in Hartshorne's book). Concretely, this means that F_x is the set of equivalence classes of pairs (U, s), with U an open neighborhood of x and s in $F(U)$, for the following equivalence relation. Two such pairs (U_1, s_1) and (U_2, s_2) are equivalent if and only if there is an open neighborhood V of x contained in $U_1 \cap U_2$, such that the elements $s_1|_V$ and $s_2|_V$ are equal. The elements of F_x are called germs of sections of F, and the class s_x of (U, s) is called the germ of s at x. If F is a sheaf of $\mathbb{R}$-algebras, then F_x is an $\mathbb{R}$-algebra.

Let us now go back to our tangent spaces. So X is again a manifold and x is in X. For c a parametrized curve at x, we get a map $\partial_c \colon C_{X,x} \to \mathbb{R}$ defined by: $\partial_c(f) = (fc)'0$ (note that this makes sense). This map is clearly $\mathbb{R}$-linear, and it turns out to be a derivation, i.e., it satisfies the product rule for differentiation:

$$(4.1.3) \qquad \partial_c(fg) = ((fg)\circ c)'0 = ((f\circ c)(g\circ c))'0 = (\partial_c f)g(x) + f(x)(\partial_c g).$$

Let us now consider the set $\mathrm{Der}_{\mathbb{R}}(C_{X,x}, \mathbb{R})$ of all $\mathbb{R}$-linear derivations from the $\mathbb{R}$-algebra $C_{X,x}$ to $\mathbb{R}$ (here we view $\mathbb{R}$ as a $C_{X,x}$-module via the map $f \mapsto f(x)$). (For A a commutative ring, B a commutative A-algebra and M a B-module, an A-linear derivation from B to M is an A-linear map $D \colon B \to M$ satisfying $D(bb') = D(b)b' + b'D(b)$ for all b and b' in B. The set $\mathrm{Der}_A(B, M)$ of such maps is clearly a B-module: $(bD + D')b' = bD(b') + D'(b')$.) We claim that, in the C^∞ and C^ω-cases, $\mathrm{Der}_{\mathbb{R}}(C_{X,x}, \mathbb{R})$ is canonically isomorphic, as $\mathbb{R}$-vector space, to $\mathrm{T}_X(x)$ (the cases C^k, $k \geq 1$ will be explained in an exercise). To prove this, let us first show that $c \mapsto \partial_c$ defines a map from $\mathrm{T}_X(x)$ to $\mathrm{Der}_{\mathbb{R}}(C_{X,x}, \mathbb{R})$. Suppose that c_1 and c_2 are equivalent. Then by the definition of this equivalence, we have $\partial_{c_1} = \partial_{c_2}$, hence we have our map. By the definition of the $\mathbb{R}$-vector space structures, this map is $\mathbb{R}$-linear. To show that it is an isomorphism is a local question, so we may and do assume that X is an open subset of $\mathbb{R}^n$ and that $x = 0$. For $1 \leq i \leq n$ let ∂_i be the element of $\mathrm{Der}_{\mathbb{R}}(C_{X,0}, \mathbb{R})$ that sends f to its ith partial derivative at 0. The elements ∂_i are linearly independent because of the relations $\partial_i x_j = \delta_{i,j}$ (where x_j is the jth coordinate function, and $\delta_{i,j}$ the Kronecker symbol). Let ∂ be in $\mathrm{Der}_{\mathbb{R}}(C_{X,0}, \mathbb{R})$. We claim that $\partial = \sum_{i=1}^n \partial(x_i)\partial_i$. To prove this, let f be in $C_{X,0}$. In the C^ω-case, we can clearly write

$$(4.1.4) \qquad f = f(0) + \sum_{i=1}^n x_i g_i, \quad \text{with } g_i \text{ in } C_{X,0}.$$

Note that $\partial_i(f) = g_i(0)$. It follows that in this case we have $\partial(f) = \sum_i \partial(x_i)\partial_i(f)$ as desired. Now consider the C^∞-case. Rewriting the equality:

$$(4.1.5) \qquad \int_0^1 \left(\frac{d}{dt} f(tx)\right) dt = f(x) - f(0),$$

with x in some neighborhood of 0, gives:

$$(4.1.6) \qquad f(x) = f(0) + \sum_{i=1}^n x_i \int_0^1 (D_i f)(tx)dt,$$

with D_i the ith partial derivative. Since the $D_i f$ are C^∞-functions, the last formula shows that once again we have an identity as in (4.1.4). The proof is then finished as before. So now we know that the ∂_i form a basis of $\mathrm{Der}_{\mathbb{R}}(C_{X,x}, \mathbb{R})$. It follows that our map from $\mathrm{T}_X(x)$ to $\mathrm{Der}_{\mathbb{R}}(C_{X,x}, \mathbb{R})$ is an isomorphism.

4.1.7 Exercise. Let k be a field, A a k-algebra and $\phi: A \to k$ a morphism of k-algebras. We view k as an A-module via ϕ. Let $m := \ker(\phi)$; note that m is a maximal ideal of A. Show that for all ∂ in $\mathrm{Der}_k(A, k)$ and all f in m^2 we have $\partial(f) = 0$; let $\overline{\partial}: m/m^2 \to k$ be the map such that $\partial(f) = \overline{\partial}(\overline{f})$ for all f in m, where $\overline{f}$ is the image of f in m/m^2. Show that $\partial \mapsto \overline{\partial}$ is an isomorphism of k-vector spaces from $\mathrm{Der}_k(A, k)$ to $(m/m^2)^\vee$, the dual of m/m^2.

Now let $k \geq 0$ be an integer. Let A be the $\mathbb{R}$-algebra $C^k_{\mathbb{R},0}$ of germs of C^k functions on neighborhoods of 0 in $\mathbb{R}$. Let $\phi: A \to \mathbb{R}$ be defined by $\phi(f) = f(0)$, and let m be its kernel. Show that $m/m^2 = 0$ for $k = 0$, and that m/m^2 is infinite dimensional if $k > 0$.

Let $k \geq 1$, $k = \infty$ or $k = \omega$. Let X be a C^k-manifold and $x \in X$. Let $\mathrm{Der}_{\mathbb{R}}(C^k_{X,x}, \mathbb{R})'$ be the set of ∂ in $\mathrm{Der}_{\mathbb{R}}(C^k_{X,x}, \mathbb{R})$ such that $\partial(f) = 0$ for all f in $C^k_{X,x} \cap m_k m_{k-1}$, where m_k (resp., m_{k-1}) is the maximal ideal of $C^k_{X,x}$ (resp., $C^{k-1}_{X,x}$) (the intersection takes place in $C^{k-1}_{X,x}$). Show that there is a canonical isomorphism as above from $\mathrm{T}_X(x)$ to $\mathrm{Der}_{\mathbb{R}}(C^k_{X,x}, \mathbb{R})'$. □

Let now $f: X \to Y$ be a morphism of C^k-manifolds, with $k \geq 1$, and x in X. Then we get an $\mathbb{R}$-linear map $\mathrm{T}_f(x): \mathrm{T}_X(x) \to \mathrm{T}_Y(fx)$ as follows. In the first description of tangent spaces, the map is $(D(\psi_j^{-1} f \phi_i))(\phi_i^{-1} x)$ (in the notation of Definition 1.3). In the second description it is the map that sends $[c]$ to $[fc]$. In the last description it is the map that sends ∂ to $\partial \circ f^*$, with $f^*: C_{Y,fx} \to C_{X,x}$ given by $g \mapsto gf$. It is left to the reader to show that these three maps are compatible with respect to the identifications between the various kinds of tangent spaces. The map $\mathrm{T}_f(x)$ is called the tangent map of f at x, or also the derivative of $f$at x. One can say that the main purpose of defining tangent spaces is just to have these tangent maps. In the same way, the main purpose of defining the tangent bundle is to have a tangent map T_f for all x at once.

4.1.8 Exercise. Let n and m be positive integers, and $k \geq 1$. Let $U \subset \mathbb{R}^n$ be open, and $f: U \to \mathbb{R}^m$ be a C^k-map. As in Section 1, let $X := f^{-1}0$, suppose that f is a submersion at all x in X, and consider X as a C^k-manifold. Let $i: X \to U$ be the inclusion map. Show that for x in X the map $\mathrm{T}_i(x)$ identifies $\mathrm{T}_X(x)$ with $\ker(\mathrm{T}_f(x))$. □

4.2 Vector bundles, the tangent bundle

Let X be a C^k-manifold with $k \geq 1$. We want to make an object T_X, called the tangent bundle of X, that combines all the $\mathrm{T}_X(x)$. As a set, T_X is just the disjoint union of all the $\mathrm{T}_X(x)$, $x \in X$. But in order to have something useful, for example second derivatives of morphisms of manifolds, we need to equip T_X with the structure of a manifold, reflecting the fact that it is a disjoint union of vector spaces. The notion of vector bundle is made exactly for doing this. Note that we have a canonical map $p: \mathrm{T}_X \to X$, such that the fibre $p^{-1}x$ over x is $\mathrm{T}_X(x)$. The following definition is meant to be a warming up for what comes after it.

4.2.1 Definition. *Let $p: E \to X$ be a morphism of manifolds. Then p is called a fibration if for every x in X there exists an open neighborhood U, a manifold F and an isomorphism $\phi: F \times U \to p^{-1}U$ such that $p \circ \phi$ is the projection $F \times U \to U$. The triple (U, F, ϕ) is called a trivialization over U. A fibration is called trivial if there exists such an isomorphism with $U = X$.*

4.2.2 Remark. The reader is supposed to understand what a product such as $F \times U$ is. See Section 1. □

4.2.3 Example. The Möbius strip with its map to the circle is a non-trivial fibration with fibre the closed interval $[-1, 1]$. □

Roughly speaking, a vector bundle is a fibration in which all the fibres are vector spaces (say over $\mathbb{R}$), such that there are local trivializations compatible with the vector space structures.

4.2.4 Definition. *Let X be a manifold. A real vector bundle over X is a five tuple $(E, p, 0, +, \cdot)$ with $p: E \to X$ a fibration, $(0, +, \cdot)$ the structure of $\mathbb{R}$-vector space on all fibres of p, such that for all x in X there exists a local trivialization $\phi: F \times U \to p^{-1}U$, with $U \ni x$, respecting the vector space structures. A complex vector bundle is defined analogously.*

4.2.5 Remark. If one wants to give a set theoretic meaning to the triple $(0, +, \cdot)$ above, it is the following. The element 0 is a section of p, i.e., it is a map from X to E such that $p \circ 0 = \mathrm{id}_X$. The element $+$ is a map from the fibred product $E \times_X E$ to E. The fibred product $E \times_X E$ is the subset of elements (e_1, e_2) of $E \times E$ with $p(e_1) = p(e_2)$, i.e., it is the set of pairs of elements of E that lie in the same fibre over X. The map $+$ is then of course the sum map of the vector space structure on the fibres. The element $\cdot$ is a map from $\mathbb{R} \times E \to E$ that gives the multiplication in the fibres.

It is certainly possible to equip the sets above with the structure of manifolds, in a natural way. The maps 0, + and $\cdot$ are then morphisms of manifolds. □

Now that we know what a vector bundle is, let us construct tangent bundles. So let X be a C^k-manifold with $k \geq 1$. As a set, T_X is the disjoint union of the $\mathrm{T}_X(x)$, for x in X. The map p from T_X to X is the unique map such that $p^{-1}x = \mathrm{T}_X(x)$ for all x. Suppose that we have an atlas (X, I, n, U, ϕ). Then we get an atlas for T_X as follows. For i in I, let $\mathrm{T}_{X,i} := p^{-1}X_i$. Put $V_i := \mathbb{R}^{n_i} \times U_i$ and define $\psi_i \colon V_i \to \mathrm{T}_{X,i}$ by: $\psi_i(v, x) = [(i, v)] \in \mathrm{T}_X(x)$, where $[(i, v)]$ denotes the compatible system corresponding to (i, v) as in our first description of $\mathrm{T}_X(\phi_i x)$. These ψ_i are easily seen to form a C^{k-1}-atlas, since

$$\psi_j^{-1}\psi_i \colon \mathbb{R}^{n_i} \times U_{i,j} \to \mathbb{R}^{n_j} \times U_{j,i}, \quad (v, x) \mapsto (((D(\phi_j^{-1}\phi_i))(x))v, (\phi_j^{-1}\phi_i)x). \tag{4.2.6}$$

The C^{k-1}-manifold thus obtained does not depend on the choice of the atlas (verification left to the reader). It remains now to show that the five tuple $(\mathrm{T}_X, p, 0, +, \cdot)$ is a vector bundle, i.e, that it has local trivializations as in Definition 4.2.4. But such trivializations are given by our maps ψ_i.

Suppose that $f \colon X \to Y$ is a morphism of C^k-manifolds, with $k \geq 1$. Let $\mathrm{T}_f \colon \mathrm{T}_X \to \mathrm{T}_Y$ be the map that is $\mathrm{T}_f(x)$ on $\mathrm{T}_X(x)$. Then T_f is a morphism of C^{k-1}-manifolds, it induces a commutative diagram:

$$\begin{array}{ccc} \mathrm{T}_X & \xrightarrow{\mathrm{T}_f} & \mathrm{T}_Y \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{f} & Y \end{array} \tag{4.2.7}$$

and it is $\mathbb{R}$-linear on the fibres. This motivates the following definition.

4.2.8 Definition. *Let $f \colon X \to Y$ be a morphism of manifolds, $p_E \colon E \to X$ and $p_F \colon F \to Y$ vector bundles. A morphism from E to F over f is then a morphism of manifolds $g \colon E \to F$ with $p_F g = f p_E$ that is a morphism of vector spaces on the fibres.*

It is clear that morphisms of vector bundles can be composed, and that we get a category of vector bundles. So finally we can say that associating to a manifold its tangent bundle, and to morphisms their derivative, is a functor T from the category of manifolds to the category of vector bundles.

4.3 Vector bundles as sheaves of modules

In this section we are interested in vector bundles over a fixed manifold X. A morphism of vector bundles over X is a morphism as in (4.2.7) with $f = \mathrm{id}_X$.

Let $p: E \to X$ be a real C^l-vector bundle over a C^k-manifold X (with $k \geq l$, of course). In practice one is very often more interested in the sections of E then in E itself. A C^l-section of E over an open subset U of X is a morphism of C^l-manifolds $s: U \to E$ such that $ps = \mathrm{id}_U$. The set of C^l-sections of E over U will be denoted $E(U)$. (For example, $\mathrm{T}_X(X)$ is the set of C^{k-1}-vector fields on X.) If $V \subset U$ is an inclusion of open subsets of X, then we have a restriction map from $E(U)$ to $E(V)$. These restriction maps clearly make the system of $E(U)$'s into a sheaf that we will still denote by E. Consider a set $E(U)$. It has the structure of $\mathbb{R}$-vector space, and as such it is usually infinite dimensional. But it also has the structure of a module over the $\mathbb{R}$-algebra $C^l_X(U)$. This structure is compatible with the restriction maps, hence the following definition says that the sheaf E is a sheaf of C^l_X-modules.

4.3.1 Definition. *Let X be a C^k-manifold, and $l \leq k$. A sheaf of C^l_X-modules (or just a C^l_X-module) is then a sheaf $\mathcal{M}$ on X together with the structure, for all open U in X, of $C^l_X(U)$-module on $\mathcal{M}(U)$, compatible with the restriction maps. A morphism of C^l_X-modules is a morphism of sheaves such that on each open subset U of X it gives a morphism of $C^l_X(U)$-modules.*

The category of C^l_X-modules is in fact an abelian category with sufficiently many injectives, but for the moment we do not need this (it will probably be shown in Berthelot's course). Associating to a vector bundle its sheaf of sections is a functor from the category of C^l-vector bundles to the category of C^l_X-modules. We will show that this functor induces an equivalence of categories from the category of C^l-vector bundles to the full subcategory of C^l_X-modules that are "locally free of finite rank". The category of vector bundles on X is not an abelian category if $\dim_X(x) > 0$ for some x. These facts show that the category of C^l_X-modules is very useful.

If $\mathcal{M}$ is a C^l_X-module and $U \subset X$ an open subset, then $\mathcal{M}|_U$ is a C^l_U-module. For $\mathcal{M}_1$ and $\mathcal{M}_2$ two C^l_X-modules we define a presheaf $\mathcal{M}_1 \oplus \mathcal{M}_2$ by: $(\mathcal{M}_1 \oplus \mathcal{M}_2)U = \mathcal{M}_1(U) \oplus \mathcal{M}_2(U)$. This presheaf is in fact a sheaf (exercise), of C^l_X-modules (trivial), with canonical morphisms from $\mathcal{M}_1$ and $\mathcal{M}_2$ to it, such that it is the direct sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ in the category of C^l_X-modules (exercise). The same then works of course for arbitrary finite direct sums. A finite direct sum of C^l_X-modules has projection morphisms to its components, making it into the direct product of those (exercise).

Suppose now that $\mathcal{M}$ is a C^l_X-module, and that we have global sections $m_1, \ldots, m_r$ of it. Then we get a morphism from $(C^l_X)^r := \oplus_{i=1}^r C^l_X$ to $\mathcal{M}$ such that for all open $U \subset X$ and all $f_1, \ldots, f_r$ in $C^l_X(U)$ the element $(f_1, \ldots, f_r)$ of $(C^l_X)^r(U)$ is sent to $f_1 m_1 + \cdots + f_r m_r$ in $\mathcal{M}(U)$ (we omit the restriction maps). The sequence $(m_1, \ldots, m_r)$ is said to be a basis of $\mathcal{M}$ if this morphism is an isomorphism. A C^l_X-module $\mathcal{M}$ is called free of rank r if it is isomorphic to $(C^l_X)^r$, or, equivalently, if it has a basis with r elements. A C^l_X-module $\mathcal{M}$ is called locally free of rank r (with r a locally constant function on X) if every x in X has an open neighborhood U such that $\mathcal{M}|_U$ is a free C^l_U-module of rank $r(x)$.

4.3.2 Exercise. Let E be a real C^l-vector bundle on X. Show that its sheaf of sections E is locally free of finite rank as C^l_X-module. Show that for $U \subset X$ open and $e_1, \ldots, e_r$ in $E(U)$ the sequence $(e_1, \ldots, e_r)$ is a basis for $E|_U$ if and only if for all x in U the sequence $(e_1(x), \ldots, e_r(x))$ is a basis for the fibre $E(x)$ of E at x. $\square$

4.3.3 Theorem. *Let X be a C^k-manifold, and let $l \leq k$. The functor that sends a C^l-vector bundle to its sheaf of sections, viewed as a C^l_X-module, is an equivalence from the category of C^l-vector bundles to the full subcategory of the category of C^l_X-modules consisting of the C^l_X-modules that are locally free of finite rank. A quasi-inverse of this functor is described in the proof below.*

Proof. We will first describe a functor G from the category of locally free C^l_X-modules of finite rank to the category of C^l-vector bundles on X and then show that it is a quasi-inverse of the functor F mentioned in the Theorem (i.e., FG and GF are isomorphic to the identity funtors of the two categories in question).

So let $\mathcal{M}$ be a locally free C^l_X-module of finite rank r. Let x be in X. We consider the stalk $C^l_{X,x}$ of C^l_X at x and the stalk $\mathcal{M}_x$ of $\mathcal{M}$ at x. It follows from the definition of stalk that $\mathcal{M}_x$ is a $C^l_{X,x}$-module. Since $\mathcal{M}$ is locally free of rank r, its stalk $\mathcal{M}_x$ is a free $C^l_{X,x}$-module of rank $r(x)$. Let $C^l_{X,x} \to \mathbb{R}$ be the map that sends f to $f(x)$. It is a morphism of $\mathbb{R}$-algebras; let $m_x \subset C^l_{X,x}$ be its kernel. Then we define:

$$E(x) := \mathcal{M}_x / m_x \mathcal{M}_x = \mathbb{R} \otimes_{C^l_{X,x}} \mathcal{M}_x, \quad E := \coprod_{x \in X} E(x).$$

By construction, we get a map $p\colon E \to X$, such that $p^{-1}x = E(x)$. The $E(x)$ are clearly $\mathbb{R}$-vector spaces. Let U be an open subset of X on which $\mathcal{M}$ is trivial; let $m := (m_1, \ldots, m_d)$ be a basis of $\mathcal{M}|_U$. For x in U and s in $\mathcal{M}(U)$, let $s(x)$ be the image of s in $E(x)$. Then for all x in U, the $m_i(x)$ form an $\mathbb{R}$-basis of $E(x)$. Hence we get a bijection:

$$\phi_{U,m}\colon \mathbb{R}^d \times U \to p^{-1}U, \quad (\lambda, x) \mapsto (\sum_i \lambda_i m_i, x).$$

This bijection gives $p^{-1}U$ the structure of a C^l-manifold. One checks that this structure does not depend on the choice of the basis m, since if m' is another basis, one has a (unique) element g in $\mathrm{GL}_d(C^l_X(U))$ such that $m'_i = g m_i$ for all i. It is now clear that $p\colon E \to X$, with the $\mathbb{R}$-vector space structures on the $E(x)$, is a C^l-vector bundle. From the construction it is also clear that a morphism between C^l_X-modules that are locally free of finite rank induces a morphism of C^l-vector bundles on X. We have thus defined our functor G.

To finish the proof, we have to show that GF (resp., FG) is isomorphic to the identity functor of the category of C^l-vector bundles (resp., the category of locally free C^l_X-modules of finite rank).

Let E be a C^l-vector bundle on X. Then $F(E)$ is the sheaf of sections of E. From the definition of $F(E)$ and the definition of stalk, it follows that we have a map $F(E)_x \to E(x)$ that sends s to $s(x)$. This map of $C^l_{X,x}$-modules is surjective and its kernel is $m_x F(E)_x$ (use a local trivialization of E at x). Hence $E(x)$ is canonically isomorphic to $F(E)_x / m_x F(E)_x = (GF(E))(x)$. It is left to the reader to verify that this fibre wise isomorphism between E and $(GF)E$ is an isomorphism of C^l-vector bundles, and that it is functorial.

Let $\mathcal{M}$ be a locally free C^l_X-module of finite rank. Let U be an open subset of X, and s in $\mathcal{M}(U)$. For x in U, let $s(x)$ be the image of s in $(G\mathcal{M})(x)$. Then $x \mapsto s(x)$ is a C^l-section of $G\mathcal{M}$ over U (the verification of this, which can be done locally, is left to the reader). Hence we have a map from $\mathcal{M}(U)$ to $(FG\mathcal{M})U$. It is again left to the reader to check that these maps define an isomorphism of C^l_X-modules, and that this isomorphism is functorial. $\square$

5 Tensor constructions

In multi-linear algebra, there are constructions that associate, to a given collection of vector spaces, a vector space. For example, to a k-vector space V (k a field) one can associate its dual $V^\vee := \mathrm{Hom}_k(V,k)$. For k-vector spaces V and W, one has $\mathrm{Hom}_k(V,W)$ and $V\otimes_k W$. We will show that the constructions in the examples carry over to vector bundles, and to their sheaves of sections. Before we do that, we recall some facts about tensor products, the symmetric algebra and the exterior algebra, mainly for free modules over a ring (that is assumed to be commutative, as usual). As a reference for multi-linear algebra one can consult any algebra book, for example Lang's "Algebra", Bourbaki, or Jacobson's "Basic algebra I and II".

5.1 Multi-linear algebra

Let A be a commutative ring. For A-modules M and N we have the A-module $M\otimes_A N$, called the tensor product of M and N over A. This A-module $M\otimes_A N$ is defined as follows: we have a universal A-bilinear map $M\times N\to M\otimes_A N$, denoted by $(m,n)\mapsto m\otimes n$. (This means that for all A-bilinear maps $b\colon M\times N\to P$, there exists a unique A-linear map $\overline{b}\colon M\otimes_A N\to P$ such that $b(m,n)=\overline{b}(m\otimes n)$ for all (m,n).) If M and N are free, with bases m_i, $i\in I$, and n_j, $j\in J$, then $M\otimes_A N$ is free and $m_i\otimes n_j$, $(i,j)\in I\times J$, is a basis. For an A-module M we define $M^\vee$ to be the A-module $\mathrm{Hom}_A(M,A)$.

5.1.1 Proposition. *Let A be a commutative ring, and M and N A-modules. Then we have an A-linear map $M^\vee\otimes_A N\to \mathrm{Hom}_A(M,N)$ that sends $l\otimes n$ to $m\mapsto l(m)n$. If M is free of finite rank, then this map is an isomorphism of A-modules.*

Proof. Since $m\mapsto l(m)n$ is bilinear in l and n, the required map exists and is unique. Assume now that M is free of some rank r. To prove that the map is an isomorphism, we may suppose that $M=A^r$, because the map is functorial in M. But then we may identify $M^\vee$ with A^r, via the dual basis of the standard basis. Hence $M^\vee\otimes_A N=A^r\otimes_A N=N^r$. On the other hand, $\mathrm{Hom}_A(A^r,N)=N^r$. We leave it to the reader to see that our map is this identification. □

5.1.2 The tensor algebra

Let A be a commutative ring, and M an A-module. For $i\geq 0$ let $\mathrm{T}^i(M):=M^{\otimes i}$ be the ith tensor power of M. One way to define $\mathrm{T}^i(M)$ is to say that we have a universal i-linear map $M^i\to \mathrm{T}^i(M)$, sending $(m_1,\ldots,m_i)$ to $m_1\otimes\cdots\otimes m_i$. We define the tensor algebra of M to be the A-module $\mathrm{T}(M):=\oplus_{i\geq 0}\mathrm{T}^i(M)$, with the A-algebra structure defined as follows. Let i and j be ≥ 0. Consider the map $M^i\times M^j=M^{i+j}\to \mathrm{T}^{i+j}(M)$. Since this map is i-linear in the first variable, it induces a map $\mathrm{T}^i(M)\times M^j\to \mathrm{T}^{i+j}(M)$ that is linear in the first variable. This last map is j-linear in the second variable, hence induces a map $\mathrm{T}^i(M)\times \mathrm{T}^j(M)\to \mathrm{T}^{i+j}(M)$ that is bilinear and defines our multiplication map. The reader will check that $\mathrm{T}(M)$ becomes an associative graded A-algebra. We have $\mathrm{T}^0(M)=A$ and $\mathrm{T}^1(M)=M$. Let B be an associative A-algebra, and $\phi\colon M\to B$ a morphism of A-modules. Then there exists a unique morphism of

A-algebras $\tilde{\phi}\colon \mathrm{T}(M) \to B$ such that the restriction of $\tilde{\phi}$ to $\mathrm{T}^1(M)$ is ϕ. This situation gives us an example of adjoint functors: we have a functor F from the category of associative A-algebras to the category of A-modules, that sends B to B viewed as an A-module, and the functor T in the other direction, such that

$$\mathrm{Hom}_{A-\mathrm{mod}}(M, F(B)) = \mathrm{Hom}_{\mathrm{ass}-A-\mathrm{alg}}(\mathrm{T}(M), B),$$

functorially in M and B. If M is a free A-module of rank n, then $\mathrm{T}^i(M)$ is free of rank n^i (the reader will provide a basis).

5.1.3 The symmetric algebra

Let again A be a commutative ring, and M an A-module. We define the symmetric algebra $\mathrm{S}(M)$ to be the quotient of $\mathrm{T}(M)$ by the ideal generated by all $x \otimes y - y \otimes x$ with x and y in M. Since $\mathrm{T}(M)$ is generated, as A-algebra, by M, $\mathrm{S}(M)$ is a commutative A-algebra. Since the kernel of $\mathrm{T}(M) \to \mathrm{S}(M)$ is generated by homogeneous elements, the grading on $\mathrm{T}(M)$ induces a grading on $\mathrm{S}(M)$. For $i \geq 0$, $\mathrm{S}^i(M)$ is called the ith symmetric product of M, and denoted $\mathrm{Sym}^i_A(M)$. As in the case of the tensor algebra, the functor S, from the category of A-modules to the category A-alg of commutative A-algebras, is the left-adjoint of the forget functor in the opposite direction:

$$\mathrm{Hom}_{A-\mathrm{mod}}(M, B) = \mathrm{Hom}_{A-\mathrm{alg}}(\mathrm{S}(M), B).$$

If M is free of rank n, say with basis $m_1, \ldots, m_n$, then the morphism of A-algebras from the polynomial ring $A[x_1, \ldots, x_n]$ to $\mathrm{S}(M)$, sending x_i to m_i, is an isomorphism (use the universal property of $\mathrm{S}(M)$ to define the inverse). In particular, $\mathrm{S}^i(M)$ has basis $m_1^{i_1} \cdots m_n^{i_n}$, $i_1 + \cdots + i_n = i$, hence $\mathrm{S}^i(M)$ is free of rank $\binom{i+n-1}{n-1}$. It is left as an exercise to see that, for any A-module M, the map $M^i \to \mathrm{S}^i(M)$ that sends $(m_1, \ldots, m_i)$ to $m_1 \cdots m_i$ is a universal symmetric i-linear map.

5.1.4 The exterior algebra

This section is somewhat less trivial than the previous two, since as a special case we construct the determinant of a square matrix. Let again A be a commutative ring, and M an A-module. The exterior algebra $\Lambda(M)$ of M is then defined to be the quotient of $\mathrm{T}(M)$ by the ideal generated by the $x \otimes x$, with x in M. Then $\Lambda(M)$ is a graded associative A-algebra. We have $\Lambda^0(M) = A$ and $\Lambda^1(M) = M$. We claim that $\Lambda(M)$ is what is called "graded-commutative":

$$yx = (-1)^{ij} xy, \quad \text{for all } i, j \geq 0,\ x \text{ in } \Lambda^i(M),\ y \text{ in } \Lambda^j(M).$$

To see this, first note that it is true when i and j are one, since then $0 = (x+y)(x+y) - xx - yy = xy + yx$. Then it follows for all i and j, since M generates $\Lambda(M)$. The product in $\Lambda(M)$ is called the wedge product, and is sometimes denoted $(x, y) \mapsto x \wedge y$. The exterior algebra $\Lambda(M)$ has the following universal property: if $\phi\colon M \to B$ is an A-linear map from M to an A-algebra B, such that $\phi(m)\phi(m) = 0$ for all m in M, then there exists a unique morphism of A-algebras

$\tilde{\phi}$ from $\Lambda(M)$ to B whose restriction to M is ϕ. We leave it as an exercise to show that the map $M^i \to \Lambda^i(M)$ that sends $(m_1, \ldots, m_i)$ to $m_1 \cdots m_i$ is a universal alternating i-linear map. If M is free of rank n, say with basis $m_1, \ldots, m_n$, then $\Lambda^i(M)$ is free of rank $\binom{n}{i}$, and $m_{j_1} \cdots m_{j_i}$, $j_1 < \cdots < j_i$, is a basis. Since this is not so obvious, we will give a proof.

So assume that $m_1, \ldots, m_n$ is a basis of M. Let $i \geq 0$. It is clear that the $m_{j_1} \cdots m_{j_i}$, $j_1 < \cdots < j_i$, generate $\Lambda^i(M)$, so it remains to show that they are linearly independent. This follows if we can construct, for all $j = (j_1, \ldots, j_i)$, with $j_1 < \cdots < j_i$, an A-linear map ϕ_j from $\Lambda^i(M)$ to A, such that for all $k = (k_1, \ldots, k_i)$ with $k_1 < \cdots < k_i$ we have $\phi_j(m_{k_1}, \ldots, m_{k_i}) = \delta_{j,k}$. Note that $\mathrm{T}^i(M)$ has basis $m_{l_1} \cdots m_{l_i}$, $l_k \in \{1, \ldots, n\}$. Define an A-linear map ϕ_j from $\mathrm{T}^i(M)$ to A by: $\phi_j(m_{l_1} \cdots m_{l_i}) = 0$ if $\{l_1, \ldots, l_i\} \neq \{j_1, \ldots, j_i\}$, and $\phi_j(m_{l_1} \cdots m_{l_i}) = \varepsilon(\sigma)$ if $\{l_1, \ldots, l_i\} = \{j_1, \ldots, j_i\}$ and σ is the permutation sending j_k to l_k (ε is the sign of a permutation). Then ϕ_j induces the desired alternating i-linear form on M.

5.1.5 Exercise. Let k be a field, V a k-vector space and $d \geq 0$ an integer. Recall that we have defined the Grassmannian $\mathrm{Gr}_d(V)$ of d-dimensional subspaces of V, with the structure of algebraic variety over k if k is algebraically closed and V finite dimensional.

Let W be a d-dimensional subspace of V, and let $w_1, \ldots, w_d$ be a basis of W. Then we get an element $w_1 \cdots w_d$ of $\Lambda^d(V)$. Show that this element is non-zero, and that its image in $\mathbb{P}(\Lambda^d(V))$ only depends on W. Hence we have a map ϕ from $\mathrm{Gr}_d(V)$ to $\mathbb{P}(\Lambda^d(V))$. Show that this map is injective. It is called the Plücker embedding.

Suppose now that k is algebraically closed and that V is finite dimensional. Show that f is a morphism of algebraic varieties. Try to describe the image of f in the case $d = 2$ and $\dim(V) = 4$. (It is possible to describe the image by just one homogeneous equation of degree two.) More on this in the seminar! □

5.2 Tensor products of vector bundles and locally free sheaves

Let X be a C^k-manifold ($k \geq 0$), and $p\colon E \to X$ a real (or complex) C^l-vector bundle on it ($l \leq k$). Then we define the dual $E^\vee$ of E, which is also a real (or complex) C^l-vector bundle, as follows. As a set, $E^\vee$ is the disjoint union of the $E(x)^\vee$, x in X. Let $q\colon E^\vee \to X$ be the map with $q^{-1}x = E(x)^\vee$. To give $E^\vee$ the structure of a C^l-vector bundle, choose a covering of X by open subsets X_i, and trivializations $\phi_i\colon F_i \times X_i \to p^{-1}X_i$ of E. Then, for x in X_i, we have an isomorphism of vector spaces $\phi_i(x)\colon F_i \to E(x_i)$. Consequently, we have an isomorphism $\psi_i(x) := (\phi_i(x)^\vee)^{-1}$ from $F_i^\vee$ to $E^\vee(x_i)$. For all i, we have a bijection $\psi_i\colon F_i^\vee \times X_i \to q^{-1}X_i$. These bijections give $E^\vee$ the structure of a C^l-vector bundle. One checks that this structure does not depend on the choice of the X_i and the ϕ_i. Defined like this, we have, for each x in X, a bilinear map $\langle\cdot,\cdot\rangle_x$ from $E^\vee(x) \times E(x)$ to $\mathbb{R}$ (or $\mathbb{C}$), given by evaluation. For $U \subset X$ open, s in $E(U)$ and t in $E^\vee(U)$, the function $\langle t, s\rangle_U$ on U that sends x to $\langle t(x), s(x)\rangle_x$ is in $C^l_X(U)$. We want to show that these maps define an isomorphism from the C^l_X-module $E^\vee$ to the dual of E, as C^l_X-module. But in order to do this, we first have to define what such a dual is supposed to be.

5.2.1 Definition. *Let $(Y, \mathcal{O}_Y)$ be a ringed space, $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_Y$-modules. Then we define the presheaf $\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})$ as follows:*

$$\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})(U) = \mathrm{Hom}_{\mathcal{O}_Y|_U}(\mathcal{M}|_U, \mathcal{N}|_U), \quad U \subset Y \text{ open},$$

with the obvious restriction maps. This presheaf is actually a sheaf (the rather long verification is left to the reader), and moreover a $\mathcal{O}_Y$-module. The $\mathcal{O}_Y$-module $\mathbf{Hom}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{O}_Y)$ is called the dual of $\mathcal{M}$ and will be denoted $\mathcal{M}^\vee$.

We will define a morphism of C^l_X-modules from the sheaf of sections of $E^\vee$ to $\mathbf{Hom}_{C^l_X}(E, C^l_X)$, and then show that it is an isomorphism. So let $U \subset X$ be open, and t in $E^\vee(U)$. Let $V \subset U$ be open, and s in $E(V)$. Then we have $\langle t|_V, s\rangle_V$ in $C^l_X(V)$. This defines a morphism of $C^l_X(U)$-modules from $E^\vee(U)$ to $\mathrm{Hom}_{C^l_X|_U}(E^\vee|_U, C^l_X|_U)$. One verifies that we get indeed a morphism from $E^\vee$ to $\mathbf{Hom}_{C^l_X}(E, C^l_X)$ as desired. Let us now show that it is an isomorphism. So let $U \subset X$ be open and let ϕ be in $\mathbf{Hom}_{C^l_X}(E, C^l_X)(U)$. Then, for each x in U, ϕ induces, via the equivalence between vector bundles and sheaves of modules (that is, Theorem 4.3.3), an element $\phi(x)$ of $E^\vee(x)$. It is easy to check that these $\phi(x)$ form an element of $E^\vee(U)$.

So we have now seen that the operation $E \mapsto E^\vee$ on C^l-vector bundles corresponds to $E \mapsto \mathbf{Hom}_{C^l_X}(E, C^l_X)$ on C^l_X-modules. Our next goal is to define tensor products, on both sides. Let us first treat the case of vector bundles. So let E and F be C^l-vector bundles on X. We define a vector bundle $E \otimes F$ as follows. As a set, it is the disjoint union of the vector spaces $E(x) \otimes F(x)$, for x in X. The map from it to X is clear. Then $E \otimes F$ is given the structure of C^l-manifold via local trivializations of E and F. Let $U \subset X$ be open, s in $E(U)$ and t in $F(U)$. For each x in U we get $s(x) \otimes t(x)$ in $(E \otimes F)(x)$. These define an element $s \otimes t$ of $(E \otimes F)(U)$. The map from $E(U) \times F(U)$ to $(E \otimes F)(U)$ sending (s, t) to $s \otimes t$ is $C^l_X(U)$-bilinear. Varying U, we get a C^l_X-bilinear map from $E \times F$ to $E \otimes F$. We will prove a bit further that this map is the universal C^l_X-bilinear map from $E \times F$ to C^l_X-modules. Before that, we define the tensor product on the side of sheaves.

5.2.2 Definition. *Let $(Y, \mathcal{O}_Y)$ be a ringed space, $\mathcal{M}$ and $\mathcal{N}$ locally free $\mathcal{O}_Y$-modules of finite rank. By definition, every y in Y has an open neighborhood U on which $\mathcal{M}$ and $\mathcal{N}$ are free, say of ranks m and n. For such open subsets U we put: $\mathcal{T}(U) := \mathcal{M}(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{N}(U)$. For V an open subset of such a U we have a restriction map $\mathcal{T}(U) \to \mathcal{T}(V)$ (note that indeed $\mathcal{M}$ and $\mathcal{N}$ are free on V). Choosing, for such a U, isomorphisms $(\mathcal{O}_Y|_U)^m \to \mathcal{M}|_U$ and $(\mathcal{O}_Y|_U)^n \to \mathcal{N}|_U$, we see that $V \mapsto \mathcal{T}(V)$ is a $\mathcal{O}_Y|_U$-module. The next lemma implies, among other things, that there is a unique $\mathcal{O}_Y$-module $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ on Y such that for all U as above we have $(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U) = \mathcal{T}(U)$. This $\mathcal{O}_Y$-module is locally free of finite rank.*

5.2.3 Lemma. *Let X be a topological space, $\mathcal{U}$ a collection of open subsets of X that covers X and that is a sieve on X (i.e., $U \in \mathcal{U}$ and $V \subset U$ open imply $V \in \mathcal{U}$). A presheaf on $\mathcal{U}$ is defined to be a contravariant functor from $\mathcal{U}$ (morphisms are just the inclusions) to the category of sets. A presheaf F on $\mathcal{U}$ is called a sheaf if, for all U in $\mathcal{U}$, F defines a sheaf on U. Let $\mathrm{Sh}(X)$ and $\mathrm{Sh}(\mathcal{U})$ denote the categories of sheaves on X and $\mathcal{U}$, respectively. Then the functor*

$F \mapsto F|_{\mathcal{U}}$ *from* $\mathrm{Sh}(X)$ *to* $\mathrm{Sh}(\mathcal{U})$ *is an equivalence of categories. A quasi-inverse is described in the proof.*

Proof. Let F be a sheaf on $\mathcal{U}$. Let V be an open subset of X. Let $\mathcal{U}|_V$ be the set of U in $\mathcal{U}$ with $U \subset V$. We define $F^+(V)$ to be the projective limit $\varprojlim F(U)$, taken over the U in $\mathcal{U}|_V$. Concretely, this means that an element of $F^+(V)$ is a compatible system of s_U in $F(U)$, indexed by $\mathcal{U}|_V$. Since the restriction of F to any U in $\mathcal{U}$ is a sheaf, we see that $F^+(U) = F(U)$ for such U. It is left to the reader to define the restriction maps for F^+, and to verify that F^+ is a sheaf. The verifications that $F \mapsto F^+$ is a functor, and that it is a quasi-inverse of $G \mapsto G|_{\mathcal{U}}$, are left to the reader. □

5.2.4 Proposition. *Let* $(Y, \mathcal{O}_Y)$ *be a ringed space and let* $\mathcal{M}$ *and* $\mathcal{N}$ *be* $\mathcal{O}_Y$*-modules. Let* $\mathcal{U}$ *be the sieve on* X *consisting of the* U *on which both* $\mathcal{M}$ *and* $\mathcal{N}$ *are trivial. Lemma 5.2.3 implies that the maps:*

$$\mathcal{M}(U) \times \mathcal{N}(U) \to \mathcal{M}(U) \otimes_{\mathcal{O}_Y(U)} \mathcal{N}(U) = (\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U)$$

define a morphism of sheaves $\mathcal{M} \times \mathcal{N} \to \mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ *on* X*. This morphism is a universal* $\mathcal{O}_Y$*-bilinear map from* $\mathcal{M} \times \mathcal{N}$ *to* $\mathcal{O}_Y$*-modules.*

Proof. Everything but the universality is clear. So let $\mathcal{P}$ be a $\mathcal{O}_Y$-module and b a bilinear map from $\mathcal{M} \times \mathcal{N}$ to $\mathcal{P}$. Then, for each U in $\mathcal{U}$, we get a unique morphism of $\mathcal{O}_Y(U)$-modules from $(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N})(U)$ to $\mathcal{P}(U)$ that, composed with the universal bilinear map from $\mathcal{M}(U) \times \mathcal{N}(U)$, is $b(U)$. Lemma 5.2.3 shows that these maps give the desired unique morphism of $\mathcal{O}_Y$-modules from $\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{N}$ to $\mathcal{P}$ that, composed with the bilinear map from $\mathcal{M} \times \mathcal{N}$, is b. □

Let us now go back to our manifolds and tensor products of vector bundles: we had a C^k-manifold X and C^l-vector bundles E and F. We have already defined the C^l-vector bundle $E \otimes F$. It is easy to check that, for an open subset U of X on which E and F are trivial, we have $(E \otimes F)U = E(U) \otimes_{C^l_X(U)} F(U)$ (or, more precisely, the natural map between them is an isomorphism). Lemma 5.2.3 tells us that (the sheaf of sections of) $E \otimes F$ is the same as (more precisely, uniquely isomorphic to) $E \otimes_{C^l_X} F$.

From what we have done up to now, it is clear how to define, for a C^l-vector bundle E on a manifold X, the bundle analogs $\mathrm{T}(E)$, $\mathrm{S}(E)$ and $\Lambda(E)$ of the tensor algebra, the symmetric algebra and the exterior algebra, and that these constructions coincide with their analogs for C^l_X-modules that are locally free of finite rank. (Here we forget for a moment that vector bundles, as we have defined them, have finite dimension.) It is clear that the approach via sheaves of modules works in the context of analytic and algebraic varieties. (This is also true for the approach via bundles.) In order to consider complex C^l-vector bundles on a C^k-manifold X it suffices to consider the sheaf $C^l_{X,\mathbb{C}}$ of complex valued C^l-functions on X, where, as explained in §1, a $\mathbb{C}$-valued function on X is called C^l if both its real and imaginary part are. Before indulging in differential forms and de Rham cohomology we take a brief look at metrics on vector bundles.

5.3 Metrics on vector bundles

Let X be a C^k-manifold, with $k \geq 0$, and E a C^l-vector bundle on X. There are various ways to describe what a metric on E is. Viewing E as a bundle, a metric on E is a collection of non-degenerate symmetric bilinear forms $\langle \cdot, \cdot \rangle_x$ on the $E(x)$, "varying C^l with x". This last condition means that after local trivialization, the coefficients of the matrix describing the $\langle \cdot, \cdot \rangle_x$ are C^l-functions. Equivalently, for U open in X and s and t in $E(U)$, $\langle s, t \rangle_U : x \mapsto \langle s(x), t(x) \rangle_x$ is in $C^l_X(U)$. Viewing E as a locally free C^l_X-module, a metric is a symmetric bilinear map $b: E \times E \to C^l_X$ such that, for all x in X, the symmetric bilinear form $b(x)$ on $E(x)$ is non-degenerate. Equivalently, b induces an isomorphism of C^l_X-modules from E to $E^\vee$. Or also: a metric on E is a symmetric isomorphism of C^l_X-modules from E to $E^\vee$. Considering the universal symmetric bilinear form on E, one sees that a metric on E is an element b of $\mathrm{S}^2(E)^\vee(X)$ such that all $b(x)$ are non-degenerate. To conclude: all various equivalent descriptions of symmetric bilinear forms that one sees in a linear algebra course work in the contexts of vector bundles and C^l_X-modules. For example, a metric b on E has a signature, which is a locally constant function $s: X \to \mathbb{Z}^2$ such that $s(x)_1$ (resp., $s(x)_2$) is the number of positive (resp., negative) coefficients of $b(x)$ in any diagonal form.

Usually when working with vector bundles with a metric, the metric comes naturally with the vector bundle. But sometimes it is useful to just choose a metric on a given vector bundle, if one exists (for example, if one wants to split short exact sequences of vector bundles). So a natural question to ask is: under what conditions does a vector bundle E admit a metric b (say with a fixed signature)? We will see, when discussing partitions of unity, that, for X paracompact (i.e., X is separated and every open cover has a locally finite refinement), every vector bundle has a positive definite metric. This has to do with the fact that the set of positive definite symmetric bilinear forms on $\mathbb{R}^n$ is convex. On the other hand, there are topological obstructions against the existence of metrics of signature $(1, 1)$, because the set of symmetric bilinear forms on $\mathbb{R}^2$ of that signature is homotopically equivalent to the circle. For example, it can be seen that the tangent bundle of the two-sphere S^2 does not admit a metric of signature $(1, 1)$ (namely, from such a metric one can construct a nowhere zero vector field on S^2, and everybody knows that the sphere can't be combed).

Before going on, let us look a bit at what is happening here. So let E be a vector bundle of constant rank r on a manifold X, and let s be a fixed signature. For x in X, the set of metrics of signature s on $E(x)$ is an open subset $Y(x)$ of $\mathrm{S}^2(E(x))^\vee$. The $Y(x)$ are all (non-canonically) isomorphic, as manifolds, to the open subset F of $\mathrm{S}^2(\mathbb{R}^r)^\vee$ consisting of metrics of signature s. Using that E is locally trivial, it is easy to equip the disjoint union Y of the $Y(x)$ with the structure of a fibration over X with fibre F. The question of whether or not E has a metric of signature s is then the same as the question of whether or not this fibration has a section. Let us now give two fibrations, with non-empty fibre, that have no section. The first example is the complement of the zero section of the Möbius strip, viewed as a fibration over S^1 with fibre $[-1, 1] - \{0\}$. The other example is the Hopf fibration of S^3 over S^2; it is obtained as follows. View $\mathbb{P}^1(\mathbb{C})$ as the two-sphere S^2. Then S^2 is the quotient of $\mathbb{C}^2 - \{0\}$ by the action of $\mathbb{C}^*$. View S^3 as the quotient of $\mathbb{C}^2 - \{0\}$ by the subgroup $\mathbb{R}^*_{>0}$ of $\mathbb{C}^*$. Then we see that S^2 is the quotient of S^3 by the group $\mathbb{C}^*/\mathbb{R}^*_{>0}$, which is isomorphic to the subgroup S^1 of $\mathbb{C}^*$. Since the action of $\mathbb{C}^*$ on $\mathbb{C}^2 - \{0\}$ is free, the action of S^1 on S^3 is so too. Hence we have our fibration. To see that there is no section, note that if there is a section, we get an isomorphism from $S^1 \times S^2$ to S^3, but the first of these two is not simply connected whereas the second is.

5.3.1 Definition. *Let X be a C^k-manifold, with $k \geq 1$. A Riemannian metric on X is then a positive definite metric on T_X.*

As we have said, we will show later that every paracompact manifold has a Riemannian metric. Suppose now that X is a C^k-manifold and that $\langle\cdot,\cdot\rangle$ is a Riemannian metric on it. Suppose that $c\colon I \to X$ is C^1, with $I = [a, b]$ some non-empty closed interval in $\mathbb{R}$. Then we can define the length of c as follows:

$$\text{length}(c) := \int_a^b \|c'(t)\|_{c(t)}\, dt, \tag{5.3.2}$$

with $\|\cdot\|$ the norm associated to $\langle\cdot,\cdot\rangle$. The reader should note that when X is $\mathbb{R}^n$ and $\langle\cdot,\cdot\rangle$ is the standard Riemannian metric, this definition of length coincides with the standard one. An important fact is that if $c_1 := c\circ\phi$ with $\phi\colon J \to I$ a diffeomorphism of closed intervals, say with $J = [a_1, b_1]$, then the length of c_1 equals that of c:

$$\int_{a_1}^{b_1} \|c_1'(s)\|_{c_1(s)}\, ds = \int_{a_1}^{b_1} |\phi'(s)|\, \|c'(\phi(s))\|_{c(\phi(s))}\, ds = \int_a^b \|c'(t)\|_{c(t)}\, dt. \tag{5.3.3}$$

This implies that the length of a curve c is independent of the choice of the parametrization. Suppose now moreover that X is connected. Then X is arcwise connected, hence we can define a real valued function d on $X \times X$ by:

$$d(x, y) := \inf\{\text{length}(c) \mid c \text{ a smooth curve from } x \text{ to } y\}. \tag{5.3.4}$$

It is quite clear that d is symmetric and that it satisfies the triangle inequality. One can show without too much pain that if X is separated, then one has $d(x, y) = 0$ if and only if $x = y$. See for example Chapter 9 of Spivak, Volume 1. The problem of finding the shortest path between two given points leads to variational calculus (see Baird's course in the second semester) and to the definition of a geodesic.

Not all metrics that arise naturally are positive definite. For example, in the theory of general relativity one studies four-dimensional manifolds with a metric of signature $(1, 3)$; so-called Lorentzian manifolds. The path from x to y that corresponds to a free fall is then a path of maximal length from x to y (of course one only considers paths that do respect the speed limit imposed by the speed of light, because otherwise the square root in the definition of $\|\cdot\|$ becomes imaginary). There is an excellent book on this matter, by Sachs and Wu, with the title "General relativity for mathematicians".

To finish this section: not even all bilinear forms that occur naturally on vector bundles are symmetric. For example, anti-symmetric bilinear forms, also called symplectic forms, play an important role in classical mechanics (Hamilton systems). Here the reader should think of formulas such as $\sum_i dp_i \wedge dq_i$.

5.4 Differential forms

Let X be a C^k-manifold, with $k \geq 1$. Then we have the C^{k-1}-vector bundle T_X on X. The dual $\mathrm{T}_X^\vee$ of T_X is called the bundle of 1-forms on X, and is denoted Ω_X^1. Note that with our conventions, Ω_X^1 also denotes the sheaf of sections of Ω_X^1. We define Ω_X to be $\Lambda(\Omega_X^1)$. This Ω_X) is a graded-commutative C_X^{k-1}-algebra, and its degree i component Ω_X^i is called the sheaf or bundle of i-forms. We will do two important things in the next two sections: we define the usual morphisms of sheaves $d\colon \Omega_X^i \to \Omega_X^{i+1}$ that give us the de Rham complex of X, and we define what integration of forms of top degree is.

Our first step is to define a map $d\colon C_X^k \to \Omega_X^1$. But even to do this, we have to go back to T_X itself: namely, we have to give an interpretation of $\mathrm{T}_X(U)$ for $U \subset X$ open. So let $U \subset X$ open, and let ∂ in $\mathrm{T}_X(U)$. We will show that ∂ defines a derivation from $C_X^k|_U$ to $C_X^{k-1}|_U$. So let $V \subset U$ be open, and let f be in $C_X^k(V)$. Let x be in V. Then we have $\partial(x)$ a tangent vector at x, and f_x in the stalk $C_{X,x}^k$. We define: $(\partial f)x := \partial(x)f_x$, which is in $\mathbb{R}$. No matter how we view tangent vectors, this number is simply the derivative at x of f in the direction $\partial(x)$. Looking in a chart, it is clear that the function ∂f from V to $\mathbb{R}$ is in $C_X^{k-1}(V)$, that ∂ is indeed a morphism of sheaves from $C_X^k|_U$ to $C_X^{k-1}|_U$ and that it is a derivation: $\partial(fg) = f\partial(g) + \partial(f)g$.

5.4.1 Question. Let X be a C^k-manifold with $k \geq 1$. Does the construction above give an isomorphism of C^{k-1}-modules between T_X and $\mathbf{Der}_{\mathbb{R}}(C_X^k, C_X^{k-1})$? I don't know. If it is not true, and one still wants an interpretation of T_X of this kind, it seems a good idea to impose some local continuity condition on the derivations themselves. □

We can now define our map $d\colon C_X^k \to \Omega_X^1$. Let $U \subset X$ be open and let f be in $C_X^k(U)$. By construction, $\Omega_X^1(U)$ is equal to $\mathbf{Hom}_{C_X^{k-1}}(\mathrm{T}_X, C_X^{k-1})(U)$. We define df to be the element in $\Omega_X^1(U)$ that sends ∂ in $\mathrm{T}_X(V)$, with $V \subset U$ open, to ∂f. The reader will verify that d is a morphism of sheaves, that it is $\mathbb{R}$-linear and that it satisfies:

$$d(fg) = f\,dg + g\,df, \tag{5.4.2}$$

for U open in X and f and g in $C_X^k(U)$. Intuitively, the expression $(df)x$ can be thought of as a measure for the infinitesimal rate of change of f at x in an unspecified direction, and that, when evaluated on a tangent vector at x, it gives the derivative at x of f in that direction. The map d itself can be thought of as a universal derivation (this is really so in the C^∞ and the C^ω-cases). One should note that the morphism of sheaves $d\colon C_X^k \to \Omega_X^1$ is not a morphism of vector bundles (except in the case where Ω_X^1 is zero, of course), because it is not C_X^k-linear (it is a derivation, after all).

5.4.3 Proposition. *Let X be a C^k-manifold with $k \geq 1$. Let $U \subset X$ be an open set and $x_1, \ldots, x_n$ local coordinates on U, i.e., the x_i are in $C_X^k(U)$ and the map $x\colon U \to \mathbb{R}^n$ sending u to $(x_1(u), \ldots, x_n(u))$ is an isomorphism of C^k-manifolds from U to an open subset V of $\mathbb{R}^n$. Then $\Omega_X^1|_U$ is a free C_U^{k-1}-module and $(dx_1, \ldots, dx_n)$ is a basis. This basis is the dual basis* *of the basis $D_1, \ldots, D_n$ of $\mathrm{T}_X|_U$ given by the partial derivatives. For f in $C^k_X(U)$ we have the formula:*

$$df = \sum_{i=1}^{n} (D_i f) dx_i.$$

Proof. The fact that the D_i form a basis of $\mathrm{T}_X|_U$ was proved, point-wise, in §4.1. By construction, the dx_i form the dual basis. The formula above follows from evaluating both sides on the D_j. □

5.4.4 Corollary. *Let X be a C^k-manifold with $k \geq 1$. Let $U \subset X$ be open, $x_1, \ldots, x_n$ local coordinates on U, and $r \geq 0$ an integer. Then $\Omega^r_X|_U$ is a free C^{k-1}_U-module and the $dx_{i_1} \cdots dx_{i_r}$ with $i_1 < \cdots < i_r$ form a basis.*

5.4.5 Proposition. *Let X be a C^k-manifold with $k = \infty$ or ω. There exists a unique morphism of sheaves $d\colon \Omega_X \to \Omega_X$ such that:*

1. *d is $\mathbb{R}$-linear and maps Ω^i_X to Ω^{i+1}_X;*
2. *the restriction of d to Ω^0_X is $d\colon C_X \to \Omega^1_X$;*
3. *for $U \subset X$ open, x in $\Omega^r_X(U)$ and y in $\Omega^s_X(U)$, we have $d(xy) = (dx)y + (-1)^r x dy$;*
4. *$d^2 = 0$.*

Proof. Since Ω^1_X generates Ω_X as a C_X-algebra, there exists at most one such morphism. Because of this uniqueness, it suffices to prove the existence locally. So we may and do assume that X is an open subset of $\mathbb{R}^n$. Let $r \geq 0$ be an integer. The $dx_{i_1} \cdots dx_{i_r}$ with $i_1 < \cdots < i_r$ form a C_X-basis of Ω^r_X. Conditions 2 and 4 of the proposition we are proving imply that we must define

$$d(f dx_{i_1} \cdots dx_{i_r}) = df\ dx_{i_1} \cdots dx_{i_r} = \sum_{i=1}^{n} (D_i f) dx_i dx_{i_1} \cdots dx_{i_r}. \tag{5.4.6}$$

Let us now show that the morphism d defined by this formula satisfies all the conditions of the proposition. Conditions 1 and 2 are clearly satisfied. Let us now do 4. One computes:

$$\begin{aligned} d(d(f dx_{i_1} \cdots dx_{i_r})) &= d(\sum_i (D_i f) dx_i dx_{i_1} \cdots dx_{i_r}) = \sum_i d(D_i f) dx_i\ dx_{i_1} \cdots dx_{i_r} = \\ &= \sum_i \sum_j D_j(D_i f) dx_j\ dx_i\ dx_{i_1} \cdots dx_{i_r} = \\ &= \left(\sum_{i,j} D_j(D_i f) dx_i\ dx_j \right) dx_{i_1} \cdots dx_{i_r} = 0. \end{aligned}$$

To prove 3, we may write $x = f dx_{i_1} \cdots dx_{i_r}$ and $y = g dx_{j_1} \cdots dx_{j_s}$. One computes:

$$\begin{aligned} d(xy) &= d(fg dx_{i_1} \cdots dx_{i_r}\ dx_{j_1} \cdots dx_{j_s}) = (f dg + g df) dx_{i_1} \cdots dx_{i_r}\ dx_{j_1} \cdots dx_{j_s} = \\ &= df\ dx_{i_1} \cdots dx_{i_r}\ g dx_{j_1} \cdots dx_{j_s} + (-1)^r f dx_{i_1} \cdots dx_{i_r}\ dg\ dx_{j_1} \cdots dx_{j_s} = \\ &= dx\ y + (-1)^r x dy. \end{aligned}$$

□

5.4.7 Remark. Note that the previous Proposition only talks about the C^∞ and the analytic cases. It is certainly possible to formulate an analogous result for $k \geq 2$, but I find that too much of a hassle. Let us consider the case where X is open in $\mathbb{R}^3$. For f in $C_X(X)$ one has $df = (D_1 f)dx_1 + (D_2 f)dx_2 + (D_3 f)dx_3$, which is just an expression for the gradient of f. The reader should verify that $d(f_1 dx_1 + f_2 dx_2 + f_3 dx_3)$ gives the curl and $d(f_1 dx_2 dx_3 + f_2 dx_3 dx_1 + f_3 dx_1 dx_2)$ the divergence. Usually, in calculus, the gradient of a function is a vector field, and not a one-form; this comes from the identification between T_X and Ω^1_X given by the standard Riemannian metric. Likewise, in calculus one applies divergence to vector fields, not to two-forms; here one uses that the multiplication $\Omega^2_X \times \Omega^1_X \to \Omega^3_X$ is a perfect pairing (i.e., it identifies both sides with the dual of the other). A similar remark holds for the curl. □

5.4.8 Exercise. Let X be a C^k-manifold with $k \geq 0$. Show that X is the disjoint union of its connected components; let $\pi_0(X)$ denote the set of connected components of X. Assume now that k is in $\{\infty, \omega\}$. Show that the kernel of $d\colon C_X \to \Omega^1_X$ is the constant subsheaf $\mathbb{R}_X$ of C_X. Conclude that:

$$\ker(d\colon C_X(X) \to \Omega^1_X(X)) = \mathbb{R}^{\pi_0(X)}.$$

□

5.4.9 Remark. We will see later that the complex of sheaves (Ω_X, d) is exact in all degrees $i > 0$. This means that (Ω_X, d) is a resolution of the sheaf $\mathbb{R}_X$. We will show that the sheaves Ω^i_X are acyclic for the functor $\Gamma(X, \cdot)$ when X is paracompact. It follows that under that condition, the complex $(\Omega_X(X), d)$ computes the cohomology of $\mathbb{R}_X$. Since the cohomology of this complex is by definition the de Rham cohomology, we see that the de Rham cohomology of X is the cohomology of $\mathbb{R}_X$. □

5.5 Volume forms, integration and orientation

Thinking about what kind of objects one can expect to be able to integrate over manifolds, one comes to the following definition.

5.5.1 Definition. *Let V be a finite dimensional $\mathbb{R}$-vector space, say of dimension n. Let W be an $\mathbb{R}$-vector space. A volume form on V with values in W is then a map $v\colon V^n \to W$ such that:*

1. *$v(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = v(v_1, \ldots, v_n)$, for all $v_1, \ldots, v_n$ in V and σ in S_n;*
2. *$v(\lambda v_1, v_2 \ldots, v_n) = |\lambda| v(v_1, \ldots, v_n)$, for all $v_1, \ldots, v_n$ in V and λ in $\mathbb{R}$;*
3. *$v(v_1 + v_2, v_2, \ldots, v_n) = v(v_1, \ldots, v_n)$, for all $v_1, \ldots, v_n$ in V.*

All volume forms as in the definition are obtained as follows (proof left to the reader). Let w be in W and l in $\Lambda^n(V)^\vee$. Then the map v defined by $v(v_1, \ldots, v_n) := |l(v_1, \ldots, v_n)|\, w$ is a volume form. It follows that for v a volume form, $v_1, \ldots, v_n$ in V and g in $\mathrm{GL}(V)$, we have $v(g(v_1), \ldots, g(v_n)) = |\det(g)|\, v(v_1, \ldots, v_n)$. The set $\mathrm{Vol}(V, W)$ of W-valued volume forms on V his itself an $\mathbb{R}$-vector space (sum and scalar multiplication are defined as usual). For $f: W \to W'$ an $\mathbb{R}$-linear map of $\mathbb{R}$-vector spaces, we get an $\mathbb{R}$-linear map f_* from $\mathrm{Vol}(V, W)$ to $\mathrm{Vol}(V, W')$, that sends v to $f \circ v$. In fact, $\mathrm{Vol}(V, \cdot)$ is a covariant functor.

5.5.2 Lemma. *Let V be as above. The functor $\mathrm{Vol}(V, \cdot)$ is representable, by a one-dimensional $\mathbb{R}$-vector space that we denote $|\Lambda^n(V)|$. Equivalently, we have a universal volume form $V^n \to |\Lambda^n(V)|$.*

Proof. Take l to be a non-zero element of $\Lambda^n(V)$, and consider the $\mathbb{R}$-valued volume form $|l|$. The discussion above shows that this volume form is universal. □

The set of $\mathbb{R}$-valued volume forms on V is $|\Lambda^n(V)|^\vee$. An $\mathbb{R}$-valued volume form is called positive if all its values are ≥ 0. We have a map $\Lambda^n(V)^\vee \to |\Lambda^n(V)|^\vee$ that sends l to $|l| := |\cdot| \circ l$. The image of this map is the set of positive volume forms.

5.5.3 Definition. *An orientation on a one-dimensional $\mathbb{R}$-vector space L is a connected component of $L - \{0\}$. The union of this component with $\{0\}$ will be denoted L^+ and it will be called the positive component.*

It is clear from the definition that $|\Lambda^n(V)|^\vee$ has a given orientation, for which the positive component consists of the positive volume forms. We are now ready to apply the notion of a volume form to manifolds.

5.5.4 Definition. *Let X be a C^k-manifold, for some $k \geq 1$. We define the vector bundle Vol_X of volume forms on X to be the C^{k-1}-vector bundle with $\mathrm{Vol}_X(x) = |\Lambda^{\dim_X(x)} \mathrm{T}_X(x))|^\vee$ for all x in X, with local trivializations induced by those of T_X. For W a finite dimensional $\mathbb{R}$-vector space $W \otimes_{\mathbb{R}} \mathrm{Vol}_X$ is defined to be the C^{k-1}-vector bundle with $(W \otimes_{\mathbb{R}} \mathrm{Vol}_X)(x) = W \otimes_{\mathbb{R}} \mathrm{Vol}_X(x)$ for all x in X, with local trivializations induced by those of T_X. For $U \subset X$ open and $l \leq k-1$, a C^l-section of $W \otimes_{\mathbb{R}} \mathrm{Vol}_X$ over U is called a W-valued volume form on U.*

5.5.5 Remark. The sheaf of C^{k-1}-sections of $W \otimes_{\mathbb{R}} \mathrm{Vol}_X$ is the tensor product $W_X \otimes_{\mathbb{R}_X} \mathrm{Vol}_X$, where W_X and $\mathbb{R}_X$ denote the constant sheaves on X associated to W and $\mathbb{R}$.

It follows immediately from the definitions that, for $U \subset X$ open with local coordinates $x_1, \ldots, x_n$, every element of $(W \otimes_{\mathbb{R}} \mathrm{Vol}_X)(U)$ can be uniquely written in the form $w \cdot |dx_1 \cdots dx_n|$, with $w: U \to W$ a C^{k-1}-function.

The finite dimensionality of W is there just because we have decided that vector bundles should have finite rank (they have to be manifolds themselves). Working with sheaves, there is no problem whatsoever to allow W to have infinite dimension. □

5.5.6 Definition. *Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X and f an element of $\mathcal{F}(X)$. The support of f, denoted $\mathrm{Supp}(f)$, is defined to be the set $\{x \in X \mid f_x \neq 0\}$; it is a closed subset of X.*

We will now be concerned with defining the integral of continuous W-valued volume forms with quasi-compact support (recall that a topological space is called quasi-compact if every open cover of it has a finite subcover). Intuitively, the integral of such a form is the sum of all its values (which are elements of W). More precisely, one should think of Riemann sums; the volume form itself tells us how to measure the size of "small cubes". But the term "cube", or "block", do not make much sense in X. So first we explain what we want to do locally; after that we set up an administration system to make sure that everything gets counted exactly once. I should also admit that the usual definition of integration of volume forms is in terms of paritions of unity. In this course I want to show that there is another definition that is closer to the way one actually computes integrals, and which does not need that the manifold is separated. Of course, both definitions give the same result for separated manifolds. We will probably need the usual definition do prove some general results on integration (such as Stokes's theorem).

So let X be a C^k-manifold, with $k \geq 1$. Let W be a finite dimensional $\mathbb{R}$-vector space and v a W-valued volume form on X. Let $\phi\colon U \to X$ be a chart, with $U \subset \mathbb{R}^n$ open. For all u in U, $\mathrm{T}_\phi(u)$ is an isomorphism from $\mathrm{T}_U(u)$ to $\mathrm{T}_X(\phi(u))$. This gives us isomorphisms from $(W \otimes_\mathbb{R} \mathrm{Vol}_U)(u)$ to $(W \otimes_\mathbb{R} \mathrm{Vol}_X)(\phi(u))$, and hence a W-valued volume form $\phi^* v$ on U. We have, uniquely, $\phi^* v = w|dx_1 \cdots dx_n|$, with $w\colon U \to W$ a C^{k-1}-map. Let $V \subset U$ be a bounded measurable (in the sense of Lebesgue) subset of U (for example, a bounded open or closed subset in U). Then we define:

$$\int_V \phi^* v := \int_V w,$$

where the last integral is in the sense of Lebesgue. To make things a bit more concrete: if $(w_1, \ldots, w_d)$ is a basis for W, and $w = \sum_i f_i w_i$, then $\int_V w = \sum_i (\int_V f_i) w_i$. We are now ready to define the integral on X itself.

5.5.7 Construction. Let X be a C^k-manifold with $k \geq 1$. Let W be a finite dimensional $\mathbb{R}$-vector space. Let v be a W-valued volume form on X with quasi-compact support. Suppose that we have an integer $m \geq 0$, charts $\phi_i\colon U_i \to X$, $1 \leq i \leq m$, $V_i \subset U_i$ bounded and measurable, such that $\mathrm{Supp}(v) \subset \cup_i \phi_i V_i$. Then we define the integral of v over X, with respect to these data, to be:

$$\int_X v := \sum_{r=1}^{m} (-1)^{r+1} \sum_{i_1 < \cdots < i_r} \int_{V_{i_1, \ldots, i_r}} \phi_{i_1}^* v,$$

where $V_{i_1,\ldots,i_r} := \phi_{i_1}^{-1} \cap_{j=1}^r \phi_{i_j} V_{i_j}$. □

The sum over r, and the signs in it, are there to make that we do not count the intersections twice, etc.; it is called the inclusion exclusion principle, easily understood in terms of the characteristic functions of the V_i. Note that the $\phi_{i_1}^{-1} V_{i_j} \cap U_{i_1}$ are measurable subsets of U_{i_1}, hence that $V_{i_1,\ldots,i_r}$ is indeed a measurable subset of U_{i_1}. Let us show how charts ϕ_i and measurable subsets V_i of U_i with $\mathrm{Supp}(v) \subset \cup_i \phi_i V_i$ can be obtained. For each x in $\mathrm{Supp}(v)$, choose a chart $\phi_x\colon U_x \to X$ and a measurable neighborhood V_x of $\phi_x^{-1} x$ in U_x (for example, a compact neighborhood). Since $\mathrm{Supp}(v)$ is quasi-compact, it is covered by a finite number of the V_x.

Numbering those x gives the desired charts and measurable subsets. Of course, in practice one usually tries to take the V_i disjoint, or at least such that the $V_i \cap V_j$ have measure zero for $i \neq j$.

5.5.8 Proposition. *The integral of v as defined in Construction 5.5.7 does not depend on the choice of the charts ϕ_i and the sets V_i.*

Proof. Suppose we have two sets of data: m, m', ϕ_i, ϕ'_j, etc. Then we construct two new sets of data as follows: $m'' := mm'$, $U_{i,j} := \phi_i^{-1}(\phi_i U_i \cap \phi'_j U'_j)$, $\phi_{i,j} := \phi_i|_{U_{i,j}}$, $U'_{i,j} := (\phi'_i)^{-1}(\phi_i U_i \cap \phi'_j U'_j)$, $\phi'_{i,j} := \phi'_j|_{U'_{i,j}}$, $V_{i,j} := \phi_i^{-1}(\phi_i V_i \cap \phi'_j V'_j)$, and $V'_{i,j} := (\phi'_i)^{-1}(\phi_i V_i \cap \phi'_j V'_j)$. Let us first argue that these new two sets of data give the same integral. For that purpose, consider a pair (i,j). Let $U := U_{i,j}$ and $U' := U'_{i,j}$. Then $f := \phi'^{-1}_j \phi_i$ defines an isomorphism from U to U', such that $V := V_{i,j}$ has image $V' := V'_{i,j}$. Let us write $\phi_i^* v = v|dx_1 \cdots dx_n|$ and $\phi'^*_j v = v'|dx'_1 \cdots dx'_n|$. The partial derivatives $\partial/\partial x_i$, $1 \leq i \leq n$ form a basis of T_U, and likewise for $\partial/\partial x'_j$ for $\mathrm{T}_{U'}$. Written in this basis, the tangent map T_f is given by the matrix whose (i,j)th coefficient is $\partial f_j/\partial x_i$, where f is written $(f_1, \ldots, f_n)$. Note that $|dx_1 \cdots dx_n|$ is a basis for Vol_U, and that $|dx'_1 \cdots dx'_n|$ is one for $\mathrm{Vol}_{U'}$. Let u be in U and put $u' := f(u)$. Then $\mathrm{T}_f(u)$ induces an isomorphism from $\mathrm{Vol}_U(u)$ to $\mathrm{Vol}_{U'}(u')$. Using the defitions of $\mathrm{Vol}_U(u)$ and $\mathrm{Vol}_{U'}(u')$, one sees that under this isomorphism $|dx'_1 \cdots dx'_n|$ is mapped to $|\det(\mathrm{T}_f(u))|\,|dx_1 \cdots dx_n|$. By construction, this isomorphism sends $v'(u')|dx'_1 \cdots dx'_n|$ to $v(u)|dx_1 \cdots dx_n|$, hence we get $v'(u')|\det(\mathrm{T}_f(u))| = v(u)$. The "change of variables formula" from vector calculus says:

$$\int_{V'} v' = \int_{V'} v'|dx'_1 \cdots dx'_n| = \int_V (v' \circ f)|df_1 \cdots df_n| = \int_V (v' \circ f)|\det(\mathrm{T}_f)|.$$

So, from what we have just seen, it follows that $\int_{V'} v' = \int_V v$. This equality is also valid for $V := V_{(i_1,j_1),\ldots,(i_r,j_r)}$ and $V' := V'_{(i_1,j_1),\ldots,(i_r,j_r)}$. That means that indeed our two new sets of data for integration give the same result.

It remains now to be shown that two sets of data, one of which is a refinement of the other, give the same integral. In order to see this, let us reconsider what happens for just one set of data m, ϕ_i and V_i. Considering all possible intersections of the V_i and their complements gives us 2^m subsets that partition X. All these subsets are contained in some V_i, except one: the complement of the union of the V_i. Note that on this last set v is zero. So on each of our subsets we can integrate v, and the sum of these integrals equals the integral of v over X relative to the set of data m, ϕ_i, V_i. (To see that, use that $f \mapsto \int_{\mathbb{R}^n} f$ is additive.) Of course, what we are doing here is the standard game with the boolean algebra generated by the characteristic functions of the V_i. Now suppose that we have a refinement m', ϕ'_j, V'_j. The $2^{m'}$ subsets of X obtained from the V'_j give a partition of X that refines the partition obtained from the V_i. Then our claim is clear. □

Now that we know how to integrate volume forms (i.e., sections of Vol_X), let us discuss the relation between volume forms and differential forms of top degree, i.e., sections of $\Omega_X^{\dim_X}$.

5.5.9 Definition. *Let X be a manifold and L a rank one vector bundle on X. An orientation of L is a collection of orientations of all $L(x)$, $x \in X$, which is locally constant. If X is C^k with $k \geq 1$, then an orientation of X is an orientation of $\Omega_X^{\dim_X}$.*

Some examples. The trivial line bundle $\mathbb{R} \times X$ has a standard orientation. The same is true for Vol_X, but not always for $\Omega_X^{\dim_X}$. If ω is a global section of $\Omega_X^{\dim_X}$ such that $\omega(x) \neq 0$ for all x in X, then ω is a basis for $\Omega_X^{\dim_X}$, hence gives an isomorphism from $\mathbb{R} \times X$ to $\Omega_X^{\dim_X}$, hence gives an orientation on $\Omega_X^{\dim_X}$.

5.5.10 Remark. It is not true in general that all orientations on a line bunlde L come from a trivialization of it (example: take X to be two copies of $\mathbb{R}$, glued via the identity along $\mathbb{R} - \{0\}$, and take an ugly line bundle). If X is paracompact, then all orientations come indeed from trivializations. □

5.5.11 Proposition. *Let X be a C^k-manifold with $k \geq 1$. An orientation on X induces a unique isomorphism from Vol_X to $\Omega_X^{\dim_X}$, such that, at each x in X, it coincides with the map from $\Omega_X^{\dim_X}(x)$ to $\mathrm{Vol}_X(x)$ that sends l to $|l|$. Conversely, an isomorphism from Vol_X to $\Omega_X^{\dim_X}$ with this property induces an orientation on X and the two constructions are inverses.*

Proof. Let x be in X, $V := \mathrm{T}_X(x)$, $n := \dim(V)$, $L := \Omega_X^{\dim_X}(x) = (\Lambda^n(V))^\vee$, $L' := \mathrm{Vol}_X(x) = |\Lambda^n(V)|^\vee$. Recall that we have the map $|\cdot| \colon L \to L'$ that sends l to $|l|$. This map is, of course, not linear. We get a linear map as follows: choose l in L^+ non-zero and send λl, for λin $\mathbb{R}$, to $\lambda|l|$. Check that this map does not depend on the choice of l, that it coincides with $|\cdot|$ on L^+ and that it is the only linear map with that property. □

It is now clear that for X an oriented C^k-manifold we can integrate sections of $\Omega_X^{\dim_X}$ that have quasi-compact support, by using the isomorphism corresponding to the orientation to transform these sections in volume forms. The procedure to integrate a differential form of top degree ω with quasi-compact support is then the same as in Construction 5.5.7, except that one should take charts that are compatible with the orientation on X and the standard orientation on $\mathbb{R}^n$. The standard orientation on $\mathbb{R}^n$ is the one such that $dx_1 \cdots dx_n$ (in this order!) is positive. Of course, if W is an $\mathbb{R}$-vector space, then we can also integrate sections with quasi-compact support of $W \otimes_{\mathbb{R}} \Omega_X^{\dim_X}$.

To finish this section, let us define a canonical volume form on a Riemannian manifold. So suppose that X is a C^k-manifold, with $k \geq 1$, and that $\langle \cdot, \cdot \rangle$ is a metric on T_X. Let x be in X. Suppose that $v_1, \ldots, v_n$ is an orthonormal basis of $\mathrm{T}_X(x)$. Let $v(x)$ be the volume form on $\mathrm{T}_X(x)$ such that $(v(x))(v_1, \ldots, v_n) = 1$. One checks that this does not depend on the orthonormal basis chosen. Hence it defines a volume form v on X.

To give some example where one uses this, note that a submanifold of a Riemannian manifold inherits the structure of Riemannian manifold. For example, consider the group $\mathrm{SO}_3(\mathbb{R})$ as a compact submanifold of $\mathbb{R}^9$ with its standard Riemannian metric. Then one can ask: what is the volume of $\mathrm{SO}_3(\mathbb{R})$?

5.6 Pullback of vector bundles and of differential forms

Let us first discuss pullback of vector bundles. Let $f \colon X \to Y$ be a morphism of manifolds, and let E be a vector bundle on Y. Then we define a vector bundle f^*E on X by: $(f^*E)(x) :=$

$E(f(x))$ for all x in X, and the local trivializations of f^*E are induced by those of E. For $U \subset Y$ open, and s in $E(U)$, we get an element f^*s of $(f^*E)(f^{-1}U)$, defined by: $(f^*s)(x) := s(f(x))$ for all x in $f^{-1}U$. (Note the special case $E = \mathbb{R} \times Y$, where s is just a function and $f^*s = s \circ f$.) The sheaf of sections of f^*E is the tensor product $C_X \otimes_{f^{-1}C_Y} f^{-1}E$, where f^{-1} is pullback of sheaves. For an arbitrary C_Y-module $\mathcal{M}$, its pullback as a module is defined as $f^*\mathcal{M} := C_X \otimes_{f^{-1}C_Y} f^{-1}\mathcal{M}$. Hence on the side of locally free sheaves of modules this operation corresponds to the pullback of vector bundles.

Suppose now moreover that F is a vector bundle on X, and that $g\colon F \to E$ is a morphism of vector bundles (see Definition 4.2.8). We claim that such a g corresponds naturally to a morphism from F to f^*E of vector bundles on X. The proof is trivial, because, for all x in X, $(f^*E)(x) = E(f(x))$.

In particular, the morphism T_f from T_X to T_Y corresponds to a morphism, also written T_f, from T_X to $f^*\mathrm{T}_Y$. Dualizing gives us a morphism $f^*\colon f^*\Omega^1_Y \to \Omega^1_X$. Doing our tensor operations gives us $f^*\colon f^*\Omega_Y \to \Omega_X$. One easily verifies that this f^* is a morphism of sheaves of graded algebras, and that for $U \subset Y$ open and ω in $\Omega_Y(U)$ one has $f^*d\omega = d(f^*\omega)$ (it suffices to prove this for forms of degree one; one has: $(f^*dg)\partial = (dg)(\mathrm{T}_f\partial) = (dg)(\partial \circ f^*) = (\partial \circ f^*)g = \partial(g \circ f) = (d(f^*g))\partial$). Suppose now that for all x in X one has $\dim_X(x) = \dim_Y(f(x))$. Then one has $f^*\colon f^*\mathrm{Vol}_Y \to \mathrm{Vol}_X$ and $f^*\colon \Omega_Y^{\dim_Y} \to \Omega_X^{\dim_X}$. If f is an isomorphism from X to an open subset U of Y, and v is in $\mathrm{Vol}_Y(Y)$ with quasi-compact support contained in U, then one has $\int_Y v = \int_X f^*v$.

5.7 Some exercises

Let G be a Lie group, i.e., G is a C^k-manifold, with $k \geq \infty$, with a C^k-group structure. Let e be its unit element. We consider the following group actions. The (left) action of G on itself by left translations: for x in G we have $l_x\colon G \to G$ sending y to xy. The (right) action of G on itself by right translations: $r_x\colon y \mapsto yx$. The action of G on itself by conjugation: $c_x\colon y \mapsto xyx^{-1}$. The action of $G \times G$ on G by translations on both sides: $b_{x,y}\colon z \mapsto xzy^{-1}$. Let l denote the morphism of groups from G to $\mathrm{Aut}_{\mathbf{Man}}(G)$ given by the action by left translations. Similarly, we have the anti-morphism r from G to $\mathrm{Aut}_{\mathbf{Man}}(G)$ given by the right translations, the morphism b from $G \times G$ to $\mathrm{Aut}_{\mathbf{Man}}(G)$ and the morphism c from G to the group $\mathrm{Aut}_{\mathrm{Lie}}(G)$ of automorphims of G as Lie group.

For every x in G we have the two isomorphisms $\mathrm{T}_{l_x}(e)$ and $\mathrm{T}_{r_x}(e)$ from $L := \mathrm{Lie}(G) := \mathrm{T}_G(e)$ to $\mathrm{T}_G(x)$. These two isomorphisms need not be the same. Show that in fact $\mathrm{T}_{r_x}(e)^{-1}\mathrm{T}_{l_x}(e)$ is the automorphism $\mathrm{T}_{c_x}(e)$ of L. (By the way, L is called the Lie algebra of G; we will discuss the structure of Lie algebra on L a bit further.) Show that $x \mapsto \mathrm{T}_{c_x}(e)$ defines an action of G on L by linear maps; this action is called the adjoint representation of G. Show that both l and r define isomorphisms, still denoted l and r, from the trivial vector bundle $L \times G$ to T_G. In particular, all Lie groups have a trivial tangent bundle, and are orientable.

Let us consider the vector space $\mathrm{T}_G(G)$ of vector fields on G. The group G acts on it via l, r and c; $G \times G$ acts via b. A vector field ∂ on G is called left-invariant if it is invariant for the action given by l; similarly, it is called right-invariant if it is invariant under r, and bi-invariant if invariant under b. Explicitly: ∂ is left-invariant if and only if for all x in G one has $\partial(x) = \mathrm{T}_{l_x}(e)\partial(e)$. Show that $\partial \mapsto \partial(e)$ gives an isomorphism from the vector space of left-invariant vector fields on G to L. Show the same with left replaced by right. Show that the space of bi-invariant forms is isomorphic in this way to the subspace of L on which G acts trivially via its adjoint representation.

Before we define the Lie algebra structure on L we need a general result on derivations. Let k be a field and A a k-algebra. Then we have the A-module of k-derivations $\mathrm{Der}_k(A) := \mathrm{Der}_k(A, A)$. One verifies immediately that for ∂_1 and ∂_2 in $\mathrm{Der}_k(A)$, the commutator $[\partial_1, \partial_2] := \partial_1\partial_2 - \partial_2\partial_1$ is in $\mathrm{Der}_k(A)$. We apply this to vector fields. Let X be a C^k-manifold (with $k \geq \infty$, remember?), $U \subset X$ open, ∂_1 and ∂_2 vector fields on U. Then we may view ∂_1 and ∂_2 as elements of $\mathrm{Der}_{\mathbb{R}}(C_U^k)$ (recall that we have a canonical isomorphism between T_X and $\mathbf{Der}_{\mathbb{R}}(C_X)$). Hence we get $[\partial_1, \partial_2]$ in $\mathrm{T}_X(U)$. (Assuming that U is an open subset $\mathbb{R}^n$, compute explicitly what this operation looks like.)

We go back to our Lie group G. Show that for ∂_1 and ∂_2 two left-invariant vector fields on G, $[\partial_1, \partial_2]$ is left-invariant too. (Of course the same holds for right-invariance; one should in fact prove a lemma concerning ∂_1 and ∂_2 on a manifold X that are invariant under an automorphism σ of X.) Because the space of left-invariant vector fields is just L (via $\partial \mapsto \partial(e)$), we get a map $[\cdot,\cdot]$ from $L \times L \to L$. This map is called the Lie bracket. Show that it is bilinear, alternating and that it satisfies Jacobi's identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

A vector space L with such an operation is called a Lie algebra. We will compute the Lie algebras of the Lie groups that we have seen in §1. We start with the group $G := \mathrm{GL}_n(\mathbb{R})$ (some $n \geq 0$). Since G is an open subset of $\mathrm{M}_n(\mathbb{R})$, we identify the $\mathrm{T}_G(x)$ with $\mathrm{M}_n(\mathbb{R})$. Let a be in $L = \mathrm{T}_e(G) = \mathrm{M}_n(\mathbb{R})$. We wish to describe explicitly the left-invariant vector field ∂_a on G such that $\partial_a(e) = a$. Verify that for g in G we have $\partial_a(g) = ga$. Now we compute $\partial_a x_{i,j}$, where the $x_{i,j}$ are the coordinate functions on $\mathrm{M}_n(\mathbb{R})$. For g in G, $(\partial_a x_{i,j})(g)$ is by definition the derivative of $x_{i,j}$ at g in the direction given by $\partial_a(g)$, i.e., in the direction ga. So we compute:

$$x_{i,j}(g + \varepsilon ga) = (g + \varepsilon ga)_{i,j} = g_{i,j} + \varepsilon(ga)_{i,j} = x_{i,j}(g) + \varepsilon \sum_k g_{i,k}a_{k,j}.$$

It follows that $\partial_a(x_{i,j}) = \sum_k x_{i,k}a_{k,j}$. Applying this formula twice gives:

$$(\partial_a\partial_b)x_{i,j} = \partial_a(\sum_k x_{i,k}b_{k,j}) = \sum_{k,k'} x_{i,k'}a_{k',k}b_{k,j} = \sum_{k'} x_{i,k'}(ab)_{k',j} = \partial_{ab}x_{i,j}.$$

From this we get:

$$[\partial_a, \partial_b]x_{i,j} = \partial_{[a,b]}x_{i,j}, \text{ and } [\partial_a, \partial_b] = \partial_{[a,b]},$$

since the $x_{i,j}$ are linearly independent over $\mathbb{R}$. So the Lie bracket for $\mathrm{GL}_n(\mathbb{R})$ is just the ordinary commutator of matrices. The reader should check that if we had used right-invariant vector fields to define the Lie bracket, we would have found the opposite result (use that $x \mapsto x^{-1}$

induces multiplication by -1 on L). Let us now reconsider the subgroups of $\mathrm{GL}_n(\mathbb{R})$ that we considered in §1. From the computations we did there, it follows that $\mathrm{Lie}(\mathrm{SL}_n(\mathbb{R}))$ is the subspace of $\mathrm{M}_n(\mathbb{R})$ consisting of the elements with trace zero, that $\mathrm{Lie}(\mathrm{SO}_n(\mathbb{R})) = \mathrm{M}_n(\mathbb{R})^-$, the space of anti-symmetric matrices, and that $\mathrm{Lie}(\mathrm{Sp}_{2n}(\mathbb{R}))$ is the space of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c^t = c$, $b^t = b$ and $d = -a^t$. From the construction of the Lie bracket it follows that the Lie bracket for any of these subgroups is just the restriction of the one for $\mathrm{M}_n(\mathbb{R})$ in the first two cases and $\mathrm{M}_{2n}(\mathbb{R})$ in the last.

Let us now look at differential forms on Lie groups. Just as for vector fields, we have the notions of left-invariant, right-invariant and bi-invariant elements of Ω^i_G. Of particular importance are bi-invariant differential forms of top degree, since those give us bi-invariant volume forms. Show, by pure thought, that $\mathrm{SO}_n(\mathbb{R})$ has such a non-zero bi-invariant form (use that $\mathrm{SO}_n(\mathbb{R})$ is compact and connected). Show that $\mathrm{O}_2(\mathbb{R})$ has a non-zero bi-invariant volume form, but not a non-zero bi-invariant 1-form. Show that $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{R})$ both have non-zero bi-invariant forms of top degree (use that the commutator subgroup of $\mathrm{SL}_n(\mathbb{R})$ is $\mathrm{SL}_n(\mathbb{R})$ itself).

Of particular fun should be the following exercise. Compute explicitly the bi-invariant volume form v on $G := \mathrm{SO}_3(\mathbb{R})$ for which G has volume one. Compute the distribution g, with respect to v, of the angles of rotation, say in the interval $[0, \pi]$, of the elements of G. More precisely, let f be the function $G \to [-1, 1]$ that sends x to $(\mathrm{tr}(x) - 1)/2$; determine the continuous function g on $]-1, 1[$ such that for every continuous $h\colon [-1, 1] \to \mathbb{R}$ one has:

$$\int_G f^*(h)v = \int_{-1}^{1} gh.$$

In the same way, compute the distribution g_2 for the function f_2 from G to $[-1, 1]$ that sends x to $(\mathrm{tr}(x^2) - 1)/2$. Why is the result so remarkable? (Hint: it might be useful to use the following chart for G; let $U := \{x \in \mathbb{R}^3 \mid 0 < \|x\| < \pi\}$ and let $\psi\colon U \to G$ be the map that sends a to the rotation of angle $|a|$ with (oriented) axis $\mathbb{R}a$.)

6 De Rham cohomology

6.1 Definition. *A differential graded-commutative* $\mathbb{R}$*-algebra is an* $\mathbb{R}$*-algebra* A *with a grading* $A = \oplus_{i\in\mathbb{Z}} A_i$*, which is graded-commutative (i.e., for* x *in* A_i *and* y *in* A_j *one has* $yx = (-1)^{ij}xy$*), and which is equipped with a differential* d *of degree one (i.e., an* $\mathbb{R}$*-linear map* d *from* A *to* A *such that* $d^2 = 0$*,* $d(A_i) \subset A_{i+1}$ *and* $d(xy) = (dx)y + (-1)^i x dy$ *for* x *in* A_i *and* y *in* A*).*

We have a contravariant functor $X \mapsto (\Omega_X(X), d)$ from the category of C^k-manifolds ($k = \infty$ or $k = \omega$) to that of differential graded-commutative $\mathbb{R}$-algebras. Note that the vector spaces $\Omega_X(X)$ tend to be very big (the typical dimension is $|\mathbb{R}|$). Although this functor does transform morphisms of manifolds into $\mathbb{R}$-linear maps, it does not really simplify the study of the category of C^k-manifolds. For example, a compact manifold X can be reconstructed from $\Omega_X^0(X)$ alone (the points of X correspond to the maximal ideals, the topology is the Zariski topology, etc.). But, if one composes the functor $X \mapsto (\Omega_X(X), d)$ with the functor that takes homology of differential graded-commutative algebras, then a miracle happens. This composed functor, called de Rham cohomology, has reasonable finiteness properties, and, most importantly, is homotopy invariant. Before proving that, let us write down the definitions in detail.

6.2 Lemma. *Let* A *be a differential graded-commutative* $\mathbb{R}$*-algebra. Then its homology* $\mathrm{H}(A)$*, defined as* $\ker(d)/\mathrm{im}(d)$*, has an induced structure of graded-commutative* $\mathbb{R}$*-algebra.*

Proof. Details are left to the reader. Show that $\ker(d)$ is a graded-commutative subalgebra of A, in which $\mathrm{im}(d)$ is a homogeneous ideal. □

6.3 Definition. *Let* X *be a* C^k*-manifold with* $k \geq \infty$*. Then the de Rham cohomology of* X *is the graded-commutative* $\mathbb{R}$*-algebra* $\mathrm{H}_{\mathrm{dR}}(X) := \mathrm{H}(\Omega_X(X), d)$*. Hence one has:*

$$\mathrm{H}^i_{\mathrm{dR}}(X) = \frac{\ker(d\colon \Omega_X^i(X) \to \Omega_X^{i+1}(X))}{\mathrm{im}(d\colon \Omega_X^{i-1}(X) \to \Omega_X^i(X))}.$$

As explained above, $\mathrm{H}_{\mathrm{dR}}(\cdot)$ *is a contravariant functor from manifolds to graded-commutative* $\mathbb{R}$*-algebras. For* $f\colon X \to Y$ *we will write* f^* *for the morphism* $\mathrm{H}_{\mathrm{dR}}(f)$ *from* $\mathrm{H}_{\mathrm{dR}}(Y)$ *to* $\mathrm{H}_{\mathrm{dR}}(X)$*.*

6.4 Remark. Be careful with de Rham cohomology defined like this for analytic manifolds, and also for C^∞-manifolds that are not paracompact. In those cases one should consider the hypercohomology $\mathbb{H}_{\mathrm{dR}}(X) := \mathbb{H}(\Omega_X, d)$ of the complex of sheaves Ω_X. Maybe we will have time to discuss this. □

6.5 Theorem. *The de Rham cohomology of* C^k*-manifolds,* $k \in \{\infty, \omega\}$*, is homotopy invariant, i.e., if* f_0 *and* f_1 *from* X *to* Y *are homotopic, then the two maps* f_0^* *and* f_1^* *from* $\mathrm{H}_{\mathrm{dR}}(Y)$ *to* $\mathrm{H}_{\mathrm{dR}}(X)$ *are equal.*

Proof. By definition of homotopy, we have a morphism of C^k-manifolds $F\colon X \times I \to Y$ with I an open interval containing 0 and 1, such that $F|_{X\times\{0\}} = f_0$ and $F|_{X\times\{1\}} = f_1$. (It would be enough to have F a C^1-morphism such that its restrictions to all $X \times i$ with i in I are C^k, but let us not bother.) We will construct a homotopy from 0 to $f_0^* - f_1^*$, i.e., a sequence of $\mathbb{R}$-linear maps $K^i\colon \Omega_Y^i(Y) \to \Omega_X^{i-1}(X)$ such that $f_0^* - f_1^* = dK + Kd$. In order to construct this K, we need to consider $\Omega_{X\times I}$ more closely. Let p_X and p_I denote the projections from $X \times I$ to X and I, respectively. (More generally, one should consider the vector bundle of differential forms on a product of two manifolds.) At a point (x,i) of $X \times I$, the tangent space $\mathrm{T}_{X\times I}(x,i)$ is canonically isomorphic to the direct sum $\mathrm{T}_X(x) \oplus \mathrm{T}_I(i)$ (note that this follows from the functorial property of tangent spaces; use p_X, p_I and the inclusions $i_{X,i}$ and $i_{I,x}$). On the level of vector bundles this gives us:

$$\mathrm{T}_{X\times I} = p_X^*\mathrm{T}_X \oplus p_I^*\mathrm{T}_I.$$

Dualizing this gives:

$$\Omega_{X\times I}^1 = p_X^*\Omega_X^1 \oplus p_I^*\Omega_I^1.$$

We suggest to the reader to prove that for a ring A and A-modules M_1 and M_2 one has

$$\Lambda^i(M_1 \oplus M_2) = \bigoplus_{j+k=i} \Lambda^j(M_1) \otimes_A \Lambda^k(M_2)$$

(in fact, the two sides are naturally isomorphic). For what we want to do it suffices of course to check this for free modules. A nice abstract proof could go like this: show that $\Lambda(M_1 \oplus M_2)$ is naturally isomorphic to the quotient of $\Lambda(M_1) \otimes_A \Lambda(M_2)$ by the ideal generated by the $m_1 \otimes m_2 + m_2 \otimes m_1$, because they have the same universal property; then look at what this means for the $\Lambda^i(M_1 \oplus M_2)$. It follows that we have a natural isomorphism:

$$\Omega_{X\times I}^i = p_X^*\Omega_X^i \oplus \left(p_X^*\Omega_X^{i-1} \otimes p_I^*\Omega_I^1\right).$$

In local coordinates, this means the following. Suppose that $x_1, \ldots, x_n$ are coordinates on $U \subset X$, and let t be the coordinate on I. Then $\Omega_{U\times I}^i$ has the basis:

$$\begin{cases} dx_{j_1}\cdots dx_{j_i}, & 1 \le j_1 < \cdots < j_i \le n, \\ dx_{j_1}\cdots dx_{j_{i-1}}dt, & 1 \le j_1 < \cdots < j_{i-1} \le n. \end{cases}$$

For ω in $\Omega_{X\times I}^i(X \times I)$ we can write uniquely $\omega = \omega_1 + \omega_2\, dt$, with ω_1 in $p_X^*\Omega_X^i(X \times I)$ and ω_2 in $p_X^*\Omega_X^{i-1}(X \times I)$. This decomposition of the $\Omega_{X\times I}^i$ also induces a decomposition of the differential $d_{X\times I}$ on $\Omega_{X\times I}$: we have, uniquely, $d_{X\times I} = d_X + d_I$, with d_X increasing the degree with respect to X by one, and d_I the same for I. (For an arbitrary product, this displays the complex $\Omega_{X\times Y}$ as the total complex associated to the double complex $p_X^*\Omega_X \otimes p_Y^*\Omega_Y$.)

We can now define our homotopy operators K_1^i from $\Omega_{X\times I}^i(X\times I)$ to $\Omega_X^{i-1}(X)$; the operator K we want to have will be $K_1 \circ F^*$. With the notation as above ($\omega = \omega_1 + \omega_2$) we define:

$$K_1^i\omega := (-1)^i \int_0^1 \omega_2\, dt. \tag{6.5.1}$$

Since the object to be integrated does not look like a function that one usually integrates, let us write it a bit more explicit. Let x be in X. Then $(K_1^i\omega)(x) = (-1)^i \int_0^1 \omega_2(x,t)\, dt$, and

$\omega_2(x,t)$ is an element of the vector space $\Omega_X^{i-1}(x)$ that does not depend on t. This means that the integral is exactly of the kind we considered in the previous section: we integrate a vector space valued volume form (namely: $\omega_2\,|dt|$) over a compact subset $[0,1]$ of the manifold I. Let i_0 and i_1 denote the inclusions of X into $X\times I$ that send x to $(x,0)$ and $(x,1)$, respectively. Then we claim that for all ω in $\Omega^i_{X\times I}(X\times I)$ we have:

$$(dK_1^i + K_1^{i+1}d)\omega = i_0^*\omega - i_1^*\omega. \tag{6.5.2}$$

To prove this identity, note that it is a local problem on X, and that both sides are additive in ω. Hence we may assume that $x_1,\ldots,x_n$ are local coordinates on X and that ω is of the form $f dx_1\cdots dx_i$ or $g dx_1\cdots dx_{i-1}dt$ with f and g in $C_{X\times I}(X\times I)$. Let us first consider the case $\omega = f dx_1\cdots dx_i$. Then $K_1^i\omega = 0$ because $\omega_2 = 0$. On the other hand,

$$d\omega = df dx_1\cdots dx_i = (d_X f)dx_1\cdots dx_i + (d_I f)dx_1\cdots dx_i.$$

It follows that:

$$K_1 d\omega = (-1)^{i+i+1}K_1((\partial f/\partial t)dx_1\cdots dx_i dt) = -\left(\int_0^1 (\partial f/\partial t)dt\right) dx_1\cdots dx_i = i_0^*\omega - i_1^*\omega.$$

One should note that the last identity is exactly the fundamental theorem of calculus. Let us now consider the second case: $\omega = g dx_1\cdots dx_{i-1}dt$. Now we have: $\omega=\omega_2$, hence:

$$K_1^i\omega = (-1)^i\left(\int_0^1 g\,dt\right) dx_1\cdots dx_{i-1} = (-1)^i G dx_1\cdots dx_{i-1},$$

where G is the function on X defined by $G(x) = \int_0^1 g(x,t)\,dt$. It follows that:

$$dK_1^i\omega = (-1)^i dG dx_1\cdots dx_{i-1}.$$

On the other hand, $d\omega = (d_X g)dx_1\cdots dx_{i-1}dt$.

$$(-1)^{i+1}K_1^{i+1}d\omega = \left(\int_0^1 (d_X g)\,dt\right) dx_1\cdots dx_{i-1} = d\left(\int_0^1 g\,dt\right) dx_1\cdots dx_{i-1} = (dG)dx_1\cdots dx_{i-1},$$

where the middle equality is the theorem in calculus that says that the derivative with respect to parameters of an integral is the integral of the derivative. So we find that $(dK_1 + K_1 d)\omega = 0$. This is just what we need, since $i_0^*(dt) = d(i_0^*t) = d(0) = 0$, and also $i_1^*(dt) = 0$.

To finish the proof of the theorem, define $K^i := K_1^i\circ F^*$ and note that, for ω in $\Omega^i_Y(Y)$, $i_0^*F^*\omega = f_0^*\omega$ and $i_1^*F^*\omega = f_1^*\omega$. □

We can now compute the de Rham cohomology for some manifolds. The empty manifold gives the zero ring, and a one point manifold gives the $\mathbb{R}$-algebra $\mathbb{R}$ itself. Now suppose that X is a contractible C^k-manifold (i.e., there is a point x in X such that the constant map $f\colon X\to X$ that sends every element of X to x is homotopic to the identity morphism id_X of X. Then f^* induces the identity endomorphism of $\mathrm{H}_{\mathrm{dR}}(X)$. But we can write $f = i_x\circ p$, with i_x the inclusion of $\{x\}$ in X and p the unique map $X\to\{x\}$. But then we see that i_x^* and p^* are inverses, hence $\mathrm{H}_{\mathrm{dR}}(X) = \mathbb{R}$.

7 Comparison between de Rham and sheaf cohomology

7.1 Proposition. *Let X be a C^k-manifold, with $k \geq \infty$. Then the de Rham complex of sheaves (Ω_X, d) is a resolution of the constant sheaf $\mathbb{R}_X$.*

This proposition is usually called Poincaré's lemma.

Proof. It suffices to show that for each x in X, the complex of $\mathbb{R}$-vector spaces $(\Omega_{X,x}, d)$ is a resolution of $\mathbb{R}$. Note that every x in X has a cofinal system of contractible open neighborhoods. Hence the result. □

7.2 Definition. *A topological space X is called quasi-paracompact if every open covering of it has a locally finite refinement (i.e., every x in X has a neighborhood meeting only finitely many elements of the refinement). A topological space is called paracompact if it is separated and quasi-paracompact.*

In Appendix A of Spivak's book (Volume one) it is shown that, for X a separated manifold, paracompactness is equivalent to X being metrizable, and also to every connected component of X having a countable basis for its topology, and also to every connected component being a countable union of compact subsets. For example, every separated manifold that has a countable atlas is paracompact. Spivak also gives examples of separated non-paracompact manifolds.

Our next step will be to establish that for X a paracompact C^∞-manifold, the sheaves Ω_X^i are acyclic for the functor that takes global sections, as well as for several other interesting functors. We will prove something more general. Note that even for X paracompact, the sheaves Ω_X^i are not, in general, flabby; it suffices to take $X := \mathbb{R}$ and to see that the function $x \mapsto x^{-1}$ on $\mathbb{R} - \{0\}$ is not the restriction of a C^∞-function on $\mathbb{R}$. We will show that, for X paracompact, all C_X-modules are "soft", and hence acyclic for the global section functor. A good reference for soft sheaves is Godement's book "Théorie des faisceaux". Or also Iversen's book "Cohomology of sheaves". Or the book "Sheaf theory" by Bredon. Most of the text below is taken more or less from Godement's book.

7.3 Definition. *Let X be a topological space. A sheaf of sets $\mathcal{F}$ on X is called soft ("mou" in french) if for every closed subset Y of X the map $\mathcal{F}(X) \to (i_Y^{-1}\mathcal{F})(Y)$ is surjective (i_Y is of course the inclusion map).*

In order to simplify the notation in what follows, we will denote, sometimes, for Y any subset of X, with its induced topology, the set $(i_Y^{-1}\mathcal{F})(Y)$ just by $\mathcal{F}(Y)$. This set will be called the set of sections of $\mathcal{F}$ over Y. Note that for Y open in X this coincides with the $\mathcal{F}(Y)$ we already had. For $Y \subset X$ an arbitrary subset, we have the following "explicit" description of $\mathcal{F}(Y)$: it is the set of maps:

$$s: Y \to \coprod_{y \in Y} \mathcal{F}_y,$$

such that, for all y in Y, $s(y)$ is in $\mathcal{F}_y$, and such that every y in Y has an open neighborhood U in X such that the restriction of s to $Y \cap U$ is induced by an element of $\mathcal{F}(U)$.

7.4 Lemma. *Let X be a quasi-paracompact topological space, $\mathcal{F}$ a sheaf of sets on X and $Y \subset X$ a closed subset. Let s be in $(i_Y^{-1}\mathcal{F})(Y)$. There exists an open subset U of X containing Y, and an element t of $\mathcal{F}(U)$, such that $s = i_Y^{-1}t$. As a consequence of this, every flabby sheaf on X is soft, and, consequently, every injective sheaf of abelian groups is soft.*

Proof. By construction, for every y in Y we have $(i_Y^{-1}\mathcal{F})_y = \mathcal{F}_y$. For every y in Y, choose an open neighborhood U_y of y in X and an element t^y in $\mathcal{F}(U_y)$ such that $t^y_z = s_z$ for all z in $U_y \cap Y$. These U_y, together with $X - Y$, form an open cover of X. Hence we get a locally finite open cover V_i, i in some set I, and $X - Y$, such that V_i is contained in some U_{y_i}. Let t_i be the restriction of t_{y_i} to V_i. Let now y be in Y. Let W_y be an open neighborhood of y, contained in the union of the V_i, meeting only finitely many of the V_i, say $V_{i_1}, \ldots, V_{i_r}$. Let U_y be the open subset of W_y consisting of the z such that the $t_{i_1,z} = \cdots = t_{i_r,z}$ (here we used the local finiteness; a finite intersection of opens is open). We have a unique t_y in $\mathcal{F}(U_y)$ that equals the restriction of the t_{i_j}. By construction, the t_y are compatible. Hence we take $U := \cup_y U_y$ and have a unique t in $\mathcal{F}(U)$ with $t|_{U_y} = t_y$ for all y. It is clear from the construction that $s = i_Y^{-1}t$. □

7.5 Lemma. (Shrinking lemma.) *Every paracompact topological space is normal (i.e., for Y and Z disjoint closed subsets of X there exist disjoint open subsets U and V with $U \supset Y$ and $V \supset Z$). For a locally finite open covering $U: I \to \mathrm{Open}(X)$ of a normal topological space X there exists an open covering $V: I \to \mathrm{Open}(X)$ such that for every i in I one has $\overline{V_i} \subset U_i$.*

Proof. Let X be paracompact, and Y and Z disjoint closed subsets of X. We want to show that Y and Z can be separated by opens. Let y be in Y. Since X is separated, we have, for every z in Z, an open neighborhood U_z of z in X such that y is not in $\overline{U_z}$. The U_z, together with $X - Z$, form an open cover of X. Hence we have open subsets U_i of X, i in some set I, with each U_i contained in some U_z, such that the U_i together with $X - Z$ form a locally finite open cover of X. Then $\cup_i \overline{U_i}$ is closed in X (because locally it is a finite union), hence $X - \cup_i \overline{U_i}$ is an open neighborhood of y, disjoint from the open neighborhood $\cup_i U_i$ of Z.

For every y in Y, let U_y be an open neighborhood of y in X such that $\overline{U_y} \cap Z$ is empty. The U_y, together with $X - Y$, form an open cover of X. Hence we have open subsets U_i, i in some set I, refining the U_y, such that the U_i together with $X - Y$ form a locally finite open cover of X. Define $U := \cup_i U_i$. Then $\overline{U} = \cup_i \overline{U_i}$ is contained in $\cup_y \overline{U_y}$, hence $\overline{U}$ does not meet Z. This shows that X is normal. If the set I is countable, we can just repeat this argument to prove the existence of V. In general, we use Zorn's Lemma.

Let us now show the second statement of the lemma. Let us first show that one can always shrink one of the U_i. More precisely, let i be in I. We will show that there exists V_i open in X such that $\overline{V_i} \subset U_i$ and such that V_i together with the U_j with $j \neq i$ cover X. Let Y_i be the union of the complements of all the U_j, $j \neq i$. Then Y_i is a closed subset of X, contained in U_i. Let Z_i be the complement of U_i. Then Y_i and Z_i are disjoint closed subsets of X. Take V_i an open subset of X that contains Y_i and that does not meet some open subset containing Z_i

(recall that we have already proved that X is normal). Then $\overline{V_i}$ is contained in U_i and together with the U_j, $j \neq i$, it covers X.

Consider pairs (V, J), with $J \subset I$ and $V: J \to \mathrm{Open}(X)$ such that for all j in J one has $\overline{V_j} \subset U_j$ and such that the V_j, $j \in J$ together with the U_i, $i \notin J$, cover X. Define a partial order on the set of such pairs as follows: $(V, J) \leq (V', J')$ means that $J \subset J'$ and that for all j in J one has $V_j = V'_j$. We want to apply Zorn's lemma to this partially ordered set, call it S. First of all, S is not empty, because it has a unique element with $J = \emptyset$. Secondly, we must show that each totally ordered subset of S has an upper bound. So suppose that $A \subset S$ is a totally ordered subset. Write the element a of A as (V_a, J_a). Take J to be the union of all the J_a, and define, for each j in J, the open subset V_j to be $(V_a)_j$, for any a such that j is in J_a. It follows from the local finiteness of the cover $(U_i)_{i \in I}$ that (V, J) is in S. By Zorn's Lemma, we have a maximal element (V, J) of E. Suppose that $J \neq I$. Take i in $I - J$. The fact that one can always shrink one element of an open cover shows that there exists a V_i open in X such that $\overline{V_i} \subset U_i$ and such that V_i, together with the V_j for j in J and the U_j with j in $I - J - \{i\}$ cover X. This shows that (V, J) is not maximal, which is a contradiction. □

7.6 Lemma. *Let X be a paracompact topological space. Let $\mathcal{F}$ be a sheaf of sets on X. Suppose that X can be covered by open subsets U with the property that for every closed subset Y contained in U the map from $\mathcal{F}(U)$ to $\mathcal{F}(Y)$ is surjective. Then $\mathcal{F}$ is soft.*

Proof. Let $Y \subset X$ be a closed subset and let s be in $\mathcal{F}(Y)$. Let U_i, $i \in I$, be an open cover such that all U_i have the property mentioned above. Since that property is stable under taking smaller open subsets, we may suppose that the U_i form a locally finite open cover. Let V_i, $i \in I$, be an open cover such that for all i one has $C_i := \overline{V_i} \subset U_i$. For $J \subset I$, let C_J be the union of the C_j for $j \in J$. Consider the set S consisting of pairs (t, J) with $J \subset I$ and t in $\mathcal{F}(C_J)$, such that t and s are equal on $Y \cap C_J$. Clearly, the set S is not empty, because it has a unique element of the form $(t, \emptyset)$. Define a partial order on S as follows: $(t, J) \leq (t', J')$ if and only if $J \subset J'$ and t is the restriction of t' to C_J. One checks that every chain in S has an upper bound. Zorn's Lemma gives us a maximal element (t, J). One checks that $J = I$. □

7.7 Lemma. *Let X be a paracompact topological space. Let $\mathcal{O}_X$ be a sheaf of (not necessarily commutative) rings on X. If $\mathcal{O}_X$ is soft, then every $\mathcal{O}_X$-module is soft. The sheaf $\mathcal{O}_X$ is soft if and only if every x in X has a neighborhood U such that for all pairs of disjoint closed subsets Y and Z of X contained in U, there exists f in $\mathcal{O}_X(U)$ with $f|_Y = 1$ and $f|_Z = 0$.*

Proof. Suppose that $\mathcal{O}_X$ is soft. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module, $Y \subset X$ closed and s in $\mathcal{M}(Y)$. By Lemma 7.4, there is an open subset U of X with $Y \subset U$, and t in $\mathcal{M}(U)$ such that $t|_Y = s$. Take V open in X with $Y \subset V$ and $\overline{V} \subset U$. Let $Z := X - V$. Then Y and Z are disjoint closed subsets of X, hence $\mathcal{O}_X(Y \cup Z) = \mathcal{O}_X(Y) \times \mathcal{O}_X(Z)$. Let f be in $\mathcal{O}_X(X)$ such that its restrictions to Y and Z are 1 and 0, respectively. Let $t' := ft$. Then t' is in $\mathcal{M}(U)$, its restriction to Y is s and its support is contained in $\overline{V}$, which is closed in X. It follows that t' can be extended by zero outside of U. The second statement to prove is a direct consequence of the argument above together with Lemma 7.6. □

7.8 Theorem. *Let X be a paracompact C^∞-manifold. Then all C_X-modules are soft.*

Proof. By Lemma 7.7, it suffices to show that every x has an open neighborhood U such that for all pairs of disjoint closed subsets Y and Z of X contained in U, there exists f in $C_X(U)$ with $f = 1$ in a neighborhood of Y and $f = 0$ in a neighborhood of Z. Let x be in X. Let $\phi: V \to X$ be a chart such that x is in ϕV, say with V an open subset of $\mathbb{R}^n$. Take U to be the image under ϕ of an open ball B with $\overline{B} \subset V$ and $\phi^{-1}(x) \in B$. Let Y and Z be disjoint closed subsets of X that are contained in U. Then $A := \phi^{-1}Y$ and $C := \phi^{-1}Z$ are disjoint closed subsets of V, contained in B, hence in $\overline{B}$. It follows that A and C are compact. We will now first produce a continuous function $f: \mathbb{R}^n \to \mathbb{R}$ that is ≥ 1 in a neighborhood of A and ≤ 0 in a neighborhood of C. For x in $\mathbb{R}^n$, let $d_A(x)$ and $d_C(x)$ be the distances of x to A and B, respectively. We take:

$$f(x) := \frac{3d_C(x)}{d_C(x) + d_A(x)} - 1.$$

(Note that this function still works for disjoint closed subsets that are not necessarily compact.) Let g be any continuous function from $\mathbb{R}$ to $\mathbb{R}$ such that $g = 0$ in a neighborhood of $]infty, 0]$, and $g = 1$ in a neighborhood of $[1, \infty[$. Then $f_1 := g \circ f$ is continuous, is 1 in a neighborhood of A and 0 in a neighborhood of C. The last step is to "smoothe" f_1 by convoluting it with a suitable C^∞-function. Let $r > 0$ be a real number such that $f_1(x) = 1$ for all x with $d_A(x) \leq r$ and such that $f_1(x) = 0$ for all x with $d_C(x) \leq r$. Let $h: \mathbb{R}^n \to \mathbb{R}$ be a C^∞-function with support contained in the ball with radius r centered at the origin, and with $\int_{\mathbb{R}^n} h = 1$. Then define $f_2: \mathbb{R}^n \to \mathbb{R}$ by:

$$f_2(x) := \int_{\mathbb{R}^n} f_1(x - y)h(y)\, dy.$$

□

Let us now prove some properties of soft sheaves on paracompact topological spaces. First of all, they have "partitions of unity", in the following sense.

7.9 Definition. *Let X be a topological space, and $\mathcal{F}$ a sheaf of abelian groups on it. Let s be in $\mathcal{F}(X)$ and let $U: I \to$ Open(X), be a cover of X. A partition of s, subject to the cover U, is a family $s: I \to \mathcal{F}(X)$ of sections of $\mathcal{F}$, such that for all i one has* Supp$(s_i) \subset U_i$*, such that every x in X has a neighborhood that meets only finitely many of the* Supp(s_i)*, and such that $s = \sum_{i \in I} s_i$ (locally this sum is finite, hence defined).*

7.10 Proposition. *Let $\mathcal{F}$ be a soft sheaf of abelian groups on a paracompact topological space X. For every cover $U: I \to$ Open(X) and every s in $\mathcal{F}(X)$ there exists a partition of s subject to U.*

Proof. The cover U has a locally finite refinement, hence we may suppose that U is locally finite. Take, using Lemma 7.5, in every U_i, a subset C_i that is closed in X, such that the C_i still cover X. For each subset J of I let C_J be the union of the C_j with j in J. Let S be the set of (t, J) with $J \subset I$ and $t: J \to \mathcal{F}(X)$ such that for all j in J one has Supp$(t_j) \subset C_j$ and such that, in $\mathcal{F}(C_J)$, one has $s = \sum_j s_j$. Applying the usual Zorn type argument finishes the proof. □

7.11 Lemma. *Let X be a paracompact topological space, and*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

a short exact sequence of sheaves of abelian groups on X. Suppose that $\mathcal{F}'$ is soft. Then the sequence:

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0$$

is exact.

Proof. Let s'' be in $\mathcal{F}''(X)$. Locally on X, s'' is the image of a section of $\mathcal{F}$, hence there exists a locally finite cover $U: I \to \mathrm{Open}(X)$ of X such that for each i we have s_i in $\mathcal{F}(U_i)$ mapping to $s|_{U_i}$. For each i, let V_i be an open subset of U_i such that the V_i cover X and that $C_i := \overline{V_i} \subset U_i$ (use Lemma 7.5). For $J \subset I$, put $C_J := \cup_{j \in J} C_j$. Let S be the set of pairs (s, J) with $J \subset I$ and s in $\mathcal{F}(C_J)$ such that s maps to $s''|_{C_J}$. Again, the usual Zorn type argument finishes the proof. □

7.12 Lemma. *Let X be a paracompact topological space, and*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

an exact sequence of sheaves of abelian groups on X. Suppose that $\mathcal{F}'$ and $\mathcal{F}$ are soft. Then $\mathcal{F}''$ is soft.

Proof. Let Y be a closed subset of X. Since i_Y^{-1} is exact, we have the exact sequence of abelian sheaves on Y:

$$0 \to i_Y^{-1}\mathcal{F}' \to i_Y^{-1}\mathcal{F} \to i_Y^{-1}\mathcal{F}'' \to 0.$$

By definition, the first two of these sheaves are soft. Hence, by Lemma 7.11, the map from $\mathcal{F}(Y)$ to $\mathcal{F}''(Y)$ is surjective. Since $\mathcal{F}$ is soft, the map $\mathcal{F}(X) \to \mathcal{F}(Y)$ is surjective. It follows that the map $\mathcal{F}''(X) \to \mathcal{F}''(Y)$ is surjective. □

7.13 Proposition. *Let X be a paracompact topological space. All soft abelian sheaves are acyclic for the functor $\Gamma(X, \cdot)$.*

Proof. We prove by induction on $i \geq 1$ that $\mathrm{H}^i(X, \mathcal{F})$ is zero for all soft abelian sheaves $\mathcal{F}$ on X. Let $\mathcal{F}$ be a soft abelian sheaf on X. Let $\mathcal{F} \to I$ be an injection into an injective sheaf, and let $\mathcal{F}' := I/\mathcal{F}$. The long exact cohomology sequence, plus Lemmas 7.4, 7.11 and 7.12 do what is needed. □

7.14 Theorem. *Let X be a paracompact C^∞-manifold. Then the complex of sheaves Ω_X is a resolution of $\mathbb{R}_X$, acyclic for $\Gamma(X, \cdot)$. Consequently, for all $i \geq 0$:*

$$\mathrm{H}^i_{\mathrm{dR}}(X) = \mathrm{H}^i(X, \mathbb{R}_X).$$

Proof. This is a standard argument in homological algebra, using the previous proposition and Prop. 7.1. □

Note that the right hand side of the last equality depends only on the underlying topological space of the C^∞-manifold X. So one consequence of the result is that the obstruction to solving the differential equation $d\omega = \eta$, for η a closed i-form on X, is of a topological nature. Secondly, let us note that on the de Rham side, we have a multiplicative structure since $\mathrm{H}_{\mathrm{dR}}(X)$ is a graded-commutative algebra. It is in fact true that there is also a multiplicative structure in sheaf cohomology, called the cup-product. The cup-product looks as follows. For X a topological space, $\mathcal{O}_X$ a sheaf of commutative rings on X and $\mathcal{F}_1$ and $\mathcal{F}_2$ two $\mathcal{O}_X$-modules, one has maps:

$$\mathrm{H}^i(X, \mathcal{F}_1) \otimes_{\mathcal{O}_X(X)} \mathrm{H}^j(X, \mathcal{F}_2) \to \mathrm{H}^{i+j}(X, \mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2).$$

The easiest way to define these maps is to view the cohomology groups as Ext groups, and to use the multiplicative structure on those. Another way is to use the canonical flabby resolution (called Godement's resolution) of abelian sheaves. Let us now look at some other applications of soft sheaves.

7.15 Proposition. *Let $f: X \to Y$ be a morphism of topological spaces, with X paracompact. Soft sheaves on X are acyclic for the functor f_*. Hence $\mathrm{R}^i f_*$ can be computed with soft resolutions.*

Proof. Let y be in Y. By definition, one has, for $i \geq 0$ and $\mathcal{F}$ an abelian sheaf on X,

$$((\mathrm{R}^i f_*)\mathcal{F})_y = \lim_{\to} \mathrm{H}^i(f^{-1}U, \mathcal{F}|_U),$$

where the limit is taken over all open neighborhood of y. So in fact the argument shows that sheaves that are acyclic for all $\Gamma(V, \cdot)$ with $V \subset X$ open, are acyclic for f_*. □

Another example is cohomology with compact support. Let X be a topological space, and $\mathcal{F}$ an abelian sheaf on it. Then we define the group $\Gamma_c(X, \mathcal{F})$ of sections with quasi-compact support to be the subgroup of $\mathcal{F}(X)$ of those elements s with $\mathrm{Supp}(s)$ quasi-compact. Note that if X is separated, this is equivalent to $\mathrm{Supp}(s)$ being compact. It is clear that $\Gamma_c(X, \mathcal{F})$ is functorial in $\mathcal{F}$. It is easy to see that this functor $\Gamma_c(X, \cdot)$ is left-exact. Its derived functors are denoted $\mathrm{H}^i_c(X, \cdot)$, and called cohomology with compact supports.

7.16 Lemma. *Let X be a paracompact topological space, and*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

a short exact sequence of abelian sheaves on X. Suppose that $\mathcal{F}'$ is soft. Then the sequence:

$$0 \to \Gamma_c(X, \mathcal{F}') \to \Gamma_c(X, \mathcal{F}) \to \Gamma_c(X, \mathcal{F}'') \to 0$$

is exact.

Proof. This is a simplified version of the proof of Lemma 7.11, so it is left to the reader. Just copy that proof, and show that the support of s'', which is compact, only meets finitely many of the U_i. □

7.17 Proposition. *Let X be a paracompact topological space. All soft abelian sheaves are acyclic for the functor $\Gamma_c(X,\cdot)$.*

Proof. Adapt the proof of Proposition 7.13. □

7.18 Definition. *Let X be a C^k-manifold with $k \geq \infty$. The de Rham cohomology with compact supports is defined to be the homology of the complex $\Gamma_c(X,\Omega_X)$. Explicitly:*

$$\mathrm{H}^i_{\mathrm{dR,c}}(X) = \frac{\ker\left(d\colon \Gamma_c(X,\Omega^i_X) \to \Gamma_c(X,\Omega^{i+1}_X)\right)}{\mathrm{im}\left(d\colon \Gamma_c(X,\Omega^{i-1}_X) \to \Gamma_c(X,\Omega^i_X)\right)}.$$

7.19 Theorem. *Let X be a paracompact C^∞-manifold. The complex Ω_X is a $\Gamma_c(X,\cdot)$-acyclic resolution of $\mathbb{R}_X$, hence, for all $i \geq 0$:*

$$\mathrm{H}^i_{\mathrm{dR,c}}(X) = \mathrm{H}^i_c(X,\mathbb{R}_X).$$

Proof. Just as for Theorem 7.14. □

In order to compute cohomology with compact suppport, even of constant sheaves on $\mathbb{R}^n$, it is useful to relate it to the ordinary cohomology.

7.20 Lemma. *Let X be a quasi-compact topological space. Then for all abelian sheaves $\mathcal{F}$ on X and all $i \geq 0$ one has $\mathrm{H}^i_c(X,\mathcal{F}) = \mathrm{H}^i(X,\mathcal{F})$.*

Proof. Just note that $\Gamma_c(X,\cdot) = \Gamma(X,\cdot)$. □

7.21 Lemma. *Let X be a topological space and U an open subset. Let j denote the inclusion from U into X. Let $\mathcal{F}$ be an abelian sheaf on U. We define the sheaf $j_!\mathcal{F}$ on X to be the sheaf associated to the presheaf that sends an open subset V of X to $\mathcal{F}(V)$ if $V \subset U$, and to $\{0\}$ if not. Then $j^{-1}j_!\mathcal{F} = \mathcal{F}$, and $i^{-1}j_!\mathcal{F} = 0$, where $Z = X - U$ and $i\colon Z \to X$ is the inclusion. The functor $j_!$ is a left adjoint of j^{-1}; it is exact. For all abelian sheaves $\mathcal{F}$ on X we have a functorial exact sequence:*

$$0 \to j_!j^{-1}\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to 0.$$

Proof. Exercise for the reader. □

7.22 Lemma. *Let X be a topological space, $U \subset X$ open and $j\colon U \to X$ the inclusion. Let $Y \subset X$ be a subset, and $\mathcal{F}$ an abelian sheaf on U. Then there is a canonical map $(j_!\mathcal{F})(Y) \to \mathcal{F}(Y \cap U)$, which is injective and whose image is the set of s in $\mathcal{F}(Y \cap U)$ such that the closure in Y of $\mathrm{Supp}(s)$, denoted $\overline{\mathrm{Supp}(s)}^Y$, is contained in U. In particular, if X is separated then $\Gamma_c(X,j_!\mathcal{F}) = \Gamma_c(U,\mathcal{F})$.*

Proof. An element s of $(j_!\mathcal{F})(Y)$ is a map from Y to the disjoint union of the $(j_!\mathcal{F})_y$, $y \in Y$, such that for every y in Y one has $s(y) \in (j_!\mathcal{F})_y$ and for every y in Y there is an open neighborhood V of y in U and an element t of $(j_!\mathcal{F})(V)$ such that $s(v) = t_v$ for all v in $V \cap Y$. Now use that $(j_!\mathcal{F})_x = \mathcal{F}_x$ for all x in U, and that $(j_!\mathcal{F})_x = 0$ for all x in $X - U$. Then it follows that we have a unique map from $(j_!\mathcal{F})(Y)$ to $\mathcal{F}(Y \cap U)$ that preserves germs. This property implies that it is injective. The support of an s in $(j_!\mathcal{F})(Y)$ is a closed subset of Y, contained in U. On the other hand, let s be an element of $\mathcal{F}(Y \cap U)$. Then s can also be seen as an element of $(j_!\mathcal{F})(Y \cap U)$. Suppose now that the support of s is closed in Y. Then s can be extended by zero to a section of $j_!\mathcal{F}$ over Y (Y is the union of the two open sets $U \cap Y$ and the complement of $\mathrm{Supp}(s)$; s on the first one and 0 on the second one are compatible). □

7.23 Lemma. *Let X be a normal topological space, $U \subset X$ open and j the inclusion, and $\mathcal{F}$ an abelian sheaf on U. If $\mathcal{F}$ is soft, then $j_!\mathcal{F}$ is soft.*

Proof. Suppose that $\mathcal{F}$ is a soft abelian sheaf on U. Let $Y \subset X$ be closed, and let s be in $(j_!\mathcal{F})(Y)$. Then, by the previous lemma, s is an element of $\mathcal{F}(Y \cap U)$ whose support S is closed in Y, hence in X. Let A and B be open disjoint subsets of X with A containing S and B containing $X - U$. Then S and $\overline{B} \cap U$ are disjoint closed subsets of U. It follows that s on $Y \cap U$ and 0 on $\overline{B} \cap U$ are compatible, hence we have an element t in $\mathcal{F}((Y \cup \overline{B}) \cap U))$ with $t|_{Y \cap U} = s$ and $t|_{\overline{B} \cap U} = 0$. Hence there exists u in $\mathcal{F}(U)$ whose restriction to $(Y \cup \overline{B}) \cap U$ is t. It is clear that u can be extended by zero to an element of $(j_!\mathcal{F})(X)$. □

7.24 Lemma. *Let X be a paracompact topological space, $U \subset X$ an open subset, and j the inclusion. Let $\mathcal{F}$ be an abelian sheaf on U. Then we have functorial isomorphisms $\mathrm{H}^i_c(U, \mathcal{F}) \to \mathrm{H}^i_c(X, j_!\mathcal{F})$.*

Proof. Since soft sheaves are Γ_c-acyclic, we may compute the H^i_c with soft resolutions. Let $\mathcal{F} \to I$ be a soft resolution. Then $j_!\mathcal{F} \to j_!I$ is a soft resolution. According to Lemma 7.22 we have $\Gamma_c(X, j_!I) = \Gamma_c(U, I)$. □

7.25 Proposition. *Let X be a paracompact topological space, and $U \subset X$ open. Define $Y := X - U$, $j\colon U \to X$ and $i\colon Y \to X$ the inclusions. Let $\mathcal{F}$ be an abelian sheaf on X. Then we have a functorial long exact sequence:*

$$\cdots \to \mathrm{H}^i_c(U, \mathcal{F}|_U) \to \mathrm{H}^i_c(X, \mathcal{F}) \to \mathrm{H}^i_c(Y, i^{-1}\mathcal{F}) \to \mathrm{H}^{i+1}_c(U, \mathcal{F}|_U) \to \cdots$$

Proof. This is just the long exact sequence of cohomology with compact support associated to the short exact sequence of sheaves on X given in Lemma 7.21. Indeed, $\mathrm{H}^i_c(X, j_!j^{-1}\mathcal{F})$ is equal to $\mathrm{H}^i_c(U, \mathcal{F}|_U)$ by Lemma 7.24, and that $\mathrm{H}^i_c(X, i_*i^{-1}\mathcal{F})$ equals $\mathrm{H}^i_c(Y, i^{-1}\mathcal{F})$ follows from the facts that i_* is exact and sends injectives to injectives. □

7.26 Theorem. *For all integers $i \geq 0$ and $n \geq 0$ and abelian groups A one has:*

$$\mathrm{H}^i_c(\mathbb{R}^n, A_{\mathbb{R}^n}) \cong \begin{cases} 0 \text{ if } i \neq n, \\ A \text{ if } i = n. \end{cases}$$

Proof. Let $n \geq 0$ be an integer. The topological space $\mathbb{R}^n$ is isomorphic to the open ball B in $\mathbb{R}^n$ centered at 0 and with radius 1. We apply Proposition 7.25 with $X := \overline{B}$, $U := B$ and hence $Y = S^{n-1}$. It follows that $\mathrm{H}^i_c(B, A_B)$ is isomorphic to $\mathrm{H}^{i-1}_c(S^{n-1}, A_{S^{n-1}})$ for all i. We assume that the reader knows how to compute the cohomology of S^n (use Mayer-Vietoris, for example). □

Concerning the functoriality of $\mathrm{H}^i_c(X, \cdot)$ in X one has the following.

7.27 Proposition. *Let $f\colon X \to Y$ be a morphism of topological spaces, such that for every $Z \subset Y$ closed and quasi-compact the subset $f^{-1}Z$ of X is quasi-compact. Let $\mathcal{F}$ be an abelian sheaf on Y. Then one has functorial morphisms:*

$$\mathrm{H}^i_c(Y, \mathcal{F}) \to \mathrm{H}^i_c(X, f^{-1}\mathcal{F}).$$

Proof. The argument is the same as for ordinary cohomology. Let $\mathcal{F} \to I$ be an injective resolution. Then $f^{-1}\mathcal{F} \to f^{-1}I$ is a resolution. Let $f^{-1}\mathcal{F} \to J$ be an injective resolution. Then there exists a morphism of complexes $f^{-1}I \to J$ inducing the identity on $f^{-1}\mathcal{F}$, unique up to homotopy. This gives morphisms of complexes $\Gamma_c(Y, I) \to \Gamma_c(X, f^{-1}I)$ and $\Gamma_c(X, f^{-1}I) \to \Gamma_c(X, J)$. The composition of the two induces the desired morphism. □

There is a lot more to say, but we are running out of time. By now, the reader should be ready to read for example the books by Bott and Tu, and Iversen. I strongly recommend the reader to learn the necessary things about spectral sequences, a technique that is almost always used when one studies the derived functors of the composition of two functors (see Lang's Algebra for the statements and proofs, and the book by Bott and Tu for some spectacular applications to topology).

The two things of which I most regret that we have to omit them are the Künneth formula and Poincaré duality. The Künneth formula, in de Rham cohomology, says that for X and Y paracompact C^∞-manifolds one has a functorial isomorphism between $\mathrm{H}_{\mathrm{dR}}(X \times Y)$ and $\mathrm{H}_{\mathrm{dR}}(X) \otimes \mathrm{H}_{\mathrm{dR}}(Y)$ (the tensor product in the sense of graded-comutative algebras). Poincaré duality says the following. Let X be an orientable paracompact connected C^∞-manifold, of dimension n. Then $\mathrm{H}^n_c(X, \mathbb{R}) = \mathrm{H}^n_{\mathrm{dR,c}}(X)$ is a one-dimensional $\mathbb{R}$-vector space. An orientation of X gives an isomorphism to $\mathbb{R}$: the class of an n-form with compact support is sent to $\int_X \omega$ (the orientation being used to define integration of n-forms). One has a pairing between $\mathrm{H}^i_{\mathrm{dR,c}}(X)$ and $\mathrm{H}^{n-i}_{\mathrm{dR}}(X)$ that simply sends two elements to their product in $\mathrm{H}^n_{\mathrm{dR,c}}$. This pairing indentifies $\mathrm{H}^{n-i}_{\mathrm{dR}}(X)$ with the dual of $\mathrm{H}^i_{\mathrm{dR,c}}(X)$ (and not the other way around, in general). The proofs are quite simple (with the tools we have now at our disposal); the interested reader can look them up in Bott and Tu, for example.

One last thing that I regret not to have talked about it Stokes's Theorem, and the pairing between singular homology and de Rham cohomology, on paracompact C^∞-manifolds, that is given by integration. This one can find in Spivak.

8 Cohomology of some Lie groups

To illustrate the methods that we have developed up to now, I think it is a good example to "compute" the de Rham cohomology of some Lie groups. Let G be a Lie group. For example, G could be one of those groups that we have seen in §1. Consider the action b of $G \times G$ on G by translations from both sides. Let G^o denote the connected component of the identity element e of G; it is a normal subgroup of G. Every (x, y) in $G^o \times G^o$ gives the automorphism $b_{x,y}$ of the C^∞-manifold G that is homotopic to the identity. By Theorem 6.5, the induced action of $G \times G$ on $\mathrm{H}_{\mathrm{dR}}(G)$ (say on the right, (x, y) acts as $b^*_{x,y}$, where $b_{x,y}(z) = xzy^{-1}$) factors through $G/G^o \times G/G^o$. Suppose now that G is connected. Then the action of $G \times G$ on $\mathrm{H}_{\mathrm{dR}}(G)$ is trivial. A natural question is then whether or not we can represent the cohomology classes by bi-invariant differential forms. This turns out to be true and rather elementary to prove if we assume moreover that G is compact, as we will now show (by elementary we mean, not using much functional analysis and representation theory). An important tool in the proof is integration over G, which give us a way of "averaging" functions, vector fields and differential forms on G.

8.1 Lemma. *Let G be a compact Lie group. Then there is a unique bi-invariant volume form v on G such that $\int_G v = 1$. This form v will be called the normalized invariant volume form on G.*

Proof. It is clear that there is a non-zero left-invariant positive volume form v on G. Since G is compact, we can integrate v over G. Locally, one sees that the integral is strictly positive. Hence $\int_G v > 0$. We multiply v by the inverse of this integral. Then we have v left-invariant, with integral one; these conditions determine v. Let us now argue that v is also right-invariant. Let x be in G. Then $r_x^* v$ is again a left-invariant volume form of integral one, hence $r_x^* v = v$. □

Before we start averaging differential forms and such, let us look at a situation that is a bit more general. Suppose that we have a C^∞-manifold X with a vector bundle E on it and a compact Lie group G that acts on (E, X). This means that G acts on the manifolds X and on E, preserving the map from E to X, such that for each x in X and g in G the map from $E(x)$ to $E(gx)$ is linear. Then we get an action, from the right, of G on the $\mathbb{R}$-vector space $E(X)$, as follows. Let s be in $E(X)$ and g in G. Then sg is the section of E over X defined by: $(sg)(x) = g^{-1}(s(gx))$, which is indeed an element of $E(x)$. This action of G on $E(X)$ allows us to define the averaging operator p on $E(X)$, defined by:

$$p\colon E(X) \to E(X), \quad (ps)(x) = \int_{g \in G} (sg)(x)\, v(g) = \int_{g \in G} g^{-1}(s(gx))\, v(g). \tag{8.2}$$

The reader should check that indeed ps is a C^∞-section of E.

8.3 Lemma. *The operator p is a projector on the space of G-invariants $E(X)^G$.*

Proof. Let s be in $E(X)$, g in G and x in X. Then we have:

$$((ps)g)(x) = g^{-1}((ps)(gx)) = g^{-1}\int_{h\in G} h^{-1}s(hgx)\,v(h) = (ps)(x),$$

where the last identity follows from the fact that v is right-invariant (consider the change of variables $h' := hg$). Hence ps in G-invariant. On the other hand, for s in $E(X)^G$, one sees from the defition of p that $ps = s$. □

Note that for s in $E(X)^G$ and x in X we have $s(x) \in E(x)^{G_x}$, where G_x denotes the stabilizer of x.

Now we go back to the original situation: G is a compact connected Lie group. We consider the action b of $G\times G$ on G as before. This action lifts to an action on Ω_G: for x in G and (y,z) in $G\times G$, note that $b_{y,z}$ sends x to yxz^{-1}, and that $b_{y,z}^*$ is an isomorphism from $\Omega_X(yxz^{-1})$ to $\Omega_G(x)$. By the considerations above, we get a projector p from $\Omega_G(G)$ to its subspace of bi-invariant forms $\Omega_G(G)^{G\times G}$ defined by:

$$(p\omega)(x) = \int_{(y,z)\in G\times G} b_{y,z}^*\omega(yxz^{-1})\,v(y)v(z).$$

8.4 Lemma. *The projector $p\colon \Omega_G(G)\to\Omega_G(G)$ is $\mathbb{R}$-linear, respects the grading and commutes with d and the $G\times G$-action.*

Proof. Everything is clear, except the statement that p commutes with d. If one writes out the definitions, it turns out that the required equality just says that the derivative of an integral over $G\times G$ with respect to a parameter is the integral of the corresponding derivative of the integrand. □

8.5 Corollary. *The complex $\Omega_G(G)$, with its $G\times G$-action, decomposes as a direct sum of complexes:*

$$\Omega_G(G) = \Omega_G(G)^{G\times G}\oplus(1-p)\Omega_G(G).$$

This induces a decomposition of graded vector spaces with $G\times G$-action:

$$\mathrm{H}_{\mathrm{dR}}(G) = \mathrm{H}(\Omega_G(G)) = \mathrm{H}(\Omega_G(G)^{G\times G})\oplus\mathrm{H}((1-p)\Omega_G(G)).$$

8.6 Lemma. *The complex $(1-p)\Omega_G(G)$ is exact.*

Proof. We would like to argue as follows: we know that the action of $G\times G$ on $\mathrm{H}_{\mathrm{dR}}(G)$ is trivial; the decomposition of $(1-p)\Omega_G(G)$ into irreducible representations of $G\times G$ only contains non-trivial irreducible representations, hence the same is true for the space $\mathrm{H}((1-p)\Omega_G(G))$, which has then to be the zero space. The problem with this argument (that can be made correct) is that we should equip $\Omega_G(G)$ with a $G\times G$-invariant inner product, and complete it, in order to apply the theory of representations of compact Lie groups on Hilbert spaces. Moreover, we would have to show that the map from $\ker(d)$ to $\mathrm{H}_{\mathrm{dR}}(G)$ is continuous, for the topology defined by the inner product. As I said, all that can be done, but I want to try to use less heavy machinery to get the result.

We choose a positive definite metric on Ω_G (left or right invariant, or bi-invariant, if you want; all metrics are equivalent, because G is compact). For x in X let $\|\cdot\|_x$ denote the norm on $\Omega_G(x)$ given by this metric. We have the corresponding sup norm $\|\cdot\|$ on $\Omega_G(G)$: $\|\omega\| := \sup\{\|\omega(x)\|_x \,|\, x \in X\}$. Let V be the $\mathbb{R}$-vector space of continuous sections of Ω_G, and let $\|\cdot\|$ be the sup norm on that space. Then V is the completion of the normed vector space $\Omega_G(G)$. The action of $G \times G$ on $\Omega_G(G)$ is continuous, in the sense that the map from $(G \times G) \times \Omega_G(G)$ to $\Omega_G(G)$ is continuous. Likewise, the operator p on $\Omega_G(G)$ is continous.

We can now reinterprete the operator p. Let P be the operator on V defined by:

$$P(\omega) := \int_{g \in G} (\omega g)\, v(g). \tag{8.6.1}$$

Here, some remarks should be made. Note that the function f_ω from G to V, given by $g \mapsto \omega g$, is continuous, and that V is a complete normed $\mathbb{R}$-vector space. Therefore, the usual theory of integration of continuous compactly supported functions still works. In this theory of integration, it is clear that, for f a continuous V-valued function on G, and $l: V \to V'$ a continuous linear map of normed $\mathbb{R}$-vector spaces, one has:

$$\int_{g \in G} l \circ f\, v(g) = l\left(\int_{g \in G} f\, v(g)\right). \tag{8.6.2}$$

Applying this to the maps $V \to \mathbb{R}$, sending ω to $\omega(x)$, for x in G, one finds that $P = p$. It follows that P preserves $\Omega_G(G)$. Let $\ker(d)$ denote the kernel of d on $\Omega_G(G)$, and let $q: \ker(d) \to \mathrm{H}_{\mathrm{dR}}(G)$ be the quotient map. We will use Poincaré duality to show that $\mathrm{H}_{\mathrm{dR}}(G)$ is finite dimensional, and that q is continuous. We choose an orientation on G. Since Poincaré duality says that $\mathrm{H}_{\mathrm{dR}}(G)$ is isomorphic to its own dual, it has to be of finite dimension (this is an interesting and not so simple exercise in linear algebra). Another way to prove that $\mathrm{H}_{\mathrm{dR}}(G)$ is finite dimensional is to show that every compact manifold has a finite good cover, i.e., a cover such that all non-trivial finite intersections are contractible. Let $i \geq 0$ and let $\ker(d)^i$ denote the kernel of the map $d: \Omega_G^i(G) \to \Omega_G^{i+1}(G)$ and let $q: \ker(d)^i \to \mathrm{H}^i_{\mathrm{dR}}(G)$ be the quotient map. Poincaré duality gives an isomorphism from $\mathrm{H}^i_{\mathrm{dR}}(G)$ to $\mathrm{H}^{n-i}_{\mathrm{dR}}(G)^\vee$, where n is the dimension of G. Choose $\eta_1, \ldots, \eta_r$ in $\Omega_G^{n-i}(G)$ such that their images $[\eta_1], \ldots, [\eta_r]$ in $\mathrm{H}^{n-i}_{\mathrm{dR}}(G)$ form a basis. This basis also gives a basis for $\mathrm{H}^i_{\mathrm{dR}}(G)$. With respect to that basis, the jth coordinate of the map q is given by:

$$q: \ker(d)^i \to \mathrm{H}^i_{\mathrm{dR}}(G), \quad \omega \mapsto \int_G \omega\eta_j. \tag{8.6.3}$$

From this it is clear that q is a continuous map, for the usual topology on the finite dimensional vector space $\mathrm{H}_{\mathrm{dR}}(G)$ and the sup norm on $\ker(d)$. Formula 8.6.2 says that the diagram:

$$\begin{array}{ccc} \ker(d) & \xrightarrow{P} & \ker(d) \\ \downarrow^q & & \downarrow^q \\ \mathrm{H}_{\mathrm{dR}}(G) & \xrightarrow{P'} & \mathrm{H}_{\mathrm{dR}}(G) \end{array} \tag{8.6.4}$$

in which P' is the averaging operator on $\mathrm{H_{dR}}(G)$, is commutative. The proof is now finished, because P' is the identity map, whereas $P = p$ is zero on $(1-p)\Omega_G(G)$. □

8.7 Lemma. *Let G be any Lie group. The differential d is zero on $\Omega_G(G)^{G\times G}$. In other words, every bi-invariant differential form on G is closed.*

Proof. Let $\psi: G \to G$ be the morphism of manifolds that sends g to g^{-1}. Then ω in $\Omega_G(G)$ is left-invariant if and only if $\psi^*\omega$ is right-invariant. The map $\mathrm{T}_\psi(e): L \to L$ is multiplication by -1, for purely formal reasons. Hence the automorphism of $\Omega^i_G(e)$ that is induced by ψ is multiplication by $(-1)^i$. Let now ω in $\Omega^i_G(G)$ be bi-invariant. Then $\psi^*\omega$ is bi-invariant too, and $(\psi^*\omega)(e) = (-1)^i\omega(e)$, hence $\psi^*\omega = (-1)^i\omega$. But ψ^* commutes with d, and $d\omega$ is bi-invariant, hence $(-1)^{i+1}d\omega = \psi^*(d\omega) = d\psi^*\omega = (-1)^i d\omega$. □

8.8 Theorem. *Let G be a compact connected Lie group. Then the inclusion of the complex of bi-invariant forms $\Omega_G(G)^{\mathrm{bi-inv}}$ in $\Omega_G(G)$ gives an isomorphism of graded-commutative $\mathbb{R}$-algebras:*

$$\Lambda(L^\vee)^G \xrightarrow{\sim} \Omega_G(G)^{\mathrm{bi-inv}} \xrightarrow{\sim} \mathrm{H_{dR}}(G),$$

where the G-action on $\Lambda(L^\vee)$ is induced by the adjoint representation of G on L.

Proof. For the second isomorphism, combine Corollary 8.5 and Lemmas 8.6 and 8.7. The first isomorphism is standard, and follows from the fact that a form is bi-invariant if and only if it is left-invariant and invariant under the action of G by itself by conjugation. □

8.9 Remark. Let G be a connected Lie group. The group of automorphisms $\mathrm{Aut}(G)$ of G as a Lie group is itself a Lie group; it is an algebraic subgroup of $\mathrm{GL}(\mathrm{Lie}(G))$. Let $\mathrm{Aut}(G)^o$ be the connected component of identity of $\mathrm{Aut}(G)$. Is it true that the complex of differential forms on G that are bi-invariant and $\mathrm{Aut}(G)^o$-invariant is isomorphic to $\mathrm{H_{dR}}(G)$? □

To finish the course, let us look at an example. For $n \geq 0$ let $\mathrm{U}_n(\mathbb{R})$ be the unitary group in dimension n. That is, $\mathrm{U}_n(\mathbb{R})$ is the subgroup of $\mathrm{GL}_n(\mathbb{C})$ fixing the standard (hermitian) inner product on $\mathbb{C}^n$. The Lie algebra L of $\mathrm{U}_n(\mathbb{R})$ is the $\mathbb{R}$-vector space of x in $\mathrm{M}_n(\mathbb{C})$ with $x^*+x=0$, where x^* denotes the complex conjugate of the transposed matrix. It has the following basis: for $1 \leq j \leq n$ one has $ie_{j,j}$, and for $1 \leq j < k \leq n$ one has $e_{j,k} - e_{k,j}$ and $i(e_{j,k}+e_{k,j})$. Since this basis is also a $\mathbb{C}$-basis for $\mathrm{M}_n(\mathbb{C})$, we see that the inclusion of L in the $\mathbb{C}$-vector space $\mathrm{M}_n(\mathbb{C})$ induces an isomorphism of Lie-algebras over $\mathbb{C}$ from $\mathbb{C}\otimes_{\mathbb{R}} L$ to $\mathrm{M}_n(\mathbb{C})$. It follows that $\mathbb{C}\otimes_{\mathbb{R}} \mathrm{H_{dR}}(\mathrm{U}_n(\mathbb{R}))$ is isomorphic to $\Lambda(\mathrm{M}_n(\mathbb{C})^\vee)^{\mathrm{GL}_n(\mathbb{C})}$, as graded-commutative $\mathbb{C}$-algebras (to see that the $\mathrm{U}_n(\mathbb{R})$-invariants are the same as the $\mathrm{GL}_n(\mathbb{C})$-invariants one has to use that the invariants are precisely the "infinitesimal invariants", i.e., invariants for the action of the Lie algebra). The next question is of course, what is this algebra? I do not know how to describe this algebra, by generators and relations, directly. But we can of course use Theorem 7.14, which gives an isomorphism of algebras between $\mathrm{H_{dR}}(\mathrm{U}_n(\mathbb{R}))$ and $\mathrm{H}(\mathrm{U}_n(\mathbb{R}), \mathbb{R}_{\mathrm{U}_n(\mathbb{R})})$. One can compute this last algebra by induction on n, using fibrations, as follows. The group $\mathrm{U}_1(\mathbb{R})$ is the circle, hence its cohomology is the graded-commutative $\mathbb{R}$-algebra $\mathbb{R}[x_1]$, with x_1 in degree one. Let $n > 1$. The orbit under $\mathrm{U}_n(\mathbb{R})$ of the vector e_n in $\mathbb{C}^n$ is the unit sphere S^{2n-1}. The stabilizer of e_n is the subgroup $\mathrm{U}_{n-1}(\mathbb{R})$ of $\mathrm{U}_n(\mathbb{R})$. Hence the map $f\colon \mathrm{U}_n(\mathbb{R}) \to S^{2n-1}$ that sends g to ge_n is a fibration with fibre $\mathrm{U}_{n-1}(\mathbb{R})$. There is a general theory, called Leray's spectral sequence, relating the cohomology of the total space (in this case $\mathrm{U}_n(\mathbb{R})$) of a fibration to the cohomology, on the base (S^{2n-1} in this case), of the higher direct images of the constant sheaf on the total space (the $\mathrm{R}^i f_*\mathbb{R}_{\mathrm{U}_n(\mathbb{R})}$ in this case). In our case the conditions are perfect, because, by the fact that S^{2n-1} is simply connected, all locally constant sheaves on it are in fact constant. It is very easy to see that the Leray spectral sequence degenerates, in this case, "at E_2". The result is the following.

8.10 Theorem. *We have an isomorphism of graded-commutative algebras:*

$$\mathbb{Z}[x_1, x_3, \ldots, x_{2n-1}] \xrightarrow{\sim} \mathrm{H}(\mathrm{U}_n(\mathbb{R}), \mathbb{Z}_{\mathrm{U}_n(\mathbb{R})}),$$

where x_i has degree i.

8.11 Corollary. *We have an isomorphism of graded-commutative algebras:*

$$\mathbb{C}[x_1, x_3, \ldots, x_{2n-1}] \xrightarrow{\sim} \Lambda(\mathrm{M}_n(\mathbb{C})^\vee)^{\mathrm{GL}_n(\mathbb{C})}.$$

Note that this Corollary is a purely algebraic statement, which has in fact also to be true over $\mathbb{Q}$. One should also ask to describe the images y_i of the x_i explicitly. I have a good candidate t_i for y_i: the i-linear alternating map that sends an $i$tuple $(a_1, \ldots, a_i)$ of elements of $\mathrm{M}_n(\mathbb{C})$ to the alternating sum of the traces of all of their products:

$$t_i\colon (a_1, \ldots, a_i) \mapsto \sum_{\sigma \in \mathrm{S}_i} \mathrm{sgn}(\sigma)\mathrm{tr}(a_{\sigma(1)} \cdots a_{\sigma(i)}), \tag{8.12}$$

where S_i is the symmetric group on $\{1, \ldots, i\}$ and sgn the sign of permuations, and tr the trace. Whether or not these candidates are the right ones is no doubt known by those who know about Lie algebra cohomology.

To finish, note that some standard decomposition (décomposition polaire in french) for $\mathrm{GL}_n(\mathbb{C})$ says that every element g of $\mathrm{GL}_n(\mathbb{C})$ can be written uniquely as ux with u in $\mathrm{U}_n(\mathbb{R})$ and with $x^* = x$ and x positive definite. The space of such x is convex, hence contractible. So $\mathrm{GL}_n(\mathbb{C})$, as a topological space, is homotopically equivalent to $\mathrm{U}_n(\mathbb{R})$, which implies that we also have:

$$\mathbb{Z}[x_1, x_3, \ldots, x_{2n-1}] \xrightarrow{\sim} \mathrm{H}(\mathrm{GL}_n(\mathbb{C}), \mathbb{Z}_{\mathrm{GL}_n(\mathbb{C})}). \tag{8.13}$$

9 The cohomology of constant sheaves

This section was not part of this course, but I include it anyway, because I want to show that the cohomology of constant sheaves on manifolds, and, more generally, on CW-complexes, say, is not more difficult to compute than singular homology or cohomology. There will be two results. The first concerns the cohomology of constant sheaves on the topological space $\mathbb{R}$. The second result is about homotopy invariance for the cohomology of constant sheaves. With these tools available, the usual Mayer-Vietoris and Cech type arguments that one finds for example in the book by Bott and Tu can be applied directly for sheaves. We begin by recalling the Mayer-Vietoris and Cech arguments. We assume that the reader is familiar with section 1 of Chapter II and sections 1 and 4 of Chapter III of Hartshorne's book.

9.1 Theorem. (Mayer-Vietoris for open subsets.) *Let X be a topological space, X_1 and X_2 two open subsets such that $X = X_1 \cup X_2$. Let $X_{1,2} := X_1 \cap X_2$, and let j_1, j_2 and $j_{1,2}$ be the inclusions. Let $\mathcal{F}$ be an abelian sheaf on X. Then we have a long exact sequence:*

$$\cdots \to \mathrm{H}^i(X, \mathcal{F}) \to \mathrm{H}^i(X_1, \mathcal{F}|_{X_1}) \oplus \mathrm{H}^i(X_2, \mathcal{F}|_{X_2}) \to \mathrm{H}^i(X_{1,2}, \mathcal{F}|_{X_{1,2}}) \to \mathrm{H}^{i+1}(X, \mathcal{F}) \to \cdots,$$

functorially in $\mathcal{F}$.

Proof. Let $\mathcal{F} \to I$ be an injective resolution. Then we have the short exact sequence of complexes:

$$0 \to I(X) \to I(X_1) \oplus I(X_2) \to I(X_{1,2}) \to 0,$$

because the I^i are sheaves (the first map takes the two restrictions, the second map the difference of the two restrictions). All we need to do now is to identify the associated long exact homology sequence as the sequence in the Theorem. This follows directly from the fact that, for $j\colon U \to X$ an open immersion, the functor j^{-1} is exact and preserves injectives (it preserves injectives because it has an exact left-adjoint, $j_!$, called extension by zero, see Hartshorne's book, Chapter II, exercise 1.19). □

9.2 Theorem. (Mayer-Vietoris for closed subsets.) *Let X be a topological space, X_1 and X_2 two closed subsets such that $X = X_1 \cup X_2$. Let $X_{1,2} := X_1 \cap X_2$, and let i_1, i_2 and $i_{1,2}$ be the injections, which are closed immersions. Let $\mathcal{F}$ be an abelian sheaf on X. Then we have a long exact sequence:*

$$\cdots \to \mathrm{H}^i(X, \mathcal{F}) \to \mathrm{H}^i(X_1, i_1^{-1}\mathcal{F}) \oplus \mathrm{H}^i(X_2, i_2^{-1}\mathcal{F}) \to \mathrm{H}^i(X_{1,2}, i_{1,2}^{-1}\mathcal{F}) \to \mathrm{H}^{i+1}(X, \mathcal{F}) \to \cdots,$$

functorially in $\mathcal{F}$.

Proof. For f a morphism of topological spaces, f_* is a left-adjoint of f^{-1}. This allows us to define a complex of sheaves:

(9.2.1) $$0 \to \mathcal{F} \to i_{1,*}i_1^{-1}\mathcal{F} \oplus i_{2,*}i_2^{-1}\mathcal{F} \to i_{1,2,*}i_{1,2}^{-1}\mathcal{F} \to 0,$$

where the first map is the sum of the two tautological maps, and where the second map is the difference of the two tautological maps. One verifies that this complex is exact, by looking at the stalks. It remains to show that the long exact cohomology sequence is the one we want. To see that, note that for $i: Y \to X$ a closed immersion, and $\mathcal{F}$ an abelian sheaf on Y, we have a natural isormorphism between $\mathrm{H}^i(Y, \mathcal{F})$ and $\mathrm{H}^i(X, i_*\mathcal{F})$ (use that i_* of an injective resolution of $\mathcal{F}$ is an injective resolution of $i_*\mathcal{F}$). □

9.3 Theorem. *Let X be a topological space, $\mathcal{F}$ an abelian sheaf on X, and $U: I \to \mathrm{Open}(X)$ an open cover of X such that for all $p \geq 0$, for all $i_0, \ldots, i_p$ in I and all $i \geq 1$ one has $\mathrm{H}^i(U_{i_0,\ldots,i_p}, \mathcal{F}) = 0$, where $U_{i_0,\ldots,i_p}$ denotes the intersection of the U_{i_j} and $\mathcal{F}$ the restriction of $\mathcal{F}$ to it. Then for all $i \geq 0$ one has a natural isomorphism between $\mathrm{H}^i(X, \mathcal{F})$ and the Cech cohomology group $\mathrm{H}^i(U, \mathcal{F})$.*

Proof. One adapts the proof of Theorem 4.5 of Chapter III of Hartshorne's book to this case. In that theorem, it is proved that Cech cohomology coincides with the real cohomology for quasi-coherent sheaves on separated schemes. Let us note that this proof does not use double complexes or spectral sequences. □

The results that we just mentioned help to reduce the computation of cohomology groups to suitable open subsets. But to really compute some cohomology groups, one has to start somewhere. There are two cases in which the cohomology of an abelian sheaf is very easy: the cases where X has at most one element. In case X is empty, all cohomology is zero, of course. In case X is just a point, taking global sections is the same as taking the stalk at that point, hence is an exact functor, so all higher cohomology is zero. Our starting point to compute more general cases is a direct computation of the $\mathrm{H}^i(\mathbb{R}, A_{\mathbb{R}})$, for all i and all abelian groups A.

9.4 Theorem. *Let A be an abelian group. Let $\mathbb{R}$ be the set of real numbers, equipped with the usual topology. Then we have:*

$$\mathrm{H}^i(\mathbb{R}, A_{\mathbb{R}}) = \begin{cases} A \text{ if } i = 0, \\ 0 \text{ if } i > 0. \end{cases}$$

Proof. Let $A_{\mathbb{R}} \to \mathcal{F}$ be an injection into a flabby sheaf. For example, one can take $\mathcal{F}$ to be the "sheaf of discontinuous sections" of $A_{\mathbb{R}}$, i.e., for U an open subset of $\mathbb{R}$, $\mathcal{F}(U)$ is the set of all $\mathbb{R}$-valued functions on U. Since $\mathcal{F}$ is flabby, we have $\mathrm{H}^i(\mathbb{R}, \mathcal{F}) = 0$ for all $i > 0$. Let $\mathcal{F}'$ be the quotient of $\mathcal{F}$ by $A_{\mathbb{R}}$. So by construction we have the short exact sequence of abelian groups on $\mathbb{R}$:

(9.4.1) $$0 \to A_{\mathbb{R}} \to \mathcal{F} \to \mathcal{F}' \to 0.$$

9.4.2 Lemma. *The map $\mathcal{F}(\mathbb{R}) \to \mathcal{F}'(\mathbb{R})$ is surjective, hence $\mathrm{H}^1(\mathbb{R}, A_{\mathbb{R}}) = 0$.*

Proof. (In fact, we prove the surjectiveness without the assumption that $\mathcal{F}$ is flabby.) Let f' be in $\mathcal{F}'(\mathbb{R})$. We have to show that f' is the image of an element of $\mathcal{F}(\mathbb{R})$. Locally this is so. Let S be the set of pairs (I, f) with $I \subset \mathbb{R}$ an open interval and f in $\mathcal{F}(I)$ with image $f'|I$ in $\mathcal{F}'(I)$. We define a partial order on S as follows: $(I, f) \leq (J, g)$ if and only if I is contained in J and $g|_I = f$. It is clear that every totally ordered subset of S has an upper bound (take the union of the I's and the section of $\mathcal{F}$ induced on it), and that S is not empty. Zorn's lemma gives us a maximal element $(]a, b[, f)$ of S. We claim that $]a, b[= \mathbb{R}$. Suppose that $b \neq \infty$. We take $\varepsilon > 0$ and g in $\mathcal{F}(]b - \varepsilon, b + \varepsilon[)$ lifting f'. Then the difference of the restrictions of f and g to $]b - \varepsilon, b[$ is in $A_{\mathbb{R}}(]b - \varepsilon, b[) = \mathbb{R} = A_{\mathbb{R}}(b - \varepsilon, b + \varepsilon)$. Let g' be the sum of g with this difference. Then f and g' are compatible, hence define a section of $\mathcal{F}$ over $]a, b + \varepsilon[$ lifting f'. Hence $(]a, b[, f)$ was not maximal. The contradiction shows that $b = \infty$. In the same way one shows that $a = -\infty$. □

9.4.3 Lemma. *The sheaf $\mathcal{F}'$ is flabby.*

Proof. Let U be an open subset of $\mathbb{R}$. We have to show that the restriction map from $\mathcal{F}'(\mathbb{R})$ to $\mathcal{F}'(U)$ is surjective. It is easy to see that for every x in U there is a maximal open interval containing x and contained in U. This implies that U is, as a topological space, a disjoint union of non-empty open intervals. Each of these intervals is isomorphic, as topological space, to $\mathbb{R}$ itself. Hence the previous lemma implies that the map $\mathcal{F}(U) \to \mathcal{F}'(U)$ is surjective. Since $\mathcal{F}(\mathbb{R}) \to \mathcal{F}(U)$ is also surjective, $\mathcal{F}'(\mathbb{R}) \to \mathcal{F}'(U)$ must be surjective, too. □

The fact that $\mathcal{F}$ and $\mathcal{F}'$ have trivial higher cohomology implies the same for $A_{\mathbb{R}}$. □

9.5 Corollary. *Let I be a non-empty interval in $\mathbb{R}$ (that is, open, closed, or half-open). Let A be an abelian group. Then $\mathrm{H}^i(I, A_I)$ is A if $i = 0$ and is 0 otherwise.*

Proof. Using Mayer-Vietoris for open subsets, one easily computes the $\mathrm{H}^i(S^1, A_{S_1})$. Then one uses Mayer-Vietoris for closed subsets to relate those to the cohomology of closed intervals. This gives the claim for closed intervals. Applying once more Mayer-Vietoris for open subsets, one gets the claim for half-open intervals. Of course, it is also possible to do some variant of the proof of Theorem 9.4. □

We can now start to work on the homotopy invariance. In order to state the result, we have to see in what way cohomology groups are functorial with respect to morphisms of topological spaces. So let for a moment $f: X \to Y$ be a morphism of topological spaces, and $\mathcal{F}$ an abelian sheaf on Y. Then we claim that there are functorial maps $f^*: \mathrm{H}^i(Y, \mathcal{F}) \to \mathrm{H}^i(X, f^{-1}\mathcal{F})$. To define these, note that for $i = 0$ we have indeed such a map, with the property that $(f^*s)_x = s_{f(x)}$ for all s in $\mathcal{F}(Y)$ and all x in X. Let $\mathcal{F} \to I$ be an injective resolution of $\mathcal{F}$. Then $f^{-1}\mathcal{F} \to f^{-1}I$ is a resolution of $f^{-1}\mathcal{F}$, and hence can be mapped to an injective resolution $f^{-1}\mathcal{F} \to J$, uniquely up to homotopy. Applying the map we have for $i = 0$ gives a map $I(Y) \to (f^{-1}I)(X)$, that we compose with $(f^{-1}I)(X) \to J(X)$. Passing to homology gives the required maps. Note that for A an abelian group we have $f^{-1}A_Y = A_X$. So in particular f induces maps f^* from $\mathrm{H}^i(Y, A_Y)$ to $\mathrm{H}^i(X, A_X)$.

Let X and Y be topological spaces, and let I be the closed interval $[0,1]$. We say that two morphisms f_0 and f_1 from X to Y are homotopic (via I), if there exists a morphism $f\colon X\times I\to Y$ whose restrictions to $X\times\{0\}$ and to $X\times\{1\}$ are f_0 and f_1, respectively.

9.6 Theorem. *(Theorem 1.1 of Chapter IV in the book by Iversen.) Let $f_0, f_1\colon X\to Y$ be homotopic morphisms of topological spaces. Let A be an abelian group and $i\geq 0$ an integer. Then the maps f_0^* and f_1^* from $\mathrm{H}^i(Y, A_Y)$ to $\mathrm{H}^i(X, A_X)$ are equal.*

Proof. The strategy is the following. Let $p\colon X\times I\to X$ be the projection. We will show that the maps p^* from the $\mathrm{H}^i(X, A_X)$ to the $\mathrm{H}^i(X\times I, p^{-1}\mathcal{F})$ are isomorphisms. For $t\in I$, let i_t denote the injection of X in $X\times I$ that sends x to (x,t). Then, for functorial reasons, the maps i_t^* from $\mathrm{H}^i(X\times I, p^{-1}\mathcal{F})$ to $\mathrm{H}^i(X, A_X)$ are the inverses of the p^*, hence are independent of t. Again by functorial reasons, it follows that the maps f_0^* and f_1^* in the Theorem are equal. Let us now carry out this program.

9.6.1 Lemma. *Let X be a topological space, Y an interval in $\mathbb{R}$ and $\mathcal{F}$ a sheaf on $X\times Y$. Let x be in X, and let $i_x\colon Y\to X$ denote the inclusion. Then the natural map:*

$$\lim_{U\supset i_xY}\mathcal{F}(U)\to\mathcal{F}(i_xY) := (i_x^{-1}\mathcal{F})(Y),$$

where lim *indicates a direct limit, is an isomorphism.*

Proof. Let us first recall a more or less explicit description of $\mathcal{F}(i_xY)$. The set $\mathcal{F}(i_xY)$ is the set of functions s from i_xY to the disjoint union of the $\mathcal{F}_y$, for y in i_xY, such that every y in i_xY has an open neighborhood U_y in $X\times Y$ such that the restriction of s to $i_xY\cap U_y$ is given by an element of $\mathcal{F}(U_y)$.

Let us prove the injectivity. So let U be an open subset of $X\times Y$ containing i_xY, and let s_1 and s_2 be in $\mathcal{F}(U)$, such that their images in $\mathcal{F}(i_xY)$ are equal. Then we have, for every y in i_xY, an open neighborhood U_y of y, contained in U, such that the images in $\mathcal{F}(i_xY\cap U_y)$ are equal. Let V be the union of the U_y. Then V is an open subset of $X\times Y$ containing i_xY and contained in U, such that the images of s_1 and s_2 in $\mathcal{F}(V)$ are equal.

It remains to prove the surjectivity. If Y is empty or just one point, the proof is obvious. So we suppose that Y has at least two elements. There are three cases to consider: Y is closed, open or half-open. Let us first do the easiest case, where Y is closed, hence compact.

So suppose that Y is compact. Let s be in $\mathcal{F}(i_xY)$. For every y in i_xY we have an open subset U_y of $X\times Y$ containing y, and an element s^y of $\mathcal{F}(U_y)$ inducing s on $i_xY\cap U_y$. Since Y is quasi-compact, i_xY is covered by finitely many of the U_y, say by $U_{y_1},\ldots,U_{y_n}$. Now consider the set U of z in the union of the U_{y_i} such that $s_z^{y_i}=s_z^{y_j}$ whenever z is in $U_{y_i}\cap U_{y_j}$. Then U is an open subset of $X\times Y$ containing i_xY, and the s^{y_i} define an element of $\mathcal{F}(U)$ whose restriction to i_xY is s.

Let us now do the case where Y is an open interval. Then Y is homeomorphic to $\mathbb{R}$, so we will work with $\mathbb{R}$. Let s be in $\mathcal{F}(i_x\mathbb{R})$. For every t in $\mathbb{R}$ we have an open neighborhood U_t of t in $\mathbb{R}$ and an open neighborhood V_t of x in X and an element s^t of $\mathcal{F}(V_t\times U_t)$ such that s^t

induces s on $i_x U_t$. The open cover of $\mathbb{R}$ by the U_t can be refined to a locally finite one, say with U_i, V_i and t^i. Then one proceeds as in the previous case where Y was compact. More precisely, the subset Z of the union of the $V_i \times U_i$ where the t^i all agree is open.

The case of a half-open interval is done in the same way, and left to the reader. $\square$

9.6.2 Lemma. *Let X be a topological space, Y an interval in $\mathbb{R}$ and $\mathcal{F}$ a sheaf on $X \times Y$. Let x be in X, and let $i_x\colon Y \to X$ denote the inclusion. Then the natural maps:*

$$\lim_{U \supset i_x Y} \mathrm{H}^i(U, \mathcal{F}) \to \mathrm{H}^i(Y, i_x^{-1}\mathcal{F})$$

are isomorphisms.

Proof. Consider the functors T^i from the category of abelian sheaves on $X \times Y$ to the category of abelian groups, given by $T^i(\mathcal{F}) := \lim_{U \supset i_x Y} \mathrm{H}^i(U, \mathcal{F})$. Since taking a filtered direct limit is exact, the T^i form a δ-functor (see Hartshorne, Chapter II, section 1 for the definitions). We also have the functors S^i, between the same categories, given by $S^i(\mathcal{F}) := \mathrm{H}^i(Y, i_x^{-1}\mathcal{F})$. They form a δ-functor too, because i_x^{-1} is exact. We have a morphism of δ-functors from the T^i to the S^i, which is an isomorphism for $i = 0$ by the previous lemma. We claim that, for $i \geq 1$, S^i and T^i are effaceable. To see it for T^i, note that one can embed every $\mathcal{F}$ into an injective one, and that the restriction to an open subset of an injective sheaf is again injective. To see it for S^i, consider again an embedding $\mathcal{F} \to I$ into an injective sheaf. We claim that $i_x^{-1}I$ is flabby. So let $U \subset \mathbb{R}$ be an open subset, and s in $\mathcal{F}(i_x U)$. We want to show that s extends to a neighborhood of $i_x U$. Note that U is a disjoint union of open intervals, hence we get such an extension from Lemma 9.6.1. By Hartshorne, Chapter II, Theorem 1.3A, both our δ-functors are universal. Since they agree for $i = 0$, they agree for all i. $\square$

9.6.3 Lemma. *Let X and Y be topological spaces, with Y a closed interval in $\mathbb{R}$. Let x be in X. Let $\mathcal{F}$ be an abelian sheaf on $X \times Y$ and let i_x denote the injection of Y in X sending y to (x, y). Then the natural maps:*

$$\lim_{U \ni x} \mathrm{H}^i(U \times Y, \mathcal{F}) \to \mathrm{H}^i(Y, i_x^{-1}\mathcal{F})$$

are isormorphisms.

Proof. This follows from Lemma 9.6.2, taking into account that the $U \times Y$ form a cofinal system of open neighborhoods of $i_x Y$ by the quasi-compactness of Y. $\square$

We are now ready to prove Theorem 9.6. In fact, we prove something that is a bit more general.

9.6.4 Lemma. *Let X be a topological space, let I be the closed interval $[0, 1]$ and let $p\colon X \times I \to X$ be the projection. Let $\mathcal{F}$ be an abelian sheaf on X. Then the maps p^* from the $\mathrm{H}^i(X, \mathcal{F})$ to the $\mathrm{H}^i(X \times I, p^{-1}\mathcal{F})$ are isomorphisms.*

Proof. Let $\mathcal{F} \to J$ and $p^{-1}\mathcal{F} \to K$ be injective resolutions. We have a morphism from $p^{-1}J$ to K inducing the identity on $p^{-1}\mathcal{F}$, unique up to homotopy. Note that, by definition, this morphism induces p^* on the cohomology.

We claim that for any abelian sheaf $\mathcal{G}$ on X, the adjunction morphism from $\mathcal{G}$ to $p_*p^{-1}G$ is an isomorphism. To see this, it suffices to check on the stalks. So let x be in X. Lemma 9.6.3, with $i = 0$, says that the natural map from $p_*p^{-1}\mathcal{G}$ to $(i_x^{-1}p^{-1}\mathcal{G})(I)$ is an isomorphism. But $(i_x^{-1}p^{-1}\mathcal{G})$ is the constant sheaf $\mathcal{G}_{x,I}$ on I, hence $(i_x^{-1}p^{-1}\mathcal{G})(I) = \mathcal{G}_x$.

The adjunction morphism identifies $p_*p^{-1}\mathcal{F} \to p_*p^{-1}J$ with $\mathcal{F} \to J$, hence $p_*p^{-1}\mathcal{F} \to p_*p^{-1}J$ is an injective resolution of $\mathcal{F}$. If we prove that $p_*p^{-1}\mathcal{F} \to p_*K$ is also an injective resolution, we have finished. By construction, the p_*K^i are injective. We have to show that the complex $p_*p^{-1}\mathcal{F} \to p_*K$ is exact. To do that, it suffices to consider the stalks. So let x be in X. Let $h^i({}_*K)$ denote the homology in degree i of p_*K; it is an abelian sheaf on X. By the definition of kernel and cokernel in the category of sheaves, we have, for all i:

$$h^i(p_*K)_x = \lim_{U \ni x} \mathrm{H}^i(U \times I, p^{-1}\mathcal{F}).$$

Lemma 9.6.3 tells us that the natural maps

$$\lim_{U \ni x} \mathrm{H}^i(U \times Y, p^{-1}\mathcal{F}) \to \mathrm{H}^i(Y, i_x^{-1}p^{-1}\mathcal{F})$$

are isomorphisms. But $i_x^{-1}p^{-1}\mathcal{F}$ is the constant sheaf $\mathcal{F}_{x,I}$ on I, hence, by Corollary 9.5, we have:

$$h^i(p_*K)_x = \begin{cases} \mathcal{F}_x & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

□

The proof of Theorem 9.6 is now finished. □

9.7 Corollary. *Let X be a contractible topological space, and A an abelian group. Then we have $\mathrm{H}^0(X, A_X) = A$ and $\mathrm{H}^i(X, A_X) = 0$ for $i > 0$.* □