# Modular parametrizations at primes of bad reduction.

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## Contents

# 1 Introduction.

## 2 Modular curves, modular forms and Hecke operators.

**2.1 Modular curves.** The aim of this section is to explain how we use the arithmetic moduli theory of elliptic curves in what follows. We also discuss how the moduli theory of Drinfelds level structures on elliptic curves, as exposed in [19], extends to the moduli theory of the generalized elliptic curves of [11]. The delicate results in this section concerning the cusps in the "bad" characteristics will not really be used in the later sections.

**2.1.1 Definition.** For $n$ a positive integer we define the following categories fibered in groupoids over the category (Sch) of schemes:

$\mathcal{M}(\Gamma(n))$**:** objects are pairs $(E/S, \alpha)$, where $E/S$ is an elliptic curve and $\alpha\colon (\mathbf{Z}/n\mathbf{Z})^2 \to E(S)$ is a Drinfeld basis of $E[n]$ (see [19] 3.1). Morphisms are cartesian squares compatible with the $\alpha$'s.

$\mathcal{M}(\Gamma_1(n))$**:** the objects are pairs $(E/S, \alpha)$, where $\alpha\colon \mathbf{Z}/n\mathbf{Z} \to E(S)$ is a $\mathbf{Z}/n\mathbf{Z}$-structure on $E$ (see [19] 3.2).

$\mathcal{M}(\Gamma_0(n))$**:** $(E/S, G)$, where $G$ is a cyclic closed subgroup scheme of $E[n]$ of rank $n$ (see [19] 3.4).

It is well known that $\mathcal{M} := \mathcal{M}(\Gamma(1))$ is an algebraic stack for the etale topology on (Sch), of finite type, separated and smooth over $\mathrm{Spec}(\mathbf{Z})$ (for example see [11] III, Thm. 2.5). Let $\mathcal{P}$ denote $\Gamma(n)$, $\Gamma_1(n)$ or $\Gamma_0(n)$. Then the obvious morphism $\mathcal{M}(\mathcal{P}) \to \mathcal{M}$ is relatively representable, finite and flat and makes $\mathcal{M}(\mathcal{P})$ into an algebraic stack, regular and of dimension two, by [19], Thm. 5.1.1.

We can compactify $\mathcal{M}$ by adding a chart at infinity as in [15] I, §4 and IV, §5.5. For example one might take the degenerate elliptic curve over $\mathbf{Z}[t, (1 + 2^6 3^3 t)^{-1}]$ defined by:

$$y^2 + xy = x^3 - 2^2 3^2 tx - t,$$

which has $c_4 = 1 + 2^6 3^3 t$, $c_6 = -c_4$, $\Delta = t(1 + 2^6 3^3 t)^2$ and $j = t^{-1} + 2^6 3^3$ ([11] VI.1.6). This gives an algebraic stack $\overline{\mathcal{M}}$ which is proper and smooth over $\mathrm{Spec}(\mathbf{Z})$ and contains $\mathcal{M}$ as an open substack. We can then obtain compactifications $\overline{\mathcal{M}}(\mathcal{P})$ of the $\mathcal{M}(\mathcal{P})$ by normalizing $\overline{\mathcal{M}}$ in $\mathcal{M}(\mathcal{P})$ ([10] page 104). This procedure of compactifying is essentially the one followed in [19]. What happens at $\infty$ can be studied using the Tate curve ([19] 8.8).

In [11] Deligne and Rapoport give a moduli interpretation for $\overline{\mathcal{M}}$: it is equivalent to the category of generalized elliptic curves whose fibres are irreducible. They also give a moduli interpretation for the $\overline{\mathcal{M}}(\mathcal{P})$ over $\mathbf{Z}[1/n]$, and in some cases over $\mathbf{Z}$. Obvious candidates for moduli interpretations for the $\overline{\mathcal{M}}(\mathcal{P})$ are the categories $\overline{\mathcal{M}}(\mathcal{P})'$ defined as follows. For $\mathcal{P} = \Gamma(n)$, $\Gamma_1(n)$ or $\Gamma_0(n)$ the objects of $\overline{\mathcal{M}}(\mathcal{P})'$ are pairs $(E/S, \alpha)$, where $E/S$ is a generalized elliptic curve and $\alpha$ is a Drinfeld basis $(\mathbf{Z}/n\mathbf{Z})^2 \to E^{\mathrm{reg}}(S)$, or a $\mathbf{Z}/n\mathbf{Z}$-structure $\mathbf{Z}/n\mathbf{Z} \to E^{\mathrm{reg}}(S)$, or a cyclic subgroup scheme of rank $n$ of $E^{\mathrm{reg}}[n]$, such that in each case $\alpha$ meets all irreducible components of all geometric fibres of $E/S$. The $\overline{\mathcal{M}}(\mathcal{P})'$ are stacks for the fpqc topology because the relevant descent data are effective; use [11] III, Lemma 2.1 and show that the level structures descend too.

**2.1.2 Proposition.** *Let $n$ be a positive integer and let $\mathcal{P}$ denote $\Gamma(n)$, $\Gamma_1(n)$ or $\Gamma_0(n)$. For $\mathcal{P}=\Gamma(n)$ and for $\mathcal{P}=\Gamma_1(n)$ the obvious morphism $\mathcal{M}(\mathcal{P})\to\overline{\mathcal{M}}(\mathcal{P})'$ extends to an isomorphism $\overline{\mathcal{M}}(\mathcal{P})\xrightarrow{\sim}\overline{\mathcal{M}}(\mathcal{P})'$. The morphism $\mathcal{M}(\Gamma_0(n))\to\overline{\mathcal{M}}(\Gamma_0(n))'$ extends to an isomorphism $\overline{\mathcal{M}}(\Gamma_0(n))\to\overline{\mathcal{M}}(\Gamma_0(n))'$ if and only if $n$ is square free. In all cases $\overline{\mathcal{M}}(\mathcal{P})$ is regular.*

**Proof.** Let us first prove the assertions concerning $\Gamma_0(n)$. For $n$ square free one uses [11] V, Thm. 1.6. If $n$ is not square free we can write $n=p^{a+b}m$ with $p$ prime, $p\not| m$, $a\geq b>0$. We claim that $\overline{\mathcal{M}}(\Gamma_0(n))$ is not an algebraic stack because it has objects with infinitesimal automorphisms. Indeed, it is not hard to check that the standard $p^a$-gon over $\overline{\mathbf{F}}_p$, equipped with a $\overline{\mathcal{M}}(\Gamma_0(n))$-structure, has automorphism functor isomorphic to the group scheme $\mu_{p^a}\times\mathbf{Z}/2\mathbf{Z}$ over $\overline{\mathbf{F}}_p$.

The proofs in the two remaining cases $\mathcal{P}=\Gamma(n)$ and $\mathcal{P}=\Gamma_1(n)$ will be given in four steps. Two of these steps depend on certain computations concerning Tate curves that will be done in §2.2.

**Step 1:** the diagonal $\overline{\mathcal{M}}(\mathcal{P})'\longrightarrow\overline{\mathcal{M}}(\mathcal{P})'\times\overline{\mathcal{M}}(\mathcal{P})'$ is representable. Let $(E_1/X,\alpha_1)$ and $(E_2/X,\alpha_2)$ be in $\overline{\mathcal{M}}(\mathcal{P})'$. By [11] III, Thm. 2.5 the functor $\mathrm{Isom}_X(E_1,E_2)$ is representable. The functor $\mathrm{Isom}_X((E_1,\alpha_1),(E_2,\alpha_2))$ is represented by the closed subscheme of $\mathrm{Isom}_X(E_1,E_2)$ defined by the compatibility of the universal isomorphism with $\alpha_1$ and $\alpha_2$.

**Step 2:** there exists an etale surjective $\overline{X}\to\overline{\mathcal{M}}(\mathcal{P})'$ with $\overline{X}$ of finite type over $\mathbf{Z}$, regular and everywhere of dimension two. Using Prop. 2.2.1 and [11] III, Thm. 2.3 (including the two remarks following it) it is straightforward to verify that $\overline{\mathcal{M}}(\mathcal{P})'$ is an algebraic stack.

**Step 3:** the diagonal $\overline{\mathcal{M}}(\mathcal{P})'\longrightarrow\overline{\mathcal{M}}(\mathcal{P})'\times\overline{\mathcal{M}}(\mathcal{P})'$ is finite. Let $X\to\overline{\mathcal{M}}(\mathcal{P})'\times\overline{\mathcal{M}}(\mathcal{P})'$ be etale surjective. We have to show that $Y:=\mathrm{Isom}_X(p_1^*(E,\alpha),p_2^*(E,\alpha))\to X$ is finite. Because $Y\to X$ is quasi-finite it suffices to prove that it is proper. For that we use the valuative criterion for properness. Because $Y\to\overline{\mathcal{M}}(\mathcal{P})'$ is etale we only need to consider morphisms $\mathrm{Spec}(D)\to X$, $D$ a complete discrete valuation ring, under which the image of the generic point is in the interior of $X$. Then use [11] IV, Prop. 1.6.

**Step 4:** $\mathcal{M}(\mathcal{P})\to\overline{\mathcal{M}}(\mathcal{P})'$ extends to an isomorphism $\overline{\mathcal{M}}(\mathcal{P})\xrightarrow{\sim}\overline{\mathcal{M}}(\mathcal{P})'$. Let $\overline{Y}\to\overline{\mathcal{M}}$ and $\overline{X}\to\overline{\mathcal{M}}(\mathcal{P})'$ be etale surjective. Let $\overline{X}':=\overline{Y}\times_{\overline{\mathcal{M}}}\overline{X}$. Then $\overline{X}'\to\overline{\mathcal{M}}(\mathcal{P})'$ is also etale surjective. Let $Y$, $X$ and $X'$ denote the pullbacks of $\overline{Y}$, $\overline{X}$ and $\overline{X}'$ to $\mathcal{M}$. Let $Z:=Y\times_{\mathcal{M}}\mathcal{M}(\mathcal{P})$ and let $\overline{Z}\to\overline{Y}$ be the normalization of $Z\to Y$. So by definition $\overline{Z}\to\overline{\mathcal{M}}(\mathcal{P})$ is etale surjective. Because $\overline{X}'$ is a chart for $\overline{\mathcal{M}}(\mathcal{P})'$ it is regular, hence normal. Also $\overline{X}'\to\overline{Y}$ is everywhere dominant, so it factors uniquely as $\overline{X}'\to\overline{Z}\to\overline{Y}$. It follows from Prop. 2.2.1 that $\overline{X}'\to\overline{Z}$ is etale surjective. So now we have two charts $\overline{X}'\to\overline{\mathcal{M}}(\mathcal{P})$ and $\overline{X}'\to\overline{\mathcal{M}}(\mathcal{P})'$. The projections from $\overline{X}'\times_{\overline{\mathcal{M}}(\mathcal{P})}\overline{X}'$ to $\overline{X}'$ are etale and the morphism $\overline{X}'\times_{\overline{\mathcal{M}}(\mathcal{P})}\overline{X}'\to\overline{X}'\times\overline{X}'$ is finite. The same holds for $\overline{X}'\times_{\overline{\mathcal{M}}(\mathcal{P})'}\overline{X}'$, hence these two fibre products are equal (both being the normalization of $\overline{X}'\times\overline{X}'$ in $X'\times_{\mathcal{M}(\mathcal{P})}X'\to X'\times X'$). It is clear that the two "compositions" in the groupoid structures coincide. □

**2.2 Computations with Tate curves.** In this section we prove some results that are needed in the proof of Prop. 2.1.2. Let $\overline{\mathbf{G}}_m^q/q^{\mathbf{Z}}$ denote the Tate curve over $\mathbf{Z}[[q]]$ as constructed in

[11] VII, §1.10; it is a generalized elliptic curve whose restriction to $\mathrm{Spec}(\mathbf{Z}[[q]]/(q))$ is the standard 1-gon. The restriction of $\overline{\mathbf{G}}_m^q/q^{\mathbf{Z}}$ to $\mathbf{Z}((q))$ is the elliptic curve denoted $\mathrm{Tate}(q)$ in [19] §8.8. By definition, $\overline{\mathbf{G}}_m^q/q^{\mathbf{Z}}$ induces a formally etale morphism $\mathrm{Spec}(\mathbf{Z}[[q]]) \to \overline{\mathcal{M}}$. Let $n \geq 1$ and let $\mathcal{P}$ denote $\Gamma(n)$, $\Gamma_1(n)$ or $\Gamma_0(n)$. By construction (see [19] §8.11), $\mathrm{Spec}(\mathbf{Z}[[q]]) \times_{\overline{\mathcal{M}}} \overline{\mathcal{M}}(\mathcal{P})$ is the normalization of $\mathrm{Spec}(\mathbf{Z}[[q]])$ in $\mathrm{Spec}(\mathbf{Z}((q))) \times_{\mathcal{M}} \mathcal{M}(\mathcal{P})$.

**2.2.1 Proposition.** *Let $\alpha$ denote the universal level-$\mathcal{P}$ structure that $\overline{\mathbf{G}}_m^q/q^{\mathbf{Z}}$ acquires over $\mathrm{Spec}(\mathbf{Z}((q))) \times_{\mathcal{M}} \mathcal{M}(\mathcal{P})$. The object $(\overline{\mathbf{G}}_m^q/q^{\mathbf{Z}}, \alpha)$ of $\mathcal{M}(\mathcal{P})$ extends uniquely to an object over $\mathrm{Spec}(\mathbf{Z}[[q]]) \times_{\overline{\mathcal{M}}} \overline{\mathcal{M}}(\mathcal{P})$. The induced morphism $\mathrm{Spec}(\mathbf{Z}[[q]]) \times_{\overline{\mathcal{M}}} \overline{\mathcal{M}}(\mathcal{P}) \to \overline{\mathcal{M}}(\mathcal{P})'$ is formally etale for $\mathcal{P} = \Gamma(n)$ and for $\mathcal{P} = \Gamma_1(n)$.*

**Proof.** The uniqueness is clear. Apart from this the proof is a long computation that, in order to save space, we will give only for $\mathcal{P} = \Gamma_1(n)$.

Let us first compute $\mathrm{Spec}(\mathbf{Z}((q))) \times_{\mathcal{M}} \mathcal{M}(\Gamma_1(n))$. As in [19] §8.7 let $T[n]$ denote the $n$-torsion subgroup scheme of $\mathrm{Tate}(q)$ over $\mathbf{Z}((q))$ (see also [11] VII §1.13). Then $T[n]$ is given as the extension

$$0 \longrightarrow \mu_n \longrightarrow T[n] \longrightarrow \mathbf{Z}/n\mathbf{Z} \longrightarrow 0$$

for which the inverse image of $b$, $0 \leq b < n$, is the $\mu_n$-torsor given by the $n$th roots of $q^b$. It follows that $T[n]$, as a scheme, is the disjoint union of the schemes $T_b = \mathrm{Spec}(\mathbf{Z}((q))[X]/(X^n - q^b))$, with $0 \leq b < n$. The scheme $\mathrm{Spec}(\mathbf{Z}((q))) \times_{\mathcal{M}} \mathcal{M}(\Gamma_1(n))$ is the closed subscheme of $T[n]_{T[n]} = T[n] \times_{\mathbf{Z}((q))} T[n]$ over which the universal point (i.e., the diagonal) defines a $\mathbf{Z}/n\mathbf{Z}$-structure. Let $0 \leq b < n$ and let $P$ denote the tautological point in $T[n]_{T_b}(T_b)$. Let $d$ be the order of $b$ in $\mathbf{Z}/n\mathbf{Z}$, let $m = n/d$ and let $a = b/m$. It is now easy to prove that for $P$ to be a $\mathbf{Z}/n\mathbf{Z}$-structure is the same as for $dP$ to be a $\mathbf{Z}/m\mathbf{Z}$-structure, i.e., for $dP$ to be a primitive $m$th root of unity ([19] Thm. 1.12.9). We find (compare [19] Thm. 13.6.4):

$$\mathrm{Spec}(\mathbf{Z}((q))) \times_{\mathcal{M}} \mathcal{M}(\Gamma_1(n)) \quad = \coprod_{\substack{0<a<d|n \\ \gcd(a,d)=1}} \mathrm{Spec}(\mathbf{Z}((q))[X]/\Phi_m(X^d q^{-a})) \tag{2.2.2}$$

We may check the statements in Prop. 2.2.1 separately for each term of the disjoint union in 2.2.2. These terms are permuted by the action of $(\mathbf{Z}/n\mathbf{Z})^*$ on $\mathcal{M}(\Gamma_1(n))$ and we only need to consider one term in each orbit. Since $(\mathbf{Z}/n\mathbf{Z})^* \to (\mathbf{Z}/d\mathbf{Z})^*$ is surjective we may restrict our atttention to terms with $a = 1$. Let $d|n$ be given. We must compute the normalization of $\mathbf{Z}[[q]]$ in $\mathbf{Z}((q))[X]/\Phi_m(X^d q^{-1})$. Let $\zeta_m$ denote the image of $X^d q^{-1}$ in the last ring; by definition $\zeta_m$ is a primitive $m$th root of unity. We get an isomorphism

$$\mathbf{Z}((q))[X]/\Phi_m(X^d q^{-1}) \xrightarrow{\sim} \mathbf{Z}[\zeta_m]((X)) \qquad q \mapsto \zeta_m^{-1} X^d$$

It is clear that the normalization of $\mathbf{Z}[[q]]$ in $\mathbf{Z}[\zeta_m]((X))$ is equal to $\mathbf{Z}[\zeta_m][[X]]$. In particular the normalization is regular. By [11] VII, §1.4 and Construction 1.14, $\overline{\mathbf{G}}_m^q/q^{\mathbf{Z}}$ and $\overline{\mathbf{G}}_m^X/q^{\mathbf{Z}}$ are isomorphic over $\mathbf{Z}[\zeta_m]((X))$ and the tautological $\Gamma_1(n)$-structure $\alpha$ on $\overline{\mathbf{G}}_m^q/q^{\mathbf{Z}}$ corresponds to the point $X$ on $\overline{\mathbf{G}}_m^X/q^{\mathbf{Z}}$. It follows that $(\overline{\mathbf{G}}_m^X/q^{\mathbf{Z}}, 1 \mapsto X)$ is an object of $\overline{\mathcal{M}}(\Gamma_1(n))'$ extending $(\overline{\mathbf{G}}_m^q/q^{\mathbf{Z}}, \alpha)$. It remains to prove that the induced morphism $\mathrm{Spec}(\mathbf{Z}[\zeta_m][[X]]) \to \overline{\mathcal{M}}(\Gamma_1(n))'$ is formally etale.

Let $(E/\mathrm{Spec}(A), \alpha)$ be an object of $\overline{\mathcal{M}}(\Gamma_1(n))'$, with $A$ local artinian. We let $k$ denote the residue field of $A$ and we suppose that after the base change to $k$ the object $(E/A, \alpha)$ becomes isomorphic to the standard $d$-gon with a $\Gamma_1(n)$-structure such that $\alpha(1)$ lies on $\mathbf{P}^1 \times \{1\}$. For $N > 0$ we define a scheme $Z_N$ by:

$$Z_N \;=\; \mathrm{Isom}_{A \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_m][[X]]/(X^N)} \Big( (E/A, \alpha), (\overline{\mathbf{G}}_m^X/q^{\mathbf{Z}}, 1 \mapsto X) \Big)$$

We want to show that for $N$ large enough $Z_N \to \mathrm{Spec}(A)$ is etale surjective. The surjectivity is obvious since $Z_N \neq \emptyset$. The rest of the proof is done in two steps.
**Step1:** $Z_N \to \mathrm{Spec}(A)$ is unramified. We may suppose that $A = k$. Then we have to compare two objects of $\overline{\mathcal{M}}(\mathcal{P})'$ over $k[\zeta_m][[X]]/(X^N)$. The first one is $(E, \alpha)$, with $E$ the standard $d$-gon. The second one is $(\overline{\mathbf{G}}_m^X/q^{\mathbf{Z}}, 1 \mapsto X)$. Let us first consider the image of the non-smooth locus for both of them. Locally at a non-smooth point the first one is given by a local equation $uv = 0$, hence the image of the non-smooth locus is all of $\mathrm{Spec}(k[\zeta_m][[X]]/(X^N))$. A local equation for the second one at a non-smooth point is $uv = X$, hence the image of the non-smooth locus is given by the equation $X = 0$. It follows that $Z_N$ lies over $\mathrm{Spec}(k[\zeta_m])$, over which both generalized elliptic curves become isomorphic to the standard $d$-gon. Let us now use the $\Gamma_1(n)$-structures. In both cases we can interpret $d \cdot \alpha(1)$ as an element of $\mu_m(k[\zeta_m])$. In the first case $z := d \cdot \alpha(1)$ lies in $\mu_m(k)$ and in the second case $d \cdot \alpha(1) = \zeta_m$. We find that $Z_N$ lies over $\mathrm{Spec}(k[\zeta_m]/(\zeta_m - z))$, i.e., over $\mathrm{Spec}(k)$. It follows that $Z_N = \mathrm{Aut}_k(E/k, \alpha)$, which is easily seen to be either $\mathrm{Spec}(k)$ or $(\mathbf{Z}/2\mathbf{Z})_k$.
**Step2:** $Z_N \to \mathrm{Spec}(A)$ is etale for $N$ large enough. Let $Z' = \mathrm{Spec}(B)$ be a connected (i.e., irreducible) component of $Z_N$. We have to show that $Z' \to \mathrm{Spec}(A)$ is etale. After replacing $A$ by a finite etale extension we may suppose that $B$ has residue field $k$. The canonical map $k \to k \otimes_A B$ is then an isomorphism since $Z' \to \mathrm{Spec}(A)$ is unramified. By Nakayama's Lemma, the map $A \to B$ is surjective. It remains to show that $A \to B$ is injective for $N$ large enough. In order to do that it is sufficient to show that there exists a faithful flat extension $A \to A'$ such that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ factors through $Z' \to \mathrm{Spec}(A)$, i.e., such that $Z' \to \mathrm{Spec}(A)$ acquires a section. That we will do now.

After a faithful finite flat base change (obtained by extracting a $d$-th root of some element of $A^*$, to be precise) there exists a $\Gamma_1(d)$-structure $\beta$ on $E/\mathrm{Spec}(A)$. By definition, $\beta$ is an embedding of $(\mathbf{Z}/d\mathbf{Z})_A$ into $E^{\mathrm{reg}}$. Let $E'$ be the quotient of $E$ by the action of $\mathbf{Z}/d\mathbf{Z}$ via $\beta$. Then $E'/\mathrm{Spec}(A)$ is a deformation of the standard 1-gon. By [11] III, §1.4.2 and Lemme 1.4.3 $E'/\mathrm{Spec}(A)$ arises by pullback from $\overline{\mathbf{G}}_m^q/q^{\mathbf{Z}}$ over $\mathbf{Z}[[q]]$. By [11] II, Prop. 1.17 $(E/\mathrm{Spec}(A), \beta)$ arises by pullback from $(\overline{\mathbf{G}}_m^q/q^{d\mathbf{Z}}, q)$ over $\mathbf{Z}[[q]]$. One now checks easily that over $A[a]/(\Phi_m(a^d))$ the pair $(E, \alpha)$ arises from $(\overline{\mathbf{G}}_m^q/q^{d\mathbf{Z}}, a \cdot q)$. But this last one is isomorphic to $(\overline{\mathbf{G}}_m^X/q^{\mathbf{Z}}, X)$ via

$$\mathbf{Z}[\zeta_m][[X]] \longrightarrow \mathbf{Z}[[q]][a]/\Phi_m(a^d) \qquad \zeta_m \mapsto a^{-d} \qquad X \mapsto qa$$

This finishes the proof of Prop. 2.2.1. $\square$

**2.3 Modular forms and $q$-expansions.** Let $N$ and $k$ be positive integers. Let $R$ be a ring in which $N$ is invertible. We define the $R$-module of modular forms over $R$ of level $N$ and weight

$k$ as:

$$M(N,k)_R := \mathrm{H}^0(\overline{\mathcal{M}}(\Gamma_1(N))_R, \underline{\omega}^{\otimes k}) \tag{2.3.1}$$

where $\underline{\omega}$ is the invertible sheaf $0^*\Omega^1_{\mathbf{E}/\overline{\mathcal{M}}}$, or, more precisely, its pullback to $\overline{\mathcal{M}}(\Gamma_1(N))_R$. In more down-to-earth terms this means that a modular form $f$ of type $(N,k)$ over $R$ is a rule, compatible with cartesian squares, that associates to each $(E/S/R,\alpha)$ in $\overline{\mathcal{M}}(\Gamma_1(N))_R$ a section $f(E/S,\alpha)$ of $\underline{\omega}^{\otimes k}_{E/S}$. Of course we could have defined $M(N,k)_R$ without $N$ being invertible in $R$, but for example for $R$ the ring of integers in a number field, $\overline{\mathcal{M}}(\Gamma_1(N))_R$ is not always the best possible model.

If $N>4$ then the stack $\overline{\mathcal{M}}(\Gamma_1(N))_R$ is represented by the curve $X_1(N)_R$ and modular forms of type $(N,k)$ over $R$ are just global sections of the invertible sheaf $\underline{\omega}^{\otimes k}$ on $X_1(N)_R$. For general $N$ we have the following description of $M(N,k)_R$. Let $n\geq 3$, then $\overline{\mathcal{M}}(\Gamma(n),\Gamma_1(N))_{R[1/n]}$ is represented by a (possibly non-connected) smooth proper curve over $R[1/n]$, say $X$, on which $\mathrm{GL}_2(\mathbf{Z}/n\mathbf{Z})$ acts. Then $M(N,k)_R\otimes_R R[1/n]$ can be identified with the submodule of $\mathrm{GL}_2(\mathbf{Z}/n\mathbf{Z})$-invariant sections of $\underline{\omega}^{\otimes k}$ over $X$.

When $R$ contains the $N$th roots of unity one obtains the $q$-expansions of $f$ at the various cusps of $\overline{\mathcal{M}}(\Gamma_1(N))_R$ by evaluating $f$ on pairs $(\mathrm{Tate}(q^d),\alpha)$, where $\mathrm{Tate}(q^d)$ is the Tate curve $\mathbf{G}_m/q^{d\mathbf{Z}}$ over $R[[q]](q^{-1})$ and $d|N$. Explicitly: the $q$-expansion $f_{d,\alpha}(q)$ of $f$ at the cusp $(\mathrm{Tate}(q^d),\alpha)$ is the power series $f(\mathrm{Tate}(q^d),\alpha)/(dt/t)^{\otimes k}$ in $R[[q]]$. A form $f$ is called a cusp form if all its $q$-expansions have constant term equal to zero.

The group $(\mathbf{Z}/N\mathbf{Z})^*$ acts on $\overline{\mathcal{M}}(\Gamma_1(N))$ by:

$$\langle a\rangle:\ (E/S,\alpha)\mapsto(E/S,a\alpha), \tag{2.3.2}$$

for $a\in(\mathbf{Z}/N\mathbf{Z})^*$. This gives an action by $(\mathbf{Z}/N\mathbf{Z})^*$ on modular forms:

$$(\langle a\rangle^*f)(E/S,\alpha)=f(E/S,a\alpha). \tag{2.3.3}$$

Let $\varepsilon\colon(\mathbf{Z}/N\mathbf{Z})^*\to R^*$ be a character. A modular form $f$ of type $(N,k)$ is said to be of type $(N,k,\varepsilon)$ if $\langle a\rangle^*f=\varepsilon(a)f$ for all $a\in(\mathbf{Z}/N\mathbf{Z})^*$. Of course, we always have $\varepsilon(-1)f=(-1)^kf$. We denote the $R$-module of forms of type $(N,k,\varepsilon)$ by $M(N,k,\varepsilon)_R$, and its submodule of cusp forms by $M^o(N,k,\varepsilon)_R$.

For more details concerning modular forms in this setting see [18], Chapter 1 or [11], VII, §3.

**2.4 Hecke operators.** The $R$-modules $M(N,k)_R$ with $k\geq 2$ are equipped with certain endomorphisms, called Hecke operators. For each $n\geq 1$ we have an operator $T_n^*$. The algebra generated by all $T_n^*$ and $\langle a\rangle^*$ acting on $M(N,k)_R$ is commutative and generated by the $T_l^*$ for $l$ prime and the $\langle a\rangle^*$. It follows that the $T_n^*$ also act on the submodules $M(N,k,\varepsilon)_R$. For a construction of the $T_n^*$ we refer to [18], §1.11; the hypothesis "$k\geq 2$" is needed to use base change: $M(N,k)_R=M(N,k)_{\mathbf{Z}[1/N]}\otimes R$. There is no problem to construct $T_n^*$ on forms of weight one if $n$ is invertible in $R$. The action of $T_n^*$ on $M(N,k)_{\mathbf{Q}}$ $(k\geq 1)$ is given by:

$$(T_n^*f)(E/\overline{\mathbf{Q}},\alpha)=n^{-1}\sum_\phi \phi^*(f(E'/\overline{\mathbf{Q}},\phi\alpha)) \tag{2.4.1}$$

where $\phi: E \to E'$ ranges over the isogenies of degree $n$ with source $E$ such that $\phi$ is injective on the image of $\alpha$. If $f$ in $M(N,k,\varepsilon)_R$ has $q$-expansion $\sum a_n q^n$ at $(\mathrm{Tate}(q), 1 \mapsto \zeta_N)$ then we have for $p$ prime:

$$(T_p^* f)(\mathrm{Tate}(q), 1 \mapsto \zeta_N) \;=\; \sum_n \left(a_{np} + \varepsilon(p) p^{k-1} a_{n/p}\right) q^n \, (dt/t)^{\otimes k} \tag{2.4.2}$$

with the conventions that $a_{n/p} = 0$ if $p \nmid n$ and $\varepsilon(p) = 0$ if $p|N$.

**2.5 Galois representations.** Let $f$ be a cusp form over $\overline{\mathbf{Q}}$ of some type $(N,k,\varepsilon)$ with $k \geq 2$ and suppose that $f$ is a common eigenvector for all the $T_p^*$, $p \nmid N$, with eigenvalues $a_p \in \overline{\mathbf{Q}}$. Let $K \subset \overline{\mathbf{Q}}$ be the field generated by the $a_p$, $p \nmid N$, and the $\varepsilon(a)$, $a \in (\mathbf{Z}/N\mathbf{Z})^*$; then $[K : \mathbf{Q}] < \infty$. Let $\lambda$ be a finite place of $K$ and let $l$ be its residue characteristic. Let $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ denote the absolute Galois group of $\mathbf{Q}$. According to a theorem of Deligne ([12], Thm. 6.1; see also [5]) there exists a unique (up to isomorphism) continuous semi-simple representation $\rho_\lambda: G_{\mathbf{Q}} \to \mathrm{GL}_2(K_\lambda)$ which is unramified outside $Nl$ and has the property that $\mathrm{trace}(\rho_\lambda(\mathrm{Frob}_p)) = a_p$ and $\det(\rho_\lambda(\mathrm{Frob}_p)) = \varepsilon(p)p^{k-1}$ for all $p \nmid Nl$. In these equalities $\rho_\lambda(\mathrm{Frob}_p)$ denotes the image of a Frobenius element at $p$ (unique up to conjugation). We will briefly describe a construction of $\rho_\lambda$; detailed constructions can be found in [5], [20] and [24].

First of all we may replace $f$ by the unique new form with eigenvalues $a_p$ for $p \nmid N$ (see [8], §2.4 and [2]); the field $K$ remains the same, the level $N$ is replaced by a divisor of it. Now $f$ is an eigenform for all $T_n^*$, $n \geq 1$, with eigenvalues $a_n \in K$, of type $(N,k,\varepsilon)$. Note that the $a_n$ $(n \geq 1)$ generate $K$ since $f$ is defined over the field generated by the $a_n$ and is an eigenvector for the $\langle a \rangle^*$.

Let $Y$ denote $\mathcal{M}(\Gamma_1(N))_{\mathbf{Q}}$ and let $\pi: \mathbf{E} \to Y$ be the universal elliptic curve. We have the sheaf $\mathcal{F}^k := \mathrm{Sym}^{k-2}(\mathrm{R}^1\pi_*\mathbf{Q})$ on $Y(\mathbf{C})$ and the sheaf $\mathcal{F}_l^k := \mathrm{Sym}^{k-2}(\mathrm{R}^1\pi_*\mathbf{Q}_l)$ on $Y_{\mathrm{et}}$. There is the Shimura isomorphism ([7], Thm. 2.10):

$$\mathrm{H}^1_{\mathrm{par}}(Y(\mathbf{C}), \mathcal{F}^k) \otimes_{\mathbf{Q}} \mathbf{C} \;\xrightarrow{\sim}\; M^o(N,k)_{\mathbf{C}} \oplus \overline{M^o(N,k)_{\mathbf{C}}} \tag{2.5.1}$$

where $\mathrm{H}_{\mathrm{par}}$ denotes "parabolic" cohomology: the image of the cohomology with compact support in the ordinary cohomology. This isomorphism is compatible with the $T_n^*$ and $\langle a \rangle^*$ acting on both sides in the usual way (see [7], Prop. 3.18).

Next let $H$ be the opposite of the $\mathbf{Q}$-algebra in $\mathrm{End}_{\mathbf{Q}}(M^o(N,k)_{\mathbf{Q}})$ generated by the $T_n^*$ and the $\langle a \rangle^*$. By definition, $M^o(N,k)_{\mathbf{Q}}$ is a right $H$-module and we denote by $h^*$ the endomorphism induced by $h \in H$. The algebra $H$ is commutative and of finite dimension over $\mathbf{Q}$. Let $M^o(N,k)_{\mathbf{Q}}^\vee$ be the left $H$-module $\mathrm{Hom}_{\mathbf{Q}}(M^o(N,k)_{\mathbf{Q}}, \mathbf{Q})$; say that $h \in H$ induces $h_* := h^{*\vee}$. We claim that $M^o(N,k)_{\mathbf{Q}}^\vee$ is free of rank 1 as $H$-module. Namely, it is faithful and for each maximal ideal $m$ of $H$, $M^o(N,k)_{\mathbf{Q}}^\vee / m M^o(N,k)_{\mathbf{Q}}^\vee = (M^0(N,k)[m])^\vee$ is a 1-dimensional $H/m$-vector space because of $q$-expansions. It follows that $\mathrm{H}^1_{\mathrm{par}}(Y(\mathbf{C}), \mathcal{F}^k)^\vee \otimes_{\mathbf{Q}} \mathbf{C}$ is a free $H \otimes_{\mathbf{Q}} \mathbf{C}$-module of rank 2. Hence $\mathrm{H}^1_{\mathrm{par}}(Y(\mathbf{C}), \mathcal{F}^k)^\vee$ is a free $H$-module of rank 2 and $\mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)^\vee$ is a free $H \otimes_{\mathbf{Q}} \mathbf{Q}_l$-module of rank 2. Let $\phi: H \to \overline{\mathbf{Q}}$ be the ring homomorphism such that for all $h \in H$: $h^*(f) = \phi(h)f$. Let $m := \ker(\phi)$; then $K = \mathrm{im}(\phi) = H/m$ and we have a surjection

$H\otimes_{\mathbf{Q}}\mathbf{Q}_l \to K_\lambda$. Let $V_\lambda := \mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)^\vee \otimes_{H\otimes \mathbf{Q}_l} K_\lambda$. Then $V_\lambda$ is a 2-dimensional $K_\lambda$-vector space with a continuous action of $G_{\mathbf{Q}}$. To be precise: we let $\sigma \in G_{\mathbf{Q}}$ act on $\mathrm{H}^1_{\mathrm{par}}(Y \times_{\mathrm{Spec}\mathbf{Q}} \mathrm{Spec}\overline{\mathbf{Q}}, \mathcal{F}_l^k)^\vee$ by $(\mathrm{id}\times \mathrm{Spec}(\sigma^{-1}))^{*\vee}$ (note that this is covariant in $\sigma$).

The action of $G_{\mathbf{Q}}$ on $V_\lambda$ is unramified at $p \not| Nl$ because $\pi\colon \mathbf{E}\to Y$ has good reduction at such $p$. Congruence formulas as in [7], §4 show that $\mathrm{Frob}_p$ $(p\not| Nl)$ satisfies the polynomial $X^2 - a_pX + \varepsilon(p)p^{k-1}$. In order to show that this polynomial is the characteristic polynomial of $\mathrm{Frob}_p$ it suffices to prove that $\mathrm{Frob}_p$ has determinant $\varepsilon(p)p^{k-1}$. This one can do by computing the action of $\mathrm{Frob}_p$ on some non-degenerate alternating $K_\lambda$-bilinear form on $V_\lambda$. Such a form is found as follows.

The Weil pairing on $\pi\colon \mathbf{E}\to Y$ gives a non-degenerate alternating pairing on $\mathrm{R}^1\pi_*\mathbf{Q}_l$ with values in the sheaf $\mathbf{Q}_l(-1)$ on $Y$. This induces a non-degenerate pairing on $\mathcal{F}_l^k$ with values in $\mathbf{Q}_l(2-k)$; this pairing is symmetric if $k$ is even and alternating if $k$ is odd. As explained in [7], §3.20, Poincaré duality then gives a non-degenerate $\mathbf{Q}_l$-bilinear form $(\cdot|\cdot)$ on $\mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)^\vee$ with values in $\mathbf{Q}_l(k-1)$; one has $(y|x) = (-1)^{k-1}(x|y)$ for all $x$ and $y$. Let $w_{\zeta_N}$ be the automorphism of $Y_{\mathbf{Q}(\zeta_N)} = \mathcal{M}(\Gamma_1(N))_{\mathbf{Q}(\zeta_N)}$ defined by:

$$w_{\zeta_N}\colon (E/S/\mathbf{Q}(\zeta_N), \alpha) \mapsto ((E'/S/\mathbf{Q}(\zeta_N), \alpha') \tag{2.5.2}$$

where $\beta\colon E\to E'$ is the $N$-isogeny with $\ker(\beta) = \mathrm{im}(\alpha)$, $\mathrm{im}(\alpha') = \ker(\beta^t)$ and $\langle \alpha(1), \alpha'(1)\rangle_\beta = \zeta_N$. Here $\beta^t\colon E'\to E$ denotes the dual of $\beta$ and $\langle\cdot,\cdot\rangle_\beta$ denotes the pairing of [19], §2.8. The use of $w_{\zeta_N}$ is that if $T_n^t$ denotes the correspondence (on $Y_{\mathbf{Q}(\zeta_N)}$) dual to $T_n$, we have $T_n^t = w_{\zeta_N} T_n w_{\zeta_N}^{-1}$ for all $n\geq 1$ and $\langle a\rangle^{-1} = w_{\zeta_N}\langle a\rangle w_{\zeta_N}^{-1}$ for all $a\in (\mathbf{Z}/N\mathbf{Z})^*$. We let $w_{\zeta_N}$ act on $\mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)$ by:

$$w_{\zeta_N}^*\colon\ \mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k) \xrightarrow{w_{\zeta_N}^*} \mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, w_{\zeta_N}^*\mathcal{F}_l^k) \xrightarrow{\mathrm{Sym}^{k-2}\beta^*} \mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k) \tag{2.5.3}$$

where $\beta\colon \mathbf{E}\to w_{\zeta_N}^*\mathbf{E}$; we let $w_{\zeta_N *} := w_{\zeta_N}^{*\vee}$. For $h\in H$ let $h_*^t$ denote the endomorphism of $\mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)^\vee$ which is dual to $h_*$ with respect to $(\cdot|\cdot)$. Then we have $h_*^t = w_{\zeta_N *} h_* w_{\zeta_N *}^{-1}$ for all $h\in H$ and $w_{\zeta_N *}^t = (-1)^k w_{\zeta_N *}$. In other words, all $h_*$ are self-adjoint with respect to the non-degenerate alternating bilinear form $\langle x|y\rangle := (x|w_{\zeta_N *}y)$ on $\mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)^\vee$.

The $K_\lambda$-vector space $V_\lambda$ is canonically isomorphic to $(\mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)^\vee \otimes_{\mathbf{Q}_l} K_\lambda)\otimes_{H\otimes_{\mathbf{Q}} K_\lambda} K_\lambda$, where $H\otimes_{\mathbf{Q}} K_\lambda \to K_\lambda$: $a\otimes b\mapsto \phi(a)b$. We extend $\langle\cdot|\cdot\rangle$ to a $K_\lambda$-bilinear form, still denoted by $\langle\cdot|\cdot\rangle$, on $\mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)^\vee \otimes_{\mathbf{Q}_l} K_\lambda$. Since $f$ is a new form the quotient $K = H/mH$ of $H$ is actually a direct factor: $H = K\times H'$ as rings, for some (unique) $H'$. It follows that $H\otimes_{\mathbf{Q}} K_\lambda = K_\lambda\times H''$ and that $\mathrm{H}^1_{\mathrm{par}}(Y_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)^\vee\otimes_{\mathbf{Q}_l} K_\lambda = V_\lambda\oplus V_\lambda'$. This direct sum decomposition is orthogonal with respect to $\langle\cdot|\cdot\rangle$ because the corresponding idempotents, lying in $H\otimes_{\mathbf{Q}} K_\lambda$, are self-adjoint. It follows that the restriction of $\langle\cdot|\cdot\rangle$ to $V_\lambda$ has all the properties we want, i.e., it is $K_\lambda$-bilinear, non-degenerate and alternating. One easily checks that $\mathrm{Frob}_p$ acts on it as desired.

# 3 Fourier expansions of new forms at various cusps.

**3.1 The general problem.** Let $M$ be a positive integer. Let $f$ be a new form on $\Gamma_1(M)$ of some weight $k$, with character $\varepsilon$ and with coefficients in $\overline{\mathbf{Q}}$. Then we ask the following question: *how are the $q$-expansions of $f$ at the various cusps related?* For $M$ square free this problem has been solved by Asai in [1]. His method uses the Atkin-Lehner operators $W_d$ and the diamond operators $\langle a \rangle$. It works because for $M$ square free these operators permute the cusps transitively, but for general $M$ this is not necessarily true. We will solve the problem in a situation described below. It might well be that the general case can be elegantly dealt with using some explicit information concerning the Kirillov models for the local factors of the automorphic representation attached to $f$.

**3.2 Our problem.** Let $N$ be a positive integer, $p$ a prime not dividing $N$ and $f$ a new form on $X_0 := \overline{\mathcal{M}}(\Gamma_0(p^2), \Gamma_1(N))_{\overline{\mathbf{Q}}}$ of some weight $k$ and character $\varepsilon \colon (\mathbf{Z}/N\mathbf{Z})^* \to \overline{\mathbf{Q}}^*$. We want to relate the $q$-expansions of $f$ at the $p{+}1$ cusps of $X_0$ lying over the cusp $(\mathrm{Tate}(q), 1{\mapsto}\zeta_N)$ of $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{Q}}}$, where $\zeta_N$ is a fixed $N$th root of unity. The most natural way to do this seems to consist of first passing to a cover $X$ of $X_0$ on which a group $G$ acts, permuting the relevant cusps transitively, and then applying the representation theory of $G$. In our case we take $X = \overline{\mathcal{M}}(\Gamma(p), \Gamma_1(N))_{\overline{\mathbf{Q}}}$. Note that by definition 2.1.1 a $\Gamma(p)$-structure is not required to be symplectic. The group $G$ will then be $\mathrm{GL}_2(\mathbf{F}_p)$ and not its subgroup $\mathrm{SL}_2(\mathbf{F}_p)$. We prefer to work with $\mathrm{GL}_2(\mathbf{F}_p)$ because its representation theory is less complicated.

Let $\pi \colon X \to X_0$ be the morphism mapping $(E/S, \phi, \alpha)$ to $(E_1/S, \ker(\phi_2 \circ \phi_1^t), \phi_1\alpha)$ where $\phi_1 \colon E \to E_1$ and $\phi_2 \colon E \to E_2$ are the $p$-isogenies with $\ker(\phi_1) = \langle \phi(1,0) \rangle$ and $\ker(\phi_2) = \langle \phi(0,1) \rangle$ (see [19] 11.3.5). We let $g$ in $G = \mathrm{Aut}(\mathbf{F}_p{}^2)$ act from the right on $X$ by sending $(E/S, \phi, \alpha)$ to $(E/S, \phi \circ g, \alpha)$. Then $\pi$ is the quotient map for the action of the diagonal subgroup $T$ of $G$. For $f_0$ in $\mathrm{H}^0(X_0, \underline{\omega}^{\otimes k})$ we define $\pi^* f_0$ in $\mathrm{H}^0(X, \underline{\omega}^{\otimes k})$ by $(\pi^* f_0)(E/S, \phi, \alpha) = \phi_1^*(f_0(E_1/S, \ker(\phi_2 \circ \phi_1^t), \phi_1\alpha))$. We define operators $T_l^*$ for $l \neq p$ and $\langle a \rangle^*$ for $a$ in $(\mathbf{Z}/N\mathbf{Z})^*$ on $\mathrm{H}^0(X, \underline{\omega}^{\otimes k})$ in the obvious way. The $T_l^*$ and the $\langle a \rangle^*$ commute with the action (from the left) of $G$. The injection $\pi^*$ of $\mathrm{H}^0(X_0, \underline{\omega}^{\otimes k})$ into $\mathrm{H}^0(X, \underline{\omega}^{\otimes k})$ intertwines the $T_l^*$ and the $\langle a \rangle^*$ on both sides. Now $\pi^* f$ generates a representation of $G$, namely the linear span $V$ of the $g\pi^* f$, $g \in G$.

**3.2.1 Lemma.** *The subspace of $T$-invariants $V^T$ of $V$ has basis $\pi^* f$ and $V$ is irreducible.*

**Proof.** The first statement follows from the multiplicity one principle on $X_0$ applied to the $T_l^*$, $l \neq p$, and the $\langle a \rangle^*$ (recall that $f$ is a new form so that we don't need $T_p^*$). Write $V = \oplus_1^r V_i$ with all $V_i$ irreducible subrepresentations. Then $\pi^* f = \sum v_i$ with $v_i \in V_i$. Since $\pi^* f$ generates $V$ we have $v_i \neq 0$ for all $i$, hence $\dim V^T \geq r$. $\square$

Let us now consider $q$-expansions. The Tate curve $\mathrm{Tate}(q) = \mathbf{G}_m / q^{\mathbf{Z}}$ over $\overline{\mathbf{Q}}((q))$ has its $p$-torsion rational over $\overline{\mathbf{Q}}((q^{1/p}))$: the $p$-torsion points are the $\zeta_p^a q^{b/p}$ where $\zeta_p$ is a fixed $p$th root of unity and $a$, $b$ are in $\mathbf{F}_p$. It follows that the set of $\Gamma(p)$-structures $\phi$ on $\mathrm{Tate}(q)$ over $\overline{\mathbf{Q}}((q^{1/p}))$

is a $G$-torsor where $g \in G$ acts by $\phi \mapsto \phi \circ g$. We have an evaluation map:

$$\mathrm{H}^0(X, \underline{\omega}^{\otimes k}) \longrightarrow \oplus_\phi \overline{\mathbf{Q}}[[q^{1/p}]], \quad f \mapsto \Big(f(\mathrm{Tate}(q), \phi, 1 \mapsto \zeta_N)/(dt/t)^{\otimes k}\Big)_\phi \tag{3.2.2}$$

We let $G$ act on $\oplus_\phi \overline{\mathbf{Q}}[[q^{1/p}]]$ by permuting the factors: an element $g$ in $G$ sends $(\sum a_{n,\phi} q^{n/p})_\phi$ to $(\sum a_{n,\phi} q^{n/p})_{\phi \circ g^{-1}}$; this action is compatible with the map 3.2.2. We identify the Galois group of $\overline{\mathbf{Q}}((q^{1/p}))$ over $\overline{\mathbf{Q}}((q))$ with $\mathbf{F}_p$ by letting $\lambda \in \mathbf{F}_p$ send $q^{1/p}$ to $\zeta_p^\lambda q^{1/p}$. Then $\lambda \in \mathbf{F}_p$ induces the automorphism $\sigma(\lambda)$ of the $p$-torsion of $\mathrm{Tate}(q)$ given by: $\zeta_p^a q^{b/p} \mapsto \zeta_p^{a+\lambda b} q^{b/p}$. Suppose that $f(\mathrm{Tate}(q), \phi, 1 \mapsto \zeta_N) = \sum a_{n,\phi} q^{n/p} (dt/t)^{\otimes k}$ and let $\lambda \in \mathbf{F}_p$. Base changing by $\overline{\mathbf{Q}}((q^{1/p})) \to \overline{\mathbf{Q}}((q^{1/p}))$, $q^{1/p} \mapsto \zeta_p^\lambda q^{1/p}$ then gives: $f(\mathrm{Tate}(q), (\sigma\lambda) \circ \phi, 1 \mapsto \zeta_N) = \sum a_{n,\phi} \zeta_p^{\lambda n} q^{n/p} (dt/t)^{\otimes k}$. In other words, the image of the map 3.2.2 is contained in the subspace consisting of $(\sum a_{n,\phi} q^{n/p})_\phi$ such that $a_{n,(\sigma\lambda)\phi} = \zeta_p^{\lambda n} a_{n,\phi}$ for all $n$, $\lambda$ and $\phi$. Note that both the condition on the image of $f$ we just found and the action of $G$ do not mix terms of different degrees in $q^{1/p}$, so that it makes sense to project on the degree $n$ parts.

For $n \in \mathbf{Z}$ let $W_n$ be the space of functions $h\colon G \to \overline{\mathbf{Q}}$ such that $h(\left(\begin{smallmatrix}1 & a\\ 0 & 1\end{smallmatrix}\right)x) = \zeta_p^{na} h(x)$ for all $a \in \mathbf{F}_p$. We let $G$ act on $W_n$ by right translations. Also let $\phi_0$ be the $\Gamma(p)$-structure on $\mathrm{Tate}(q)$ with $\phi_0(a,b) = \zeta_p^a q^{b/p}$. Then we can restate the results above as the following proposition.

**3.2.3 Proposition.** *For $f \in \mathrm{H}^0(X, \underline{\omega}^{\otimes k})$ let us define the $a(f,n,\phi)$ by*

$$f(\mathrm{Tate}(q), \phi, 1 \mapsto \zeta_N) = \Big(\sum a(f,n,\phi) q^{n/p}\Big) (dt/t)^{\otimes k}.$$

*Then, for all $n$, mapping $f$ to the function $x \mapsto a(f,n,\phi_0 \circ x)$ on $G$ gives a $G$-equivariant map $\mathrm{H}^0(X, \underline{\omega}^{\otimes k}) \to W_n$.* $\square$

The usefulness of this result clearly depends on the multiplicity of the irreducible representation $V$ in $W_n$.

**3.3 Applying the representation theory of $\mathrm{GL}_2(\mathbf{F}_p)$.** The notation is as in §3.2. We have $G = \mathrm{GL}_2(\mathbf{F}_p)$, $B \subset G$ is the upper triangular Borel subgroup, $T$ is the diagonal subgroup and $U$ is the subgroup of order $p$ of $B$. First of all we need a character table for $G$. The one given below is copied from [6].

**3.3.1 Table.** *The character table of* $\mathrm{GL}_2(\mathbf{F}_p)$.

| conjugacy class of | $\pi(\alpha,\beta),\ \alpha \neq \beta$ | $\pi(\Lambda),\ \Lambda^p \neq \Lambda$ | $\alpha \circ \det$ | $\pi^-(\alpha)$ |
|---|---|---|---|---|
| $\left(\begin{smallmatrix}x & 0\\ 0 & x\end{smallmatrix}\right)$ $x \in \mathbf{F}_p^*$ | $(p+1)\alpha(x)\beta(x)$ | $(p-1)\Lambda(x)$ | $\alpha(x)^2$ | $p\alpha(x)^2$ |
| $\left(\begin{smallmatrix}x & 0\\ 0 & y\end{smallmatrix}\right)$ $x, y \in \mathbf{F}_p^*$ $x \neq y$ | $\alpha(x)\beta(y) + \alpha(y)\beta(x)$ | $0$ | $\alpha(x)\alpha(y)$ | $\alpha(x)\alpha(y)$ |
| $\left(\begin{smallmatrix}x & 1\\ 0 & x\end{smallmatrix}\right)$ $x \in \mathbf{F}_p^*$ | $\alpha(x)\beta(x)$ | $-\Lambda(x)$ | $\alpha(x)^2$ | $0$ |
| $\left(\begin{smallmatrix}z & 0\\ 0 & z^p\end{smallmatrix}\right)$ $z \in \mathbf{F}_{p^2}^*$ $z^p \neq z$ | $0$ | $-\Lambda(z) - \Lambda(z^p)$ | $\alpha(z^{p+1})$ | $-\alpha(z^{p+1})$ |

In this table $\alpha$ and $\beta$ denote characters $\mathbf{F}_p^* \to \overline{\mathbf{Q}}^*$ and $\Lambda$ denotes a character $\mathbf{F}_{p^2}^* \to \overline{\mathbf{Q}}^*$. As is well known, the representation $W_n$ is isomorphic to the induced representation $\mathrm{Ind}_U^G \rho_n$, where $\rho_n$ is the one-dimensional representation $\rho_n(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}) = \zeta_p^{na}$ of $U$. It is now a trivial matter to compute the multiplicities $\dim \mathrm{Hom}_G(V, W_n) = \dim \mathrm{Hom}_U(\mathrm{Res}_U^G V, \rho_n)$ and the dimensions $\dim V^T$. One finds the following table.

**3.3.2 Table.**

| $V$ | $\pi(\alpha,\beta),\ \ \alpha \neq \beta$ | $\pi(\Lambda),\ \ \Lambda^p \neq \Lambda$ | $\alpha\circ\det$ | $\pi^-(\alpha)$ |
|---|---|---|---|---|
| $\dim \mathrm{Hom}_G(V, W_n)$ | 1 *if* $p \not| n$<br>2 *if* $p \mid n$ | 1 *if* $p \not| n$<br>0 *if* $p \mid n$ | 0 *if* $p \not| n$<br>1 *if* $p \mid n$ | 1 |
| $\dim V^T$ | $\delta(\alpha\beta, 1)$ | $\delta(\Lambda\vert_{\mathbf{F}_p^*}, 1)$ | $\delta(\alpha, 1)$ | $\delta(\alpha, 1) + \delta(\alpha^2, 1)$ |

In this table the symbol $\delta$ denotes the function that takes the value 1 or 0 according to when its two arguments are equal or not.

Let us go back to the situation of §3.2. So $f$ is a new form on $X_0$, $V$ is the irreducible subrepresentation of $\mathrm{H}^0(X, \underline{\omega}^{\otimes k})$ generated by $\pi^* f$ and $\pi^* f$ is a basis for $V^T$. Looking at the last row of Table 3.3.2 one sees that $V \cong \pi(\alpha, \beta)$ implies $\beta = \alpha^{-1}$ and $\alpha \neq 1$, that $V \cong \pi(\Lambda)$ implies $\Lambda|_{\mathbf{F}_p^*} = 1$, that $V \cong \alpha\circ\det$ is impossible and that $V \cong \pi^-(\alpha)$ implies that $\alpha$ has order two. Let $n$ be a positive integer and let $h_n \in W_n$ be given by $h_n(x) = a(\pi^* f, n, \phi_0 \circ x)$ as in Prop. 3.2.3. Looking at the second row in Table 3.3.2 one sees that $h_n$ is determined up to scalar multiple by the conditions that it is $T$-invariant and that it generates a quotient representation of $V$, except in the case where $V \cong \pi(\alpha, \beta)$ and $p|n$. To get such a function $h'_n$ we can use the central idempotents in the group rings of $G$, $T$ and $U$ corresponding to the representations we want $h'_n$ to be in. The central idempotent $e_\chi$ associated to an irreducible character $\chi$ of a finite group $G$ is given by $e_\chi = \chi(1)/(\#G) \sum_y \chi(y^{-1})\delta_y$, where the sum ranges over the elements of $G$ (we view the group ring of $G$ as the ring $\overline{\mathbf{Q}}[G]$ of functions on $G$ with convolution as multiplication). From now on let $\chi$ be the character of $G$ on $V$. We can start with an arbitrary element of $\overline{\mathbf{Q}}[G]$, project it into $W_n$, then into its $\chi$-part and finally into the $T$-invariants. Starting with $\delta_e$, the delta function at the unit element of $G$, gives us the function

$$h'_n(x) = \sum_{t \in T, u \in U} \rho_n(u^{-1}) \chi(uxt) \tag{3.3.3}$$

By construction we have that $h'_n \neq 0$ if $\dim \mathrm{Hom}_G(V, W_n) \neq 0$.

**3.3.4 Proposition.** *In the situation of §3.2 we define the* $a(n, x)$ *by:*

$$\pi^* f(\mathrm{Tate}(q), \phi_0 \circ x, 1 \mapsto \zeta_N) = \left(\sum a(n, x) q^{n/p}\right) (dt/t)^{\otimes k}.$$

*Suppose that* $p \not| n$ *if* $\chi = \pi(\alpha, \alpha^{-1})$ *for some* $\alpha$. *Then the vector* $(a(n, x))_{x \in G}$ *is a multiple of the vector* $(h'_n(x))_{x \in G}$ *as defined in 3.3.3.* $\square$

Using this proposition we can almost solve the problem posed in the beginning of §3.2. Let us write:

$$
\begin{aligned}
f(\mathrm{Tate}(q),\mu_{p^2},1\mapsto\zeta_N) &= \Big(\sum a_n q^n\Big)(dt/t)^{\otimes k}\\
f(\mathrm{Tate}(\zeta_p q),\langle q^{1/p}\rangle,1\mapsto\zeta_N) &= \Big(\sum a_n' q^n\Big)(dt/t)^{\otimes k}\\
f(\mathrm{Tate}(q^{p^2}),\langle q\rangle,1\mapsto\zeta_N) &= \Big(\sum a_n'' q^n\Big)(dt/t)^{\otimes k}
\end{aligned}
\tag{3.3.5}
$$

Applying the definitions of $\pi^*f$ and $\phi_0$ one finds:

$$
\begin{aligned}
a_n &= a(n,\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right))\\
a_n' &= a(n,\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right))\\
a_n'' &= \varepsilon(p)^{-1}p^{-k}a(n,\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right))
\end{aligned}
\tag{3.3.6}
$$

A direct calculation shows that for $p\not|n$ we have

$$
h_n'\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)=p(p-1),\qquad h_n'\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)=\pm p(p-1)
\tag{3.3.7}
$$

where $\pm=\alpha(-1)$ if $\chi=\pi(\alpha,\alpha^{-1})$ or $\chi=\pi^-(\alpha)$, and $\pm=-\Lambda(z_0)$ if $\chi=\pi(\Lambda)$ and $z_0^{p-1}=-1$. It follows that for $p\not|n$ we have

$$
a_n'=\pm p^{-1}(p-1)^{-1}h_n'\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)\cdot a_n,\qquad a_n''=\pm\varepsilon(p)^{-1}p^{-k}a_n
\tag{3.3.8}
$$

where the sign $\pm$ is as in 3.3.7. It remains to see what the $a_n'$ and $a_n''$ are for $p|n$. For the $a_n''$ this can be solved using the Atkin-Lehner automorphism $w_{p^2}$ of $X_0$ defined as follows:

$$
w_{p^2}\colon (E/S,\ker(\phi),\alpha)\mapsto(E'/S,\ker(\phi^t),\phi\alpha)
\tag{3.3.9}
$$

where $\phi\colon E\to E'$ is cyclic of degree $p^2$. On $\mathrm{H}^0(X_0,\underline{\omega}^{\otimes k})$ we define an operator $w_{p^2}^*$ by:

$$
(w_{p^2}^*f_0)(E/S,\ker(\phi),\alpha)=\phi^*(f_0(E',\ker(\phi^t),\phi\alpha))
\tag{3.3.10}
$$

One easily checks that $w_{p^2}^2=\langle p^2\rangle_N$, that $w_{p^2}^{*2}=p^{2k}\langle p^2\rangle_N^*$ and that $w_{p^2}^*$ commutes with all $T_l^*$ ($l\neq p$) and all $\langle a\rangle_N^*$ ($a\in(\mathbf{Z}/N\mathbf{Z})^*$). It follows that $w_{p^2}^*f=\pm\varepsilon(p)p^kf$; evaluating this on $(\mathrm{Tate}(q),\mu_{p^2},1\mapsto\zeta_N)$ gives

$$
\sum a_n''q^n=\pm\varepsilon(p)^{-1}p^{-k}\sum a_nq^n
\tag{3.3.11}
$$

So we see that the equality in 3.3.8 concerning $a_n''$ is in fact true for all $n$.

It remains to determine the $a_n'$ for $p|n$. Prop. 3.3.4 is of no help when one wants to express the $a_{pn}'$ and $a_{pn}''$ in terms of $a_{pn}$, since $a_{pn}=0$ for all $n$. For $\chi=\pi(\Lambda)$ it follows from 3.3.2 that $a(pn,x)=0$ for all $n$ and $x$, hence that $a_{pn}=0=a_{pn}'=a_{pn}''$. The other two cases, $\chi=\pi(\alpha,\alpha^{-1})$ or $\chi=\pi^-(\alpha)$, will be dealt with in the next section.

**3.4 Twisting.** Suppose that $\chi=\pi(\alpha,\alpha^{-1})$. We will show that $f$ is a twist of a new form on $X_1:=\overline{\mathcal{M}}(\Gamma_1(p),\Gamma_1(N))_{\overline{\mathbf{Q}}}$ and use this to solve our problem. For general information about twisting of modular forms one can consult [26], page 91.

Let $\beta\colon \mathbf{F}_p{}^* \to \overline{\mathbf{Q}}^*$ be a character. We will define an automorphism $\Theta_\beta$ of the $\overline{\mathbf{Q}}$-vector space $\mathrm{H}^0(X, \underline{\omega}^{\otimes k})$ such that:

$$
\begin{aligned}
T_l^* \circ \Theta_\beta &= \Theta_\beta \circ \beta(l) T_l^* & \forall l \neq p \\
g \circ \Theta_\beta &= \Theta_\beta \circ \beta(\det(g)) g & \forall g \in G
\end{aligned}
\tag{3.4.1}
$$

Let $e_p$ denote the Weil pairing on the $p$-torsion of any elliptic curve (see e.g. [19], §2.8), let $z$ be the locally constant $\mu_p(\overline{\mathbf{Q}})$-valued function $e_p(\phi(1,0),\phi(0,1))$ in $\mathrm{H}^0(X,\mathcal{O}_X)$ and let $b := \sum_i \beta(i)^{-1} z^i$, where the sum ranges over $i \in \mathbf{F}_p{}^*$. Then one easily checks that the operator $\Theta_\beta\colon f \mapsto bf$ on $\mathrm{H}^0(X, \underline{\omega}^{\otimes k})$ satisfies 3.4.1 and that $b$ is a unit if $\beta$ is non-trivial.

Let $V_1$ be the image of $V$ under $\Theta_\alpha$. It follows from 3.4.1 and Table 3.3.1 that $V_1 \cong \pi(\alpha^2, 1)$ and that $\dim V_1^H = 1$, where $H \subset G$ is the subgroup $\{(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix})\}$. Let $\pi_1\colon X \to X_1$ be the morphism sending $(E/S, \phi, \psi)$ to $(E/S, 1 {\mapsto} \phi(1,0), \psi)$. Then $\pi_1$ identifies $X_1$ with $X/H$, $\pi_1$ is compatible with the $T_l^*$ for $l \neq p$ and the $\langle a \rangle_N^*$ and $\pi_1 \circ (\begin{smallmatrix} x & a \\ 0 & y \end{smallmatrix}) = \langle x \rangle_p \circ \pi_1$. This implies that there exists a unique normalized new form $f_1$ of weight $k$ and character $\alpha^2 \varepsilon$ on $X_1$ such that $\pi_1^* f_1 \in V_1$. If we write:

$$
f_1(\mathrm{Tate}(q), 1 {\mapsto} \zeta_p, 1 {\mapsto} \zeta_N) = \left(\sum b_n q^n\right)(dt/t)^{\otimes k} \tag{3.4.2}
$$

then we have $a_n = \alpha^{-1}(n) b_n$ for all $n$ (we will always extend characters $\mathbf{F}_p{}^* {\to} \overline{\mathbf{Q}}^*$ to $\mathbf{F}_p$ by 0). The $q$-expansions of $f_1$ at other cusps can be computed in various ways, see for example [1]??? and [16]??? ; one just evaluates the identity $T_p^* f_1 = b_p f_1$ at such a cusp. First of all we have $b_p \neq 0$. We define the $b_n'$ by:

$$
f_1(\mathrm{Tate}(q)/\overline{\mathbf{Q}}((q^{1/p})), 1 {\mapsto} q^{1/p}, 1 {\mapsto} \zeta_N) = \left(\sum b_n' q^{n/p}\right)(dt/t)^{\otimes k} \tag{3.4.3}
$$

For $\beta\colon \mathbf{F}_p{}^* \to \overline{\mathbf{Q}}^*$ a character and $n \in \mathbf{Z}$ let $g(\beta, n)$ denote the Gauss sum $\sum_x \beta(x) \zeta_p^{nx}$. Then:

$$
b_n' = b_p^{-1} \varepsilon(p) p^{k-1} b_{n/p}' + p^{-1} \alpha^{-2}(n) g(\alpha^2, 1) b_p^{-1} b_n \tag{3.4.4}
$$

A convenient way to express this result is the following Euler product expansion:

$$
\sum b_n' n^{-s} = p^{-1} b_p^{-1} g(\alpha^2, 1)(1 - \lambda_p' p^{-s})^{-1} \prod_{l \neq p} \left(1 - \alpha^{-1}(l) a_l l^{-s} + \alpha^{-2}(l) \varepsilon(l) l^{k-1-2s}\right)^{-1} \tag{3.4.5}
$$

where $\lambda_p'$ is defined by: $\lambda_p' b_p = \varepsilon(p) p^{k-1}$. The $q$-expansions of $f$ at various cusps can now be computed by expressing $\pi^* f$ in terms of $\pi_1^* f_1$. By 3.4.1 we have $\Theta_{\alpha^{-1}} \pi_1^* f_1 \in V$. Recall that $\pi^* f$ is a basis for $V^T$. Unfortunately the projection of $\Theta_{\alpha^{-1}} \pi_1^* f_1$ in $V^T$ is zero because $(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}) \Theta_{\alpha^{-1}} \pi_1^* f_1 = \alpha(xy^{-1}) \Theta_{\alpha^{-1}} \pi_1^* f_1$, but one can check that the projection of $(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}) \Theta_{\alpha^{-1}} \pi_1^* f_1$ is non-zero. It follows that:

$$
\pi^* f = c \sum_x (\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix})(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}) \Theta_{\alpha^{-1}} \pi_1^* f_1 \tag{3.4.6}
$$

for some $c \in \overline{\mathbf{Q}}^*$ (note that $\sum (\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix})$ projects into $V^T$ because the central character of $V$ is trivial). Evaluating 3.4.6 on say $(\mathrm{Tate}(q), \phi_0 \circ (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), 1 {\mapsto} \zeta_N)$ one finds that $c = g(\alpha^2, 1)^{-1} b_p$. Let the $a_n'$ be as defined in 3.3.5. By evaluating 3.4.6 on $(\mathrm{Tate}(q), \phi_0 \circ (\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}), 1 {\mapsto} \zeta_N)$ one computes that:

$$
a_n' = \alpha(-1) g(\alpha^2, 1)^{-1} b_p g(\alpha, 1)\left(j(\alpha^{-1}, n) b_n' + b_{n/p}\right) \tag{3.4.7}
$$

where for any character $\beta: \mathbf{F}_p^* \to \overline{\mathbf{Q}}^*$ and $n \in \mathbf{Z}$:

$$j(\beta, n) = \sum_x \beta(x)\beta(x-1)\zeta_p^{nx} \tag{3.4.8}$$

We summarize these results in the following proposition.

**3.4.9 Proposition.** *Let $f$ be a new form as in §3.2. Suppose that $\pi^* f$ generates the representation $\pi(\alpha, \alpha^{-1})$ of $G$ for some $\alpha: \mathbf{F}_p^* \to \overline{\mathbf{Q}}^*$. Let $f_1$ be the unique normalized new form on $\overline{\mathcal{M}}(\Gamma_1(p), \Gamma_1(N))_{\overline{\mathbf{Q}}}$ such that $f$ is the twist of $f_1$ by $\alpha^{-1}$. Then the $q$-expansions of $f$ and $f_1$ at the various cusps are related by formulas 3.3.5, 3.4.2, 3.4.3, 3.4.4, 3.4.5 and 3.4.7.*

Suppose now that $\chi = \pi^-(\alpha)$. Using techniques as above one can show that there exists a unique normalized new form $g$ of weight $k$ and character $\varepsilon$, either on $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{Q}}}$ or on $\overline{\mathcal{M}}(\Gamma_1(p), \Gamma_1(N))_{\overline{\mathbf{Q}}}$, such that $a_n = \alpha(n) b_n$ for all $n$, where $T_n^* g = b_n g$. In the first case one finds:

$$a'_n = \varepsilon(p)^{-1} g(\alpha, 1)^{-1} \left(p^k \varepsilon(p) b_{n/p^2} + j(\alpha, n) b_n\right) \tag{3.4.10}$$

In the second case we define the $b'_n$ by:

$$g(\mathrm{Tate}(q), 1 \mapsto q^{1/p}, 1 \mapsto \zeta_N) = \left(\sum b'_n q^{n/p}\right)(dt/t)^{\otimes k} \tag{3.4.11}$$

Evaluating $T_p^* g = b_p g$ one obtains $b_p^2 = \varepsilon(p) p^{k-2}$ and $b'_n = -p^{-1} b_p^{-1} b_n$ for all $n$. By the way, it follows that this case cannot occur if $k = 1$ since $b_p$ has to be integral. For the $a'_n$ one finds the following expression:

$$a'_n = \varepsilon(p)^{-1} g(\alpha, 1)^{-1} \left(-p b_p b_{n/p} + j(\alpha, n) b_n\right) \tag{3.4.12}$$

**3.4.13 Proposition.** *Let $f$ be a new form as in §3.2. Suppose that $\pi^* f$ generates the representation $\pi^-(\alpha)$ of $G$, hence with $\alpha: \mathbf{F}_p^* \to \overline{\mathbf{Q}}^*$ of order 2. Then there is a unique normalized new form $g$ of weight $k$ and character $\varepsilon$, either on $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{Q}}}$ or on $\overline{\mathcal{M}}(\Gamma_1(p), \Gamma_1(N))_{\overline{\mathbf{Q}}}$, such that $a_n = \alpha(n) b_n$ for all $n$, where $T_n^* g = b_n g$. In the first case the $a'_n$ are given by 3.4.10 and in the second case by 3.4.12.*

**3.5 $p$-adic valuations.** Let $f$ be as in §3.2. In §6 we will need to know the $p$-adic valuation (in some sense) of the $q$-expansions of $f$ at certain cusps. More precisely: we fix an embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$ and we normalize the $p$-adic valuation $v_p$ on $\overline{\mathbf{Q}}_p$ by setting $v_p(p) = 1$. Then we want to know the valuations of the power series $\sum a'_n q^n$ and $\sum a''_n q^n$ in 3.3.5; by definition we have $v_p(\sum a'_n q^n) = \min\{v_p(a'_n) \mid n \geq 0\}$. Let $\overline{\mathbf{Z}}_p$ be the ring of integers in $\overline{\mathbf{Q}}_p$ and let $\overline{\mathbf{F}}_p$ be the residue field of $\overline{\mathbf{Z}}_p$. Let $\mathbf{F}$ be a finite extension of $\mathbf{F}_p$. Clearly any character $\alpha: \mathbf{F}^* \to \overline{\mathbf{Q}}_p^*$ has its image in $\overline{\mathbf{Z}}_p^*$. We call $\alpha$ fundamental if $\alpha: \mathbf{F} \to \overline{\mathbf{Z}}_p \to \overline{\mathbf{F}}_p$ is an embedding of fields. The Teichmüller character $\tau$ is the unique fundamental character $\tau: \mathbf{F}_p^* \to \overline{\mathbf{Q}}_p^*$. We denote the two fundamental characters $\mathbf{F}_{p^2}^* \to \overline{\mathbf{Q}}_p^*$ by $\tau_2$ and $\tau'_2 = \tau_2^p$.

**3.5.1 Lemma.**

1. $v_p(g(\tau^{-m}, 1)) = \frac{m}{p-1}$ *for* $0 \leq m < p-1$.

*2.* $v_p(j(\tau^m,1)) = \begin{cases} 1-\frac{2m}{p-1} & \textit{if} \;\; 0 < m \leq \frac{p-1}{2} \\ 0 & \textit{if} \;\; \frac{p-1}{2} \leq m < p-1 \end{cases}$

*3.* $v_p(j(\tau^m,0)) = \begin{cases} 1 & \textit{if} \;\; 0 < m < \frac{p-1}{2} \\ 0 & \textit{if} \;\; \frac{p-1}{2} \leq m < p-1 \end{cases}$

*4. Let* $\Lambda = \tau_2^{(p-1)m}$ *with* $0 < m < p+1$*, let* $\chi = \pi(\Lambda)$ *and let* $h'_n$ *be as in 3.3.3. Then* $h'_n(x) = 0$ *for all* $x$ *if* $p|n$*. Assume that* $p \not| n$ *and that* $p \neq 2$*. Then:*

*(a)* $v_p(h'_n(\begin{smallmatrix}1&0\\0&1\end{smallmatrix})) = v_p(h'_n(\begin{smallmatrix}0&1\\1&0\end{smallmatrix})) = 1$

*(b)* $v_p(h'_n(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})) = \begin{cases} \frac{p-m}{p-1} & \textit{if} \;\; m \leq \frac{p+1}{2} \\ \frac{m-1}{p-1} & \textit{if} \;\; m \geq \frac{p+1}{2} \end{cases}$

**Proof.** 1. Very well known, see for example ...
2. We compute the image of $j(\tau^m,1)$ in $\mathbf{F}_p[\zeta_p] := \mathbf{Z}_p[\zeta_p]/p\mathbf{Z}_p[\zeta_p]$:

$$\begin{aligned} j(\tau^m,1) \bmod p &= \sum_a a^m(a-1)^m\zeta_p^a = \sum_a a^m \sum_{k=0}^m \binom{m}{k}(-1)^{m-k}a^k\zeta_p^a \\ &= (-1)^m\sum_{k=0}^m(-1)^k\binom{m}{k}\sum_a a^{m+k}\zeta_p^a = (-1)^m\sum_{k=0}^m(-1)^k\binom{m}{k}g(\tau^{m+k},1) \end{aligned}$$

The statement now follows from part 1 of this lemma.
3. Well known because $j(\tau^m,0)$ is $\tau(-1)^m$ times the usual Jacobi sum, see ...
4. It follows from 3.3.2 that $h'_n(x) = 0$ if $p|n$. Assume that $p \not| n$. Part (a) follows from 3.3.7. The computation of $v_p(h'_n(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}))$ is more complicated. By definition 3.3.3 we have:

$$h'_n(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}) = \sum_{b,x,y}\zeta_p^{-nb}\pi(\Lambda)(\begin{smallmatrix}x+xb&by\\x&y\end{smallmatrix}), \quad b \in \mathbf{F}_p,\; x,y \in {\mathbf{F}_p}^*.$$

Write $b = a-1$. Note that $(\begin{smallmatrix}ax&(a-1)y\\x&y\end{smallmatrix})$ is never scalar, hence the value of $\pi(\Lambda)$ on it is determined by its trace ($= ax+y$) and its determinant ($= xy$). For any $\alpha \in \mathbf{F}_p$ and $\beta \in {\mathbf{F}_p}^*$ the number of solutions of

$$\begin{cases} ax+y &= \alpha \\ xy &= \beta \end{cases}$$

equals $\left(\frac{\alpha^2-4a\beta}{p}\right)+1$, where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Using this fact, Table 3.3.1 and the non-triviality of $\Lambda$, we find:

$$h'_n(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}) = -\zeta_p^n\sum_{a\in\mathbf{F}_p}\zeta_p^{-na}\sum_{z\in\mathbf{F}_{p^2}^*}\Lambda(z)\left(\frac{(z+z^p)^2-4az^{p+1}}{p}\right).$$

The image of $h'_n(\begin{smallmatrix}1&0\\1&1\end{smallmatrix})$ in $\mathbf{F}_{p^2}[\zeta_p]$ can be written as:

$$h'_n(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}) \bmod p = -\zeta_p^n\sum_{a\in\mathbf{F}_p}\zeta_p^{-na}\sum_{z\in\mathbf{F}_{p^2}^*}z^{(p-1)m}\left((z+z^p)^2-4az^{p+1}\right)^{(p-1)/2}$$

Expanding and rearranging terms gives:

$$h_n'\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right) \bmod p = -\zeta_p^n \sum_{k=1}^{(p-1)/2} (-4)^k \binom{\frac{p-1}{2}}{k} \sum_z z^{(p-1)(m+k+1)} (1+z^{p-1})^{p-1-2k} g(\tau^k,-n) \qquad (3.5.2)$$

Clearly the right hand side is of the form $\sum c_k\, g(\tau^k,-n)$. It follows from part 1 of this lemma that $v_p(g(\tau^k,-n)) = 1-k/(p-1)$, hence we have to find the largest $k$ for which $c_k \neq 0$. The exponent of $z$ in each term of 3.5.2 is positive and less than $2(p^2-1)$. Because we sum over $z$, only terms with $p^2-1$ as exponent of $z$ can give a non-zero contribution. It follows that we should look for the largest $k$ for which there is a non-zero term depending on $a$ and $z$ as $a^k z^{p^2-1}$. This means that we are looking for the largest $k$ such that there exists $i$ with $0 \leq i \leq p-1-2k$ such that $m+k+1+i = p+1$. Recall that $0 < m < p+1$ and $0 < k \leq (p-1)/2$. It is then easy to see that one should take $k = p-m$ and $i = 0$ if $m \geq (p+1)/2$, and $k = m-1$ and $i = p-1-2k$ if $m \leq (p+1)/2$. □

**3.5.3 Proposition.** *Let $N$ be a positive integer, $p > 2$ a prime not dividing $N$ and $f$ a normalized new form on $\overline{\mathcal{M}}(\Gamma_0(p^2),\Gamma_1(N))_{\overline{\mathbf{Q}}}$ of some weight $k \geq 2$ and character $\varepsilon\colon (\mathbf{Z}/N\mathbf{Z})^* \to \overline{\mathbf{Q}}^*$. Let $\chi$ denote the character of the irreducible representation associated to $f$ as in §3.2. Then we have $\chi = \pi(\alpha,\alpha^{-1})$ for some $\alpha\colon \mathbf{F}_p^* \to \overline{\mathbf{Q}}^*$ with $\alpha \neq \alpha^{-1}$, or $\chi = \pi(\Lambda)$ for some $\Lambda\colon \mathbf{F}_{p^2}^* \to \overline{\mathbf{Q}}^*$ with $\Lambda^p = \Lambda^{-1} \neq \Lambda$, or $\chi = \pi^-(\alpha)$ with $\alpha\colon \mathbf{F}_p^* \to \overline{\mathbf{Q}}^*$ of order two. Fix an embedding of $\overline{\mathbf{Q}}$ into $\overline{\mathbf{Q}}_p$ and normalize the $p$-adic valuation $v_p$ on $\overline{\mathbf{Q}}_p$ by $v_p(p) = 1$. Let $\tau$, $\tau_2$ and $\tau_2^p$ denote the fundamental characters of level 1 and 2 as in the beginning of §3.5. Let $\zeta_p$ and $\zeta_N$ be a pth and a Nth root of unity in $\overline{\mathbf{Q}}$. Let the $a_n$, $a_n'$ and $a_n''$ be as defined in 3.3.5. Then we have:*

1. $v_p(\sum a_n q^n) = 0$,
2. $v_p(\sum a_n'' q^n) = -k$,
3. *if* $\chi = \pi(\tau^m,\tau^{-m})$ *with* $0 < m < (p-1)/2$ *then* $v_p(\sum a_n' q^n) = -m/(p-1)$,
4. *if* $\chi = \pi(\tau_2^{(p-1)m})$ *with* $0 < m < (p+1)/2$ *then* $v_p(\sum a_n' q^n) = (1-m)/(p-1)$,
5. *if* $\chi = \pi^-(\tau^{(p-1)/2})$ *then* $v_p(\sum a_n' q^n) = -1/2$.

*Finally, the valuations of the $q$-expansions of $f$ at the cusps $(\mathrm{Tate}(\zeta_p^i q), \langle q^{1/p}\rangle, 1 \mapsto \zeta_N)$, $1 \leq i \leq p-1$, are all equal.*

**Proof.** Part 1 is a consequence of the definition of "normalized". Part 2 results from 3.3.11. Parts 3, 4 and 5 are an easy exercise using Lemma 3.5.1, Prop. 3.4.9, Prop. 3.3.4 and Prop. 3.4.13 (for 3 one has to use that in 3.4.5 $v_p(\lambda_p') \geq 0$). For the last statement one notes that $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ permutes those cusps transitively; that for any $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, $\sigma(f)$ generates a representation with character $\sigma(\chi)$; that $\sigma(\chi) = \chi$ (see 3.3.1), and that we have already proved that the valuation at one of those cusps depends only on that character. □

**3.5.4 Remark.** 1. Note that $\pi(\alpha,\alpha^{-1}) = \pi(\alpha^{-1},\alpha)$ and that $\pi(\Lambda) = \pi(\Lambda^p)$.

2. The relation between $\chi$ and the Galois representation corresponding to $f$ is given in Prop.4.8.

3. Note the curious case $\chi = \pi(\tau_2^{p-1})$ where $v_p(\sum a_n' q^n) = 0$.

# 4 A stable model for $\overline{\mathcal{M}}(\Gamma(p), \Gamma_1(N))$ at $p$.

**4.1 Construction of a stable model.** In order to explain the construction of a stable model for $\overline{\mathcal{M}}(\Gamma(p), \Gamma_1(N))$ at $p$, it is better to consider the following more general situation. Let $S$ be the spectrum of a discrete valuation ring with perfect residue field, let $s$ be the closed point of $S$ and let $\eta$ be the generic point. Let $\mathcal{C} \to S$ be a curve: $\mathcal{C} \to S$ is proper, flat and purely of relative dimension one. We suppose that $\mathcal{C}_\eta$ is smooth over $\eta$ and that $\mathcal{C}$ is regular.

Under the hypotheses above, the irreducible components of $\mathcal{C}_s$ are Cartier divisors on $\mathcal{C}$. After blowing up repeatedly in closed points of $\mathcal{C}_s$ we may assume that $\mathcal{C}_s$ is a Cartier divisor on $\mathcal{C}$ with normal crossings. Let $n$ be the least common multiple of the multiplicities of the irreducible components of $\mathcal{C}_s$, and let $\pi_0$ be a uniformizer on $S$. Let $T$ be $S[\pi]$, with $\pi^n = \pi_0$, thus we have $T \to S$ totally ramified of degree $n$. Now we consider $\widetilde{\mathcal{C}_T}$, the normalization of the pullback of $\mathcal{C}$ to $T$. Let $t$ be the closed point of $T$.

**4.1.1 Proposition.** *If $n$ is invertible on $S$, then the geometric fibre $\widetilde{\mathcal{C}_{T,\bar{t}}}$ of $\widetilde{\mathcal{C}_T}/T$ is a reduced curve whose singularities are ordinary double points (i.e., $\widetilde{\mathcal{C}_T} \to T$ is a semi-stable model).*

**Proof.** By replacing $S$ by its strict henselization we may suppose that $k(s)$ is algebraically closed. We will check that $\widetilde{\mathcal{C}_{T,t}}$ is reduced and that its singularities are ordinary double points. In order to do this we compute $\widetilde{\mathcal{C}_T}$ locally in the etale topology. Let $x \in \mathcal{C}(s)$. Since $\mathcal{C}_s$ is a normal crossings divisor on $\mathcal{C}$ there is an etale neighborhood $U \to \mathcal{C}$ of $x$ and a system $X, Y$ of parameters in $\mathcal{O}_{U,x}$ such that $X^aY^b = \pi_0 u$ with $u \in \mathcal{O}_{U,x}$ a unit and $a$, $b$ non-negative integers with $a \neq 0$. The numbers $a$ and $b$ are the multiplicities of the one or two irreducible components of $U_s$ passing through $x$; $b$ equals zero if and only if $x$ lies on exactly one irreducible component of $U_s$. By assumption $a$ is invertible on $U$, hence $U[u^{1/a}]$ is finite etale over $U$. Changing $X$ by an $a$th root of $u$ we have: $X^aY^b = \pi_0$. It follows that $\mathcal{C}_T$ is locally isomorphic to a subscheme of $\mathbf{A}^2_T$ defined by the equation $X^aY^b = \pi^n$. The normalizations of the rings $\mathcal{O}_T[X,Y]/(X^aY^b - \pi^n)$ can be easily computed, see for example [13], §2.2. One finds the following.

4.1.2 The normalization of $\mathrm{Spec}(\mathcal{O}_T[X,Y]/(X^a - \pi^n))$ is the disjoint union, indexed by the $\zeta_a \in \mu_a$, of copies of $\mathrm{Spec}(\mathcal{O}_T[Y])$; on the copy labeled by $\zeta_a$ we have $X = \zeta_a \pi^{n/a}$.

4.1.3 Suppose that $b \neq 0$. Let $c = \gcd(a,b)$, write $a = a'c$, $b = b'c$, $n = ca'b'm$. Write $1 = a'e + b'd$ for some integers $d$ and $e$. Then the normalization of $\mathrm{Spec}(\mathcal{O}_T[X,Y]/(X^aY^b - \pi^n))$ is the disjoint union, indexed be the $\zeta_c \in \mu_c$, of copies of $\mathrm{Spec}(\mathcal{O}_T[V,Z]/(VZ - \pi^m))$. On the copy labeled by $\zeta_c$ we have: $V = \pi^{a'em}X^dY^{-e}$, $Z = \pi^{b'md}X^{-d}Y^e$, $X = \zeta_c^e V^{b'}$ and $Y = \zeta_c^d Z^{a'}$.

From these results one concludes that indeed $\widetilde{\mathcal{C}_T} \to T$ is a semi-stable model. $\square$

**4.1.4 Remark.** 1. If $\mathcal{C}_\eta$ is geometrically irreducible and has genus at least 2, one obtains the stable model (in the sense of [10]) of $\mathcal{C}_\eta$ over $T$ by contracting the projective lines in $\widetilde{\mathcal{C}_{T,\bar{t}}}$ intersecting the rest of $\widetilde{\mathcal{C}_{T,\bar{t}}}$ in fewer than 3 points. By contracting and blowing up one also obtains the minimal model of $\mathcal{C}$ over $T$ (n.b. $\widetilde{\mathcal{C}_T}$ itself may not be regular).

2. The computations of the normalizations describe the morphism $\widetilde{\mathcal{C}_T} \to \mathcal{C}_T$ locally in the etale topology. In particular, one knows the ramification structure of $\widetilde{\mathcal{C}}_{T,t} \to \mathcal{C}_{s,\mathrm{red}}$.

3. The choice of the uniformizing element $\pi_0$ on $S$ is unimportant, since all totally ramified extensions of degree $n$ of $S$ are isomorphic over the strict henselization of $S$.

4. On the contrary, if $n$ is not invertible on $S$, there are lots of non-isomorphic (wildly ramified) extensions of the same degree. If one knows the action of the inertia subgroup of $\mathrm{Gal}(\bar\eta/\eta)$ on $\mathrm{H}^1(\mathcal{C}_{\bar\eta}, \mathbf{Q}_\ell)$ one can pick the right extension. In the case of modular curves (of arbitrary level) this action is known ([5]). The problem is the description of the rings $\widehat{\mathcal{O}}_{\mathcal{C},x}$.

**4.2 Construction of a stable model for $\overline{\mathcal{M}}(\Gamma(p), \Gamma_1(N))$.** Let $p$ be a prime, let $N$ be an integer that is not divisible by $p$ and let $X := \overline{\mathcal{M}}(\Gamma(p), \Gamma_1(N))$ be the compactification (obtained by normalization, as in §2.1) of the algebraic stack classifying triples $(E/S, \phi, \alpha)$ with $E/S$ an elliptic curve, $\phi\colon \mathbf{F}_p \times \mathbf{F}_p \to E[p](S)$ a Drinfeld basis and $\alpha\colon \mathbf{Z}/N\mathbf{Z} \to E[N](S)$ a Drinfeld $\mathbf{Z}/N\mathbf{Z}$-structure. For any elliptic curve $E/S$ let $e_p\colon E[p]\times_S E[p] \to \mu_{p,S}$ denote the Weil pairing (see [19], §2.8). The morphism $X \to \mathrm{Spec}(\mathbf{Z})$ factors through $\mathrm{Spec}(\mathbf{Z}[\zeta_p])$ by sending $(E/S, \phi, \alpha)$ to $e_p(\phi(1,0), \phi(0,1)) \in \mathrm{Spec}(\mathbf{Z}[\zeta_p])(S)$. Let $X\tilde{\otimes}_{\mathbf{Z}}\mathbf{Z}[\zeta_p]$ denote the normalization of $X \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_p]$. For $i \in {\mathbf{F}_p}^*$ let $X_i$ denote the algebraic stack $\overline{\mathcal{M}}(\Gamma(p)^{\zeta_p^i-\mathrm{can}}, \Gamma_1(N))$ over $\mathbf{Z}[\zeta_p]$ (see [19], §9.4). By definition, for a scheme $S$ over $\mathbf{Z}[\zeta_p]$ an object in $\mathcal{M}(\Gamma(p)^{\zeta_p^i-\mathrm{can}}, \Gamma_1(N))(S)$ is a triple $(E/S/\mathbf{Z}[\zeta_p], \phi, \alpha)$ with $e_p(\phi(1,0), \phi(0,1)) = \zeta_p^i$; $X_i$ is obtained from $\mathcal{M}(\Gamma(p)^{\zeta_p^i-\mathrm{can}}, \Gamma_1(N))$ by normalization at infinity. The morphism

$$\coprod_{i\in{\mathbf{F}_p}^*} X_i \;\longrightarrow\; X \otimes \mathbf{Z}[\zeta_p], \qquad (E/S/\mathbf{Z}[\zeta_p], \phi, \alpha) \mapsto ((E/S, \phi, \alpha), e_p(\phi(1,0), \phi(0,1)))$$

induces an isomorphism

$$\coprod_{i\in{\mathbf{F}_p}^*} X_i \;\xrightarrow{\sim}\; X\tilde{\otimes}\mathbf{Z}[\zeta_p] \tag{4.2.1}$$

because both sides are normal (in fact, both sides are regular).

We will now apply the construction of §4.1 to $X\tilde{\otimes}\mathbf{Z}[\zeta_p]$ over $\mathbf{Z}[\zeta_p]$. The fact that we are dealing with an algebraic stack that is not necessarily a scheme is no problem because blowing up and normalization are compatible with etale descent. We have already remarked that $X\tilde{\otimes}\mathbf{Z}[\zeta_p]$ is regular. The $X_i \otimes_{\mathbf{Z}[\zeta_p]} \mathbf{F}_p$ are described in [19], Thm. 13.7.6 and Thm. 13.8.4: $X_i \otimes_{\mathbf{Z}[\zeta_p]} \mathbf{F}_p$ has $p+1$ irreducible components, all isomorphic to $\overline{\mathcal{M}}(\mathrm{ExIg}(p,1), \Gamma_1(N))$; these irreducible components are smooth over $\mathbf{F}_p$, geometrically irreducible, have multiplicity one and intersect each other exactly at the supersingular points; the intersections are pairwise transversal.

Let $Y \to X\tilde{\otimes}\mathbf{Z}[\zeta_p]$ be the blow up in the supersingular points in characteristic $p$. Then $Y \otimes_{\mathbf{Z}[\zeta_p]} \mathbf{F}_p$ is a divisor on $Y$ with normal crossings and over each supersingular point lies an exceptional curve having multiplicity $p+1$ in $Y \otimes_{\mathbf{Z}[\zeta_p]} \mathbf{F}_p$. Let $\pi_0$ be the uniformizer $1-\zeta_p$ of $\mathbf{Z}[\zeta_p]$ at $p$. According to Prop. 4.1.1 $X^{\mathrm{st}} := Y\tilde{\otimes}_{\mathbf{Z}[\zeta_p]}\mathbf{Z}[\zeta_p, \pi]$, where $\pi^{p+1} = \pi_0$, is a model for $X$ over $\mathbf{Z}[\zeta_p, \pi]$ that is semi-stable at $p$. The computations in the proof of Prop. 4.1.1 show that $X^{\mathrm{st}}$ is regular and that the fibre $X^{\mathrm{st}}_p := X^{\mathrm{st}} \otimes_{\mathbf{Z}[\zeta_p,\pi]} \overline{\mathbf{F}}_p$ over $\overline{\mathbf{F}}_p$ has two kinds of irreducible components. First of all one has the irreducible components that map isomorphically onto an irreducible component of $(X\tilde{\otimes}\mathbf{Z}[\zeta_p])_{\overline{\mathbf{F}}_p}$; these components are isomorphic to $\overline{\mathcal{M}}(\mathrm{ExIg}(p,1),\Gamma_1(N))_{\overline{\mathbf{F}}_p}$; we call them components of Igusa type. Secondly, there are the irreducible components that map to a supersingular point in $X\tilde{\otimes}\mathbf{Z}[\zeta_p]$; over each supersingular point in characteristic $p$ there is exactly one such component; we call them components of Drinfeld type; they are all isomorphic. Each component of Drinfeld type maps with degree $p{+}1$ to some exceptional curve in $Y\otimes_{\mathbf{Z}[\zeta_p]}\overline{\mathbf{F}}_p$. This shows a component of Drinfeld type is a cover of degree $p{+}1$ of $\mathbf{P}^1_{\overline{\mathbf{F}}_p}$ ramified exactly and totally over the $p{+}1$ $\mathbf{F}_p$-rational points.

**4.3 Components of Drinfeld type.** In this section we want to get equations for the components of Drinfeld type in $X_p^{\mathrm{st}}$. To do this we follow the construction of §4.2. Let $s=(E/\overline{\mathbf{F}}_p,\phi,\alpha)$ be a supersingular point in $X_i(\overline{\mathbf{F}}_p)$ for some $i\in\mathbf{F}_p{}^*$. We denote the corresponding component of Drinfeld type by $D_{i,s}$. Let $W$ be the ring of Witt vectors of $\overline{\mathbf{F}}_p$. By [19], Thm. 13.8.4, the complete local ring of $X\tilde{\otimes}\mathbf{Z}[\zeta_p]$ at $s$ is isomorphic to $W[\zeta_p][[x,y]]/(f)$, with $f=f_0+\pi_0 f_1$, $f_0=x^py-xy^p+g$, $g\in(x,y)^{p+2}$ and $f_1$ a unit. As explained in [13], §1.3.1, the completion of $Y$ along the exceptional curve lying over $s$ can be covered by two open affines isomorphic to $\mathrm{Spf}(A_1)$ and $\mathrm{Spf}(A_2)$ with $A_1=W[\zeta_p][v][[x]]/(f(x,vx))$ and $A_2=W[\zeta_p][u][[y]]/(f(uy,y))$ respectively.

It follows that the completion of $X^{\mathrm{st}}$ along $D_{i,s}$ is covered by two open affines isomorphic to $\mathrm{Spf}(A_j\tilde{\otimes}_{W[\zeta_p]}W[\zeta_p,\pi])$, $j$ in $\{1,2\}$. The normalization $A_j\tilde{\otimes}_{W[\zeta_p]}W[\zeta_p,\pi]$ of $A_j\otimes_{W[\zeta_p]}W[\zeta_p,\pi]$ is obtained by blowing up in the ideals $(x,\pi)$ and $(y,\pi)$ respectively (this can be seen from the computations in the proof of Prop. 4.1.1). Let us write out what happens for $j=1$. Then:

$$A_1\otimes_{W[\zeta_p]}W[\zeta_p,\pi]=W[\zeta_p,\pi][v][[x]]/\left(x^{p+1}(v-v^p)+x^{p+2}h+\pi^{p+1}f_1(x,vx)\right)$$

where we have written $g(x,vx)=x^{p+2}h$. Blowing up in $(x,\pi)$ means setting $\pi=xw$. We find:

$$A_1\tilde{\otimes}_{W[\zeta_p]}W[\zeta_p,\pi]=W[\zeta_p,\pi][v][[x]][w]/\left(\pi-xw,v-v^p+xh+w^{p+1}f_1(x,vx)\right)$$

The corresponding affine part of $D_{i,s}$ is given by the equation $x=0$. Substituting this we find that $D_{i,s}$ is the smooth complete model of the affine curve in $\mathbf{A}^2_{\overline{\mathbf{F}}_p}$ given by the equation

$$aw^{p+1}=v^p-v \tag{4.3.1}$$

where $a\in\overline{\mathbf{F}}_p{}^*$ is the image under $W[\zeta_p]\to\overline{\mathbf{F}}_p$ of $f_1(0,0)$. If we put $\alpha:=w^{-1}$ and $\beta:=vw^{-1}$ then we find the equation:

$$-a=\alpha^p\beta-\alpha\beta^p \tag{4.3.2}$$

for an other affine part of $D_{i,s}$.

In order to describe various actions on the $D_{i,s}$ it is necessary to recall the construction in [19], §13.8 of the parameters $x$ and $y$ of the complete local ring $\hat{\mathcal{O}}_{X_i,s}$ of $X_i$ at $s$. By definition we have a triple $(\mathcal{E}/\hat{\mathcal{O}}_{X_i,s},\phi,\alpha)$. Let $Z$ be a parameter of the formal group of $\mathcal{E}$. Then we can take $x=Z(\phi(1,0))$ and $y=Z(\phi(0,1))$. Putting things together we have:

$$x=Z(\phi(1,0)),\;\; y=Z(\phi(0,1)),\;\; y=vx,\;\; \pi=xw,\;\; \alpha=\pi^{-1}x,\;\; \beta=\pi^{-1}y. \tag{4.3.3}$$

**4.4 Components of Igusa type.** In this section we want to give a precise description of the components of Igusa type in $X_p^{\mathrm{st}}$. We have already seen in §4.2 that these are precisely the irreducible components of $(X\tilde{\otimes}\mathbf{Z}[\zeta_p])\otimes\overline{\mathbf{F}}_p$. Because we have the isomorphism 4.2.1 between $X\tilde{\otimes}\mathbf{Z}[\zeta_p]$ and the disjoint union of the $X_i$, $i\in\mathbf{F}_p{}^*$, it is sufficient to describe the irreducible components of the $X_i\otimes_{\mathbf{Z}[\zeta_p]}\overline{\mathbf{F}}_p$. In fact, all $X_i\otimes_{\mathbf{Z}[\zeta_p]}\overline{\mathbf{F}}_p$ are the same: they classify triples $(E/S/\overline{\mathbf{F}}_p,\phi,\alpha)$ with $e_p(\phi(1,0),\phi(0,1))=1$.

In [19], §13.7and §13.8 it is shown that the irreducible components of $X_i\otimes_{\mathbf{Z}[\zeta_p]}\overline{\mathbf{F}}_p$ are the compactifications of the $p{+}1$ stacks, labeled by the $P\in\mathbf{P}^1(\mathbf{F}_p)$, classifying triples $(E/S/\overline{\mathbf{F}}_p,\phi,\alpha)$ with $P\subset\ker(\phi)$, where we view $P$ as a line in $\mathbf{F}_p{}^2$. We denote the corresponding components in $X_p^{\mathrm{st}}$ by $\mathrm{Ig}_{i,P}$, where $i\in\mathbf{F}_p{}^*$ and $P\in\mathbf{P}^1(\mathbf{F}_p)$. For every $P\in\mathbf{P}^1(\mathbf{F}_p)$ we choose a surjective linear map $L_P\colon\mathbf{F}_p^{\,2}\to\mathbf{F}_p$ such that $P=\ker(L_P)$ as in [19], §13.8: for $P=\mathbf{F}_p\cdot\binom{x}{1}$ we take $L_P\colon\binom{a}{b}\mapsto a{-}bx$ and for $P=\mathbf{F}_p\cdot\binom{1}{0}$ we take $L_P\colon\binom{a}{b}\mapsto b$. Using these $L_P$ we get isomorphisms:

$$\mathrm{Ig}_{i,P}\ \tilde{\to}\ \overline{\mathcal{M}}(\mathrm{ExIg}(p,1),\Gamma_1(N))_{\overline{\mathbf{F}}_p},\qquad (E/S/\overline{\mathbf{F}}_p,\phi,\alpha)\mapsto(E/S/\overline{\mathbf{F}}_p,\bar{\phi},\alpha)\tag{4.4.1}$$

where $\bar{\phi}\colon\mathbf{F}_p\to E(S)$ is the unique map such that $\phi=\bar{\phi}\circ L_P$. Since these isomorphisms depend on the choice of the $L_P$ we will avoid using them as much as we can.

**4.5 The action of $\mathrm{GL}_2(\mathbf{F}_p)$.** The group $G:=\mathrm{GL}_2(\mathbf{F}_p)=\mathrm{Aut}(\mathbf{F}_p{}^2)$ acts from the right on $X$ by automorphisms $r(g)$:

$$r(g)\colon\quad (E/S,\phi,\alpha)\mapsto(E/S,\phi\circ g,\alpha)$$

We let $G$ act on $X\otimes\mathbf{Q}(\zeta_p,\pi)$ via its action on $X$; by construction this action extends uniquely to an action on $X^{\mathrm{st}}$. The aim of this section is to describe the action that we get on $X_p^{\mathrm{st}}$. First of all, since $e_p(\cdot,\cdot)$ is alternating we have:

$$e_p(\phi\circ g(1,0),\phi\circ g(0,1))=e_p(\phi(1,0),\phi(0,1))^{\det(g)}$$

Hence the action of $g\in G$ on $\coprod_i X_i\otimes_{\mathbf{Z}[\zeta_p]}\overline{\mathbf{F}}_p$ is given by:

$$\Big((E/S/\overline{\mathbf{F}}_p,\phi,\alpha),i\Big)\mapsto\Big((E/S/\overline{\mathbf{F}}_p,\phi\circ g,\alpha),i\det(g)\Big)$$

From this we can easily see the action of $g\in G$ on the $\mathrm{Ig}_{i,P}$. Each $g$ induces an isomorphism $r(g)\colon\mathrm{Ig}_{i,P}\ \tilde{\to}\ \mathrm{Ig}_{i\det(g),g^{-1}P}$. The stabilizer of $\mathrm{Ig}_{i,P}$ is the Borel subgroup $B_P$ of $\mathrm{SL}_2(\mathbf{F}_p)$ that fixes $P$. Let $\chi_P\colon B_P\to\mathbf{F}_p{}^*$ be the character giving the action on the line $P$, then for $g\in B_P$ we have $L_P\circ g=\chi_P(g)^{-1}L_P$ since $\det(g)=1$. It follows that the action of $g\in B_P$ on $\overline{\mathcal{M}}(\mathrm{ExIg}(p,1),\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ that we get via the isomorphism 4.4.1 is the so-called diamond operator:

$$\langle\chi_P(g)^{-1}\rangle_p:(E/S,\bar{\phi},\alpha)\mapsto(E/S,\chi_P(g)^{-1}\bar{\phi},\alpha)\tag{4.5.1}$$

Note that any other choice of $L_P$ would lead to the same $\langle\chi_P(g)^{-1}\rangle_p$.

Let us now consider the $D_{i,s}$. Each $g\in G$ induces an automorphism $r(g)\colon D_{i,s}\ \tilde{\to}\ D_{i\det(g),s}$. The stabilizer of $D_{i,s}$ is $\mathrm{SL}_2(\mathbf{F}_p)$. Let $g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbf{F}_p)$. Then $g$ acts (from the left) on

$W[\zeta_p][[x,y]]/(f)$ by:

$$\begin{aligned} r(g)^{\#}x &= r(g)^{\#}Z(\phi(1,0)) = Z(\phi\circ g(1,0)) = Z(\phi(a,c)) = Z(a\phi(1,0)+c\phi(0,1)) = \\ &= ax+cy \pmod{(x,y)^2} \\ r(g)^{\#}y &= bx+dy \pmod{(x,y)^2} \end{aligned}$$

It follows that $g$ acts on $D_{i,s}$ by:

$$r(g)^{\#}: \alpha \mapsto a\alpha + c\beta, \qquad \beta \mapsto b\alpha + d\beta \tag{4.5.2}$$

where $\alpha$ and $\beta$ are the coordinates from 4.3.2.

**4.6 The action of inertia.** Let $G_{\mathbf{Q}}$ be the Galois group of $\overline{\mathbf{Q}}$ over $\mathbf{Q}$. Let $\overline{\mathbf{Z}}$ be the integral closure of $\mathbf{Z}$ in $\overline{\mathbf{Q}}$. We fix a surjection $\overline{\mathbf{Z}} \to \overline{\mathbf{F}}_p$ (these are permuted transitively by $G_{\mathbf{Q}}$). Then we have a sequence of subgroups:

$$I_p \lhd I \lhd G_p \subset G_{\mathbf{Q}},$$

where $G_p$ is the decomposition group, $I$ the inertia (or ramification) subgroup, and $I_p$ the wild inertia subgroup. We can identify $G_p$ with $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$, $G_p/I$ with $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$, and $I_t := I/I_p$ (the tame inertia group) with $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{tr}}/\mathbf{Q}_p^{\mathrm{unr}})$ and with $\varprojlim \mathbf{F}_{p^n}^* = \varprojlim \mu_n(\overline{\mathbf{F}}_p)$ by $\sigma \mapsto (\sigma(p^{1/n})p^{-1/n})_{p\nmid n}$.

For $\sigma \in G_{\mathbf{Q}}$ we define $\gamma(\sigma)$ to be the automorphism $\mathrm{id} \times \mathrm{Spec}(\sigma)$ of $X \times \mathrm{Spec}(\overline{\mathbf{Q}}) = X^{\mathrm{st}} \times_{\mathrm{Spec}(\mathbf{Q}(\zeta_p,\pi))} \mathrm{Spec}(\overline{\mathbf{Q}})$. Note that $\gamma$ is a right-action: $\gamma(\sigma_1\sigma_2) = \gamma(\sigma_2)\circ\gamma(\sigma_1)$. By construction, the automorphisms $\gamma(\sigma)$ extend uniquely to automorphisms $\gamma(\sigma)$ of $X^{\mathrm{st}} \times_{\mathrm{Spec}(\mathbf{Z}[\zeta_p,\pi])} \mathrm{Spec}(\overline{\mathbf{Z}})$; the diagrams:

$$\begin{array}{ccc} X^{\mathrm{st}} \times_{\mathrm{Spec}(\mathbf{Z}[\zeta_p,\pi])} \mathrm{Spec}(\overline{\mathbf{Z}}) & \xrightarrow{\gamma(\sigma)} & X^{\mathrm{st}} \times_{\mathrm{Spec}(\mathbf{Z}[\zeta_p,\pi])} \mathrm{Spec}(\overline{\mathbf{Z}}) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\overline{\mathbf{Z}}) & \xrightarrow{\mathrm{Spec}(\sigma)} & \mathrm{Spec}(\overline{\mathbf{Z}}) \end{array}$$

are commutative. For each $\sigma \in G_p$ we get an automorphism, still denoted $\gamma(\sigma)$, of $X_p^{\mathrm{st}}$, compatible with the automorphism $\mathrm{Spec}(\sigma)$ of $\mathrm{Spec}(\overline{\mathbf{F}}_p)$. Finally, for $\sigma \in I$ we get an $\overline{\mathbf{F}}_p$-automorphism $\gamma(\sigma)$ of $X_p^{\mathrm{st}}$. This right-action of $I$ on $X_p^{\mathrm{st}}$ is what we call the action of inertia.

Since $X^{\mathrm{st}}$ lives over $\mathbf{Z}[\zeta_p,\pi]$ which is of degree $p^2-1$ over $\mathbf{Z}$ and totally ramified at $p$, the inertia action of $I$ factors through an antihomomorphism $\bar{\gamma}\colon \mathbf{F}_{p^2}^* \to \mathrm{Aut}_{\overline{\mathbf{F}}_p}(X_p^{\mathrm{st}})$. The composition of $I \to \mathbf{F}_{p^2}^*$ and the norm $\mathbf{F}_{p^2}^* \to \mathbf{F}_p{}^*$ is the character giving the action of $I$ on the $p$th roots of unity. It follows that the action of $u \in \mathbf{F}_{p^2}^*$ on the left hand side of 4.2.1 is given by sending $(E/S \xrightarrow{f} \mathrm{Spec}(\mathbf{Z}[\zeta_p]), \phi, \alpha)_i$ to $(E/S \xrightarrow{f'} \mathrm{Spec}(\mathbf{Z}[\zeta_p]), \phi, \alpha)_{iu^{-p-1}}$, where $f' = \mathrm{Spec}(u^{p+1})\circ f$. We conclude that $u \in \mathbf{F}_{p^2}^*$ induces isomorphisms:

$$\bar{\gamma}(u) : \begin{cases} \mathrm{Ig}_{i,P} \xrightarrow{\sim} \mathrm{Ig}_{iu^{-p-1},P} \\ D_{i,s} \xrightarrow{\sim} D_{iu^{-p-1},s} \end{cases} \tag{4.6.1}$$

The stabilizers in $\mathbf{F}_{p^2}^*$ of the $\mathrm{Ig}_{i,P}$ and the $D_{i,s}$ are all equal to $\mu_{p+1}(\mathbf{F}_{p^2})$. Since the $\mathrm{Ig}_{i,P}$ already live in $X\tilde{\otimes}\mathbf{Z}[\zeta_p]$ the $\bar{\gamma}(u)$ with $u \in \mu_{p+1}(\mathbf{F}_{p^2})$ act trivially on them. The action of the $\bar{\gamma}(u)$ with $u \in \mu_{p+1}(\mathbf{F}_{p^2})$ on the $D_{i,s}$ can be read off from equations 4.3.2 and 4.3.3 (just note that $\gamma(u)^{\#}\pi = u\pi \bmod \pi^2$). One finds that for $u \in \mu_{p+1}(\mathbf{F}_{p^2})$ the automorphism $\bar{\gamma}(u)$ of $D_{i,s}$ is given in the coordinates from 4.3.2 by:

$$\bar{\gamma}(u)^{\#}\colon \alpha \mapsto u^{-1}\alpha, \quad \beta \mapsto u^{-1}\beta. \tag{4.6.2}$$

**4.7 Proposition.** *The actions of $G$ and $\mathbf{F}_{p^2}^*$ from §4.5 and §4.6 on $X_p^{\mathrm{st}}$ commute. The stabilizer of $\mathrm{Ig}_{i,P}$ in $\mathbf{F}_{p^2}^* \times G$ is $\{(u,g) \mid u^{p+1} = \det(g),\ \ g^{-1}P = P\}$; it acts on $\mathrm{Ig}_{i,P}$ by $\langle u^{p+1}\chi_P(g^{-1})\rangle_p$ (cf. 4.5.1). The stabilizer of $D_{i,s}$ is $\{(u,g) \mid u^{p+1} = \det(g)\}$; $(u, (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}))$ in it acts by $\alpha \mapsto u^{-1}(a\alpha + c\beta)$, $\beta \mapsto u^{-1}(b\alpha + d\beta)$ in the coordinates of 4.3.2.*

**Proof.** The two actions commute by construction. The remaining statements follow from computations as in §4.5 and §4.6. $\square$

**4.8 Proposition.** *Let $f$ and $\chi$ be as in Prop. 3.5.3. Let $K \subset \overline{\mathbf{Q}}$ be the field generated by the eigenvalues of $f$ for the $T_n^*$ $(n \geq 1)$; then $\chi$ has values in $K$. Let $\lambda \not| p$ be a finite place of $K$; let $\rho_\lambda\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(K_\lambda)$ be associated to $f$ as in §2.5. Then $\rho_\lambda|_{I_p}$ (notation as in §4.6) is trivial and:*

1. *if $\chi = \pi(\alpha, \alpha^{-1})$ with $\alpha \neq \alpha^{-1}\colon \mathbf{F}_p{}^* \to \overline{K}^*$ then $(\rho_\lambda|_I) \otimes \overline{K}_\lambda \cong \alpha \oplus \alpha^{-1}$,*
2. *if $\chi = \pi^-(\alpha)$ with $\alpha\colon \mathbf{F}_p{}^* \to K^*$ of order 2 then $\rho_\lambda|_I$ is an extension of $\alpha$ by $\alpha$,*
3. *if $\chi = \pi(\Lambda)$ with $\Lambda^p = \Lambda^{-1} \neq \Lambda\colon \mathbf{F}_{p^2}^* \to \overline{K}^*$ then $(\rho_\lambda|_I) \otimes \overline{K}_\lambda \cong \Lambda \oplus \Lambda^{-1}$.*

**Proof.** Since $f$ is defined over $K$ (use a $q$-expansion) the $G$-representation $V$ that it generates is defined over $K$ hence its character $\chi$ has values in $K$. Let $\chi_\lambda := \chi \otimes_K K_\lambda$.

Let $X^o$ denote the complement of the cusps in $X$ and $j\colon X^o \hookrightarrow X$ denote the inclusion. As in §2.5 one shows that the $K_\lambda$-representation $\rho_\lambda^\vee \otimes \chi_\lambda$ of $G_{\mathbf{Q}} \times G$ occurs in $\mathrm{H}^1_{\mathrm{par}}(X^o_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k) \otimes_{\mathbf{Q}_l} K_\lambda$; after replacing $N$ by some multiple we can assume that $N > 4$. The cokernel of the inclusion $j_!\mathcal{F}_l^k \hookrightarrow j_*\mathcal{F}_l^k$ is supported on the cusps hence the map $\mathrm{H}^1_c(X^o_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k) \to \mathrm{H}^1(X_{\overline{\mathbf{Q}}}, j_*\mathcal{F}_l^k)$ is surjective. The Leray spectral sequence for $j_*$ shows that $\mathrm{H}^1(X_{\overline{\mathbf{Q}}}, j_*\mathcal{F}_l^k) \to \mathrm{H}^1(X^o_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)$ is injective. It follows that $\mathrm{H}^1_{\mathrm{par}}(X^o_{\overline{\mathbf{Q}}}, \mathcal{F}_l^k)$ is equal to $\mathrm{H}^1(X_{\overline{\mathbf{Q}}}, j_*\mathcal{F}_l^k)$. The exact sequence of vanishing cycles (see [17], VII, Exp. XIII, Thm. 3.4(ii)) reads:

$$0 \to \mathrm{H}^1(X_p^{\mathrm{st}}, j_*\mathcal{F}_l^k) \to \mathrm{H}^1(X^{\mathrm{st}}{}_{\overline{\mathbf{Q}}_p}, j_*\mathcal{F}_l^k) \to \bigoplus_x \mathcal{F}_l^k(-1)_x \tag{4.8.1}$$

where $x$ ranges over the double points of $X_p^{\mathrm{st}}$. If $\pi$ denotes the normalization map of $X_p^{\mathrm{st}}$ then from the short exact sequence $0 \to j_*\mathcal{F}_l^k \to \pi_*\pi^*j_*\mathcal{F}_l^k \to \bigoplus_x \mathcal{F}_{l,x}^k \to 0$ (depending on the choice of an orientation of the graph associated to $X_p^{\mathrm{st}}$) we get an exact sequence:

$$\bigoplus_x \mathcal{F}_{l,x}^k \longrightarrow \mathrm{H}^1(X_p^{\mathrm{st}}, j_*\mathcal{F}_l^k) \longrightarrow \bigoplus_{i,P} \mathrm{H}^1(\mathrm{Ig}_{i,P}, j_*\mathcal{F}_l^k) \oplus \bigoplus_{i,s} \mathrm{H}^1(D_{i,s}, \mathcal{F}_{l,s}^k) \longrightarrow 0 \tag{4.8.2}$$

The exact sequences 4.8.1 and 4.8.2 are compatible with the actions of $G_p \times G$; the inertia group $I \lhd G_p$ acts via its tame quotient $I_t = \varprojlim \mathbf{F}_{p^n}^*$. The following lemma describes the terms of 4.8.1 and 4.8.2 as $I_t \times G$-modules.

**4.8.3 Lemma.** *(i).* $\oplus_x\overline{\mathbf{Q}}_l \cong \oplus_s \oplus_\alpha \alpha^{-1}\otimes(\pi^-(\alpha)\oplus\alpha\circ\det)$, *where $s$ ranges over the super singular points in $\mathcal{M}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ and $\alpha$ over all characters $\mathbf{F}_p{}^*\to\overline{\mathbf{Q}}_l^*$.*

*(ii).* $\oplus_{i,P}\mathrm{H}^1(\mathrm{Ig}_{i,P}, j_*\mathcal{F}_l^k)\otimes\overline{\mathbf{Q}}_l$ *is a direct sum of copies of* $\alpha^{-1}\otimes\pi(\alpha,\beta)$, $\alpha^{-1}\otimes\pi^-(\alpha)$ *and* $\alpha^{-1}\otimes\alpha\circ\det$.

*(iii).* $\oplus_{i,s}\mathrm{H}^1(D_{i,s},\overline{\mathbf{Q}}_l)\cong\oplus_s\oplus_\Lambda\Lambda^{-1}\otimes\pi(\Lambda)$, *with* $\Lambda^p\neq\Lambda\colon\mathbf{F}_{p^2}^*\to\overline{\mathbf{Q}}_l^*$.

**Proof.** (i). Fix an $s$ and let $V:=\oplus_{x\mapsto s}\overline{\mathbf{Q}}_l$. As a representation of $G$, $V\cong\oplus_\alpha(\pi^-(\alpha)\oplus\alpha\circ\det)$ since $V$ is the induced of the trivial representation of some Borel subgroup of $\mathrm{SL}_2(\mathbf{F}_p)$. Note that $I_t$ acts on $V$ via $\mathbf{F}_p{}^*$ so that as a representation of $\mathbf{F}_p{}^*\times G$, $V$ is the direct sum over the $\alpha$ of $\beta_\alpha\otimes\pi^-(\alpha)$ and $\beta'_\alpha\otimes\alpha\circ\det$. Since the stabilizer subgroup of some $x$ is the subgroup $H:=\{(\det(b),b)\mid b\in B\}$ of $\mathbf{F}_p{}^*\times B$ all $\beta_\alpha\otimes\pi^-(\alpha)$ and $\beta'_\alpha\otimes\alpha\circ\det$ must have non-zero $H$-invariants. From the formulas $\mathrm{Res}^G_B\pi^-(\alpha)=\mathrm{Res}^T_B(\alpha,\alpha)\oplus(\text{irr. of dim. } p-1)$ and $\mathrm{Res}^G_B(\alpha\circ\det)=\mathrm{Res}^T_B(\alpha,\alpha)$ it follows that $\beta_\alpha=\beta'_\alpha=\alpha^{-1}$.

(ii). Let $V:=\oplus_{i,P}\mathrm{H}^1(\mathrm{Ig}_{i,P},j_*\mathcal{F}_l^k)$. Note that the action of $I_t$ on $V$ factors through $\mathbf{F}_p{}^*$. As a representation of $G$, $V$ is a direct sum of non-cuspidal representations since it is the induction from $B$ to $G$ of $\oplus_i\mathrm{H}^1(\mathrm{Ig}_{i,(1:0)},j_*\mathcal{F}_l^k)$ and $B$ acts on $\coprod_i\mathrm{Ig}_{i,(1:0)}$ through $T$. Suppose that $W\cong\gamma\otimes\pi(\alpha,\beta)$ is a subrepresentation of $V\otimes\overline{\mathbf{Q}}_l$. Let $H$ be the subgroup $\{(u,(\begin{smallmatrix}u&a\\0&1\end{smallmatrix}))\mid u\in\mathbf{F}_p{}^*,a\in\mathbf{F}_p\}$ of $\mathbf{F}_p{}^*\times G$. Then we must have $W^H\neq0$ since $H$ acts trivially on the $\mathrm{Ig}_{i,(1:0)}$. One computes that $\mathrm{Res}^G_HW$ is the direct sum of the characters $\gamma\alpha$ and $\gamma\beta$ of $\mathbf{F}_p{}^*$ and an irreducible representation of dimension $p-1$. It follows that $\gamma=\alpha^{-1}$ and $\gamma=\beta^{-1}$. The cases $\pi^-(\alpha)$ and $\alpha\circ\det$ are treated in the same way.

(iii). Fix an $s$. Let $V:=\oplus_i\mathrm{H}^1(D_{i,s},\overline{\mathbf{Q}}_l)$. Note that $I_t$ acts on $V$ via $\mathbf{F}_{p^2}^*$. Using the Lefschetz trace formula one computes that, as a $\mathbf{F}_{p^2}^*$-representation, $V\cong\oplus_{\Lambda\neq\Lambda^p}(p-1)\Lambda$. Let $V_\Lambda$ denote the $\Lambda$-eigenspace of $V$. The trace of an element $g$ in $G$ on $V_\Lambda$ is equal to the trace of $g\mathrm{Pr}_\Lambda$ on $V$, where $\mathrm{Pr}_\Lambda=(p^2-1)^{-1}\sum_u\Lambda(u^{-1})u$. Using the Lefschetz trace formula one finds that $V_\Lambda\cong\pi(\Lambda^{-1})$ as $G$-module. ☐

Note that $\det\circ\rho_\lambda|_I$ is trivial. Prop. 4.8 now follows from the exact sequences 4.8.1 and 4.8.2 and Lemma 4.8.3. ☐

## 5 A stable model for $\overline{\mathcal{M}}(\Gamma_0(p^2), \Gamma_1(N))$a at $p$.

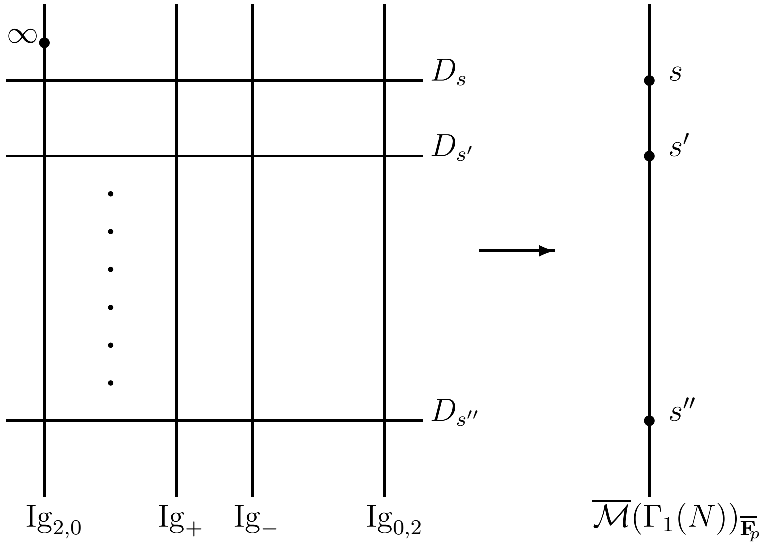
**5.1 Construction of a stable model.** We keep the notation from §4: $X = \overline{\mathcal{M}}(\Gamma(p), \Gamma_1(N))$ and $X^{\mathrm{st}}$ is the model of $X$ over $\mathbf{Z}[\zeta_p, \pi]$ which is stable at $p$, with $\pi^{p+1} = 1-\zeta_p$, as constructed in §4.2; in particular, the fibre $X^{\mathrm{st}}_p$ of $X^{\mathrm{st}}$ over $\overline{\mathbf{F}}_p$ is reduced, its irreducible components are smooth and intersect each other transversally. The right-action of the group $G = \mathrm{GL}_2(\mathbf{F}_p)$ on $X$ induces a right-action of $G$ on $X^{\mathrm{st}}$. Let $X_0 := \overline{\mathcal{M}}(\Gamma_0(p^2), \Gamma_1(N))$. The morphism $X_{\overline{\mathbf{Q}}} \to X_{0,\overline{\mathbf{Q}}}$ described in §3.2 comes from a morphism $X \to X_0$, which identifies $X_0$ as the quotient of $X$ by the diagonal subgroup $T$ of $G$ (away from the cusps this is [19] 11.3.5, then use that both $X$ and $X_0$ are regular). We define $X_0^{\mathrm{st}} := X^{\mathrm{st}}/T$; from general results about quotients (see §5.3) it follows that $X_0^{\mathrm{st}}$ is a model of $X_0$ which is stable at $p$. Note that the order of $T$ is prime to $p$, hence the fibre $X^{\mathrm{st}}_{0,p}$ of $X_0^{\mathrm{st}}$ over $\overline{\mathbf{F}}_p$ is equal to $X^{\mathrm{st}}_p/T$.

Another way to construct $X_0^{\mathrm{st}}$ is to apply the construction of §4.1 to the blow up of $\overline{\mathcal{M}}(\Gamma_0(p^2), \Gamma_1(N))$ in its supersingular points in characteristic $p$. Such a construction is given in [13].

**5.2 Description of the stable model.** From now on we assume that $p > 2$. Let us first compute the images of the irreducible components of Igusa type of $X^{\mathrm{st}}_p$ in $X^{\mathrm{st}}_{0,p}$. In §4.5 we have seen how the elements of $T$ act on the disjoint union of the $\mathrm{Ig}_{i,P}$, where $i \in {\mathbf{F}_p}^*$ and $P \in \mathbf{P}^1(\mathbf{F}_p)$. The action of $T$ on ${\mathbf{F}_p}^* \times \mathbf{P}^1(\mathbf{F}_p)$ has four orbits; representatives for them are: $(1,(1:0))$, $(1,(0:1))$, $(1,(1:1))$, and $(1,(1:d))$ where $d$ is some non-square in $\mathbf{F}_p$. The stabilizers of these representatives are $T \cap \mathrm{SL}_2(\mathbf{F}_p)$, $T \cap \mathrm{SL}_2(\mathbf{F}_p)$, $\{\pm 1\}$ and $\{\pm 1\}$ respectively. It follows that the image in $X^{\mathrm{st}}_{0,p}$ of the disjoint union of the $\mathrm{Ig}_{i,P}$ consists of exactly four irreducible components: $\mathrm{Ig}_{(2,0)} := \mathrm{Ig}_{1,(1:0)}/T \cap \mathrm{SL}_2(\mathbf{F}_p)$, $\mathrm{Ig}_{(0,2)} := \mathrm{Ig}_{1,(0:1)}/T \cap \mathrm{SL}_2(\mathbf{F}_p)$, $\mathrm{Ig}_+ := \mathrm{Ig}_{1,(1:1)}/\{\pm 1\}$, and $\mathrm{Ig}_- := \mathrm{Ig}_{1,(1:d)}/\{\pm 1\}$; we call them the components of Igusa type of $X^{\mathrm{st}}_{0,p}$. We have isomorphisms: $\mathrm{Ig}_{(2,0)} \cong \mathrm{Ig}_{(0,2)} \cong \overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ and $\mathrm{Ig}_+ \cong \mathrm{Ig}_- \cong \overline{\mathcal{M}}(\mathrm{Ig}(p)/\{\pm 1\}, \Gamma_1(N))_{\overline{\mathbf{F}}_p}$, where $\mathcal{M}(\mathrm{Ig}(p))$ is the $\mathbf{F}_p$-stack that classifies pairs $(E/S, P)$ with $P \in E^{(p)}(S)$ a generator of $\ker(V\colon E^{(p)} \to E)$ (see [19], §12.3.1). Under the morphism $X^{\mathrm{st}}_{0,p} \to \overline{\mathcal{M}}(\Gamma_0(p^2), \Gamma_1(N))_{\overline{\mathbf{F}}_p}$ the irreducible components $\mathrm{Ig}_{(2,0)}$, $\mathrm{Ig}_{(0,2)}$, $\mathrm{Ig}_+$ and $\mathrm{Ig}_-$ are mapped to the irreducible components called of type $(2,0)$, $(0,2)$, $(1,1)$ and $(1,1)$ in [19] §13.5.6. As stacks over $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ we have isomorphisms $\mathrm{Ig}_{(2,0)} \cong \overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$, $\mathrm{Ig}_+ \cong \mathrm{Ig}_- \cong \overline{\mathcal{M}}(\mathrm{Ig}(p)/\{\pm 1\}, \Gamma_1(N))_{\overline{\mathbf{F}}_p}$ and $\mathrm{Ig}_{(2,0)} \cong \overline{\mathcal{M}}(\Gamma_1(N))^{(p^2)}_{\overline{\mathbf{F}}_p}$, where $\overline{\mathcal{M}}(\Gamma_1(N))^{(p^2)}_{\overline{\mathbf{F}}_p}$ denotes the pullback of $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ via the square of the $\overline{\mathbf{F}}_p$-linear Frobenius on $\overline{\mathcal{M}}_{\overline{\mathbf{F}}_p}$.

The stabilizer in $T$ of an irreducible component $D_{i,s}$ of Drinfeld type is $T \cap \mathrm{SL}_2(\mathbf{F}_p)$. It follows that the image in $X^{\mathrm{st}}_{0,p}$ of the disjoint union of the $D_{i,s}$ consists of irreducible components $D_s := D_{1,s}/T \cap \mathrm{SL}_2(\mathbf{F}_p)$, labeled by the super singular $s$ in $\mathcal{M}(\Gamma_1(N))(\overline{\mathbf{F}}_p)$; we call them the components of Drinfeld type of $X^{\mathrm{st}}_{0,p}$. By construction, the $D_s$ are smooth irreducible curves over $\overline{\mathbf{F}}_p$; they are all isomorphic and one can easily verify that their genus is $p-1$. An element $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ acts on the coordinates of $D_{i,s}$ in 4.3.2 by: $\alpha \mapsto t\alpha$, $\beta \mapsto t^{-1}\beta$. It follows that $v_1 := \alpha\beta$ and $u_1 := \alpha^{p-1}$ are coordinates of an affine open of $D_s$; equation 4.3.2 gives: $u_1^2 v_1 + a u_1 - v_1^p = 0$.

Figure 1: A picture of $X_{0,p}^{\mathrm{st}} \to \overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$.




In other words, $D_s$ is the smooth complete model of the hyperelliptic curve given by:

$$u_2^2 = v_1^{p+1} + \tfrac{1}{4}a^2 \tag{5.2.1}$$

where $u_2 = u_1 v_1 + \frac{1}{2}a$.

The double points of $X_{0,p}^{\mathrm{st}}$ can be described as follows: the components of Igusa type are all disjoint and the $D_s$ are all disjoint; each $D_s$ meets all four components of Igusa type at the unique supersingular point lying over $s$ under the morphism to $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$. Figure 1 gives a picture of the situation. We will use the following notation for the double points in $X_{0,p}^{\mathrm{st}}(\overline{\mathbf{F}}_p)$: $x_{(2,0),s} := \mathrm{Ig}_{(2,0)} \cap D_s$, $x_{(0,2),s} := \mathrm{Ig}_{(0,2)} \cap D_s$, $x_{+,s} := \mathrm{Ig}_+ \cap D_s$ and $x_{-,s} := \mathrm{Ig}_- \cap D_s$. The complete local rings of $X_0^{\mathrm{st}}$ at the double points can be easily computed using either 4.1.2 and 4.1.3 or Prop. 5.3.3. The complete local rings at $x_{(2,0),s}$ and $x_{(0,2),s}$ are isomorphic to $W[\pi][[x,y]]/(xy - \pi^{p-1})$ where $W$ denotes the ring of Witt vectors of $\overline{\mathbf{F}}_p$. The complete local rings at $x_{+,s}$ and $x_{-,s}$ are isomorphic to $W[\pi][[x,y]]/(xy - \pi^2)$.

As in §4.6 let $I$ be an inertia subgroup of $G_{\mathbf{Q}}$ and let $\bar{\gamma}\colon \mathbf{F}_{p^2}^* \to \mathrm{Aut}_{\overline{\mathbf{F}}_p}(X_{0,p}^{\mathrm{st}})$ be the antihomomorphism giving the inertia action. The inertia action preserves the irreducible components $D_s$, $\mathrm{Ig}_{(2,0)}$ and $\mathrm{Ig}_{(0,2)}$; it interchanges $\mathrm{Ig}_+$ and $\mathrm{Ig}_-$. Let $u \in \mathbf{F}_{p^2}^*$. Then $\bar{\gamma}(u)$ is the identity on $\mathrm{Ig}_{(2,0)}$ and on $\mathrm{Ig}_{(0,2)}$; on $D_s$ it is the automorphism given by:

$$\bar{\gamma}(u)^{\#}\colon u_2 \mapsto u_2, \quad v_1 \mapsto u^{p-1} v_1 \tag{5.2.2}$$

where $u_2$ and $v_1$ are the coordinates of 5.2.1. If $u \in \mathbf{F}_{p^2}^*$ is a square then it induces the automorphism $\langle u^{(p+1)/2} \rangle_p$ on $\mathrm{Ig}_+$ and on $\mathrm{Ig}_-$. It will be useful to have a list giving the inertia action on the cotangent spaces of irreducible components of $X_{0,p}^{\mathrm{st}}$ at the double points.

**5.2.3 Table.** *Inertia action on cotangent spaces at double points.*
*The bottom row in this table gives the action of $\bar{\gamma}(u)^*$ on the cotangent space given in the top*

*row for $u \in \mathbf{F}_{p^2}^*$ fixing the corresponding double point (for $x_{(2,0),s}$ and $x_{(0,2),s}$ this is no condition on $u$; for $x_{\pm,s}$ it means that $u$ is a square).*

| $x_{(2,0),s}^*\Omega^1_{\mathrm{Ig}_{(2,0)}/\overline{\mathbf{F}}_p}$ | $x_{(2,0),s}^*\Omega^1_{D_s/\overline{\mathbf{F}}_p}$ | $x_{\pm,s}^*\Omega^1_{\mathrm{Ig}_\pm/\overline{\mathbf{F}}_p}$ | $x_{\pm,s}^*\Omega^1_{D_s/\overline{\mathbf{F}}_p}$ | $x_{(0,2),s}^*\Omega^1_{\mathrm{Ig}_{(0,2)}/\overline{\mathbf{F}}_p}$ | $x_{(0,2),s}^*\Omega^1_{D_s/\overline{\mathbf{F}}_p}$ |
|---|---|---|---|---|---|
| $u^0$ | $u^{p-1}$ | $u^{p+1}$ | $u^{1-p}$ | $u^0$ | $u^{p-1}$ |

**5.3 Some generalities on quotients of stable curves.** The following is based on [19], pp. 508–510. There it is proved that the quotient of a smooth curve over a noetherian regular scheme by a finite group acting trivially on the base is again a smooth curve. We do the same kind of computations for stable curves over discrete valuation rings.

**5.3.1 Lemma.** *Let $V$ be a complete discrete valuation ring with uniformizer $\pi$. Let $A := V[[x,y]]/(xy-\pi^e)$ and let $G$ be a finite subgroup of $\mathrm{Aut}_V(A)$ that does not interchange the two branches modulo $\pi$. Let $A^G$ be the subring of $G$-invariants in $A$. Let $B := V[[u,v]]/(uv-\pi^{eg})$, where $g := \#G$. Define $B \hookrightarrow A$ by $u \mapsto N(x)$, $v \mapsto N(y)$, where $N\colon A \to A^G$ is the map $a \to \prod_{\sigma\in G}\sigma(a)$. Then the image of $B$ in $A$ is $A^G$.*

**Proof.** Clearly we have that $B \subset A^G$. It is a well known fact that both $A$ and $B$ are normal. Therefore it suffices to show that $A$ is finitely generated as a $B$-module (then the same holds for $A^G$) and that the field of fractions $\mathrm{Frac}(B)$ of $B$ is equal to $\mathrm{Frac}(A^G)$.

We will show that 1, $x$, $y$,…,$x^{g-1}$, $y^{g-1}$ generate $A$ as $B$-module. For each $\sigma \in G$ we have $\sigma(x) = xt_\sigma$ and $\sigma(y) = yt_\sigma^{-1}$ with $t_\sigma \in A^*$. Hence $N(x) = x^g t$ and $N(y) = y^g t^{-1}$ for some $t \in A^*$. Now let $a \in A$. Using the relation $xy = \pi^e$ we can write $a = \sum a_i x^i + \sum b_i y^i$ with $a_i$ and $b_i$ in $V$. It follows that we can write $a = \sum_{i=0}^{g-1} a_i x^i + \sum_{i=0}^{g-1} b_i y^i + ua' + va''$ with $a'$ and $a''$ in $A$. Iteration of this process shows that $a$ is a linear combination over $B$ of the $x^i$ and $y^i$ with $0 \le i < g$.

Let $B'$ be the local ring of $\mathrm{Spec}(B)$ at the generic point of branch given by $u = 0$ and let $A'$ be the local ring of $\mathrm{Spec}(A)$ at the generic point of $x = 0$. Then $B' \subset A'$ is a finite extension of discrete valuation rings, both with uniformizer $\pi$. Hence the degree of the extension of fraction fields is the degree of the extension of residue fields. Let $k := V/\pi V$. The extension of the residue fields can be written as $k((v)) \subset k((y))$ and 1, $y$,…,$y^{g-1}$ is a basis of $k((y))$ over $k((v))$ because $v = y^g t^{-1}$. It follows that $\mathrm{Frac}(A^G) = \mathrm{Frac}(B)$ since $\mathrm{Frac}(A)$ has dimension $g$ over both of them. □

**5.3.2 Lemma.** *Let $V$ and $A$ be as in Lemma 5.3.1 and let $G$ be a finite subgroup of $\mathrm{Aut}_V(A)$ that interchanges the two branches mod $\pi$. Let $H$ be the subgroup of $G$ that stabilizes the two branches. Then $A^G$ is regular and as parameters one can take $\pi$ and $N_H(x) + \sigma N_H(y)$ where $\sigma \notin H$ and $N_H\colon A \to A^H$ is the map $a \mapsto \prod_{\sigma\in H}\sigma(a)$.*

**Proof.** Let us first take the quotient by $H$ and let $h := \#H$. By Lemma 5.3.1 we are reduced to the situation where $\mathbf{Z}/2\mathbf{Z}$ acts on $V[[x,y]]/(xy-\pi^{eh})$. Now we have $\sigma(x) = yu$ with $u$ a unit. We replace the coordinate $y$ by $\sigma(x)$. Then the situation is: $A = V[[x,y]]/(xy - \pi^{eh}u)$

and $\sigma(x) = y$. Any $a \in A$ can be written uniquely as $a = a_0 + \sum_{i>0} a_i x^i + \sum_{i>0} b_i y^i$ with $a_0$, $a_i$ and $b_i$ in $V$. We see that $a = \sigma(a)$ if and only if $a_i = b_i$ for all $i$. In other words, we have $A^\sigma = \{a_0 + \sum_{i>0} a_i(x^i + y^i)\}$. All the $x^i + y^i$ can be expressed (with coefficients in $\mathbf{Z}$) in $t := x + y$ and $xy = \pi^{eh} u$. Iteration shows that $A^\sigma = V[[t]]$. $\square$

**5.3.3 Proposition.** *Let $V$ be a discrete valuation ring with uniformizer $\pi$ and residue field $k$. Let $X \to \mathrm{Spec}(V)$ a stable curve whose generic fibre is smooth. Let $G$ be a finite subgroup of $\mathrm{Aut}_V(X)$. Then we have:*

*(i) The special fibre of $G\backslash X$ is reduced and its singularities are ordinary double points. In other words: $G\backslash X$ is a semi-stable curve over $V$.*

*(ii) The image in $(G\backslash X)_k$ of a smooth point $P$ of $X_k$ is smooth and if $\pi$ and $x$ are parameters at $P$ then $\pi$ and $N_{\mathrm{Stab}}(x)$ are parameters at the image of $P$.*

*(iii) Let $P \in X_k(k)$ be singular, say with complete local ring isomorphic over $V$ to the ring $V[[x_P, y_P]]/(x_P y_P = \pi^{e_P})$. Let $Q$ be the image of $P$ in $(G\backslash X)_k(k)$. Then $Q$ is singular if and only if $\mathrm{Stab}_P$ fixes the two branches at $P$. If $Q$ is singular then the complete local ring of $Q$ is isomorphic to $V[[x_Q, y_Q]]/(x_Q y_Q - \pi^{e_Q})$ with $e_Q = e_P \cdot \#\mathrm{Stab}_P$, $x_Q = N_{\mathrm{Stab}_P(x_P)}$ and $y_Q = N_{\mathrm{Stab}_P(y_P)}$. If $Q$ is smooth then $\pi$ and $N_H(x_P) + \sigma N_H(x_P)$ are parameters at $Q$, where $H \subset \mathrm{Stab}_P$ is the subgroup stabilizing the two branches at $P$ and $\sigma \in \mathrm{Stab}_P$, $\sigma \notin H$.*

*(iv) Let $Y$ be an irreducible component of $X_k$ and $Z$ the image of $Y$ in $(G\backslash X)_k$. Let $H := \mathrm{Stab}_Y$ and $K := \ker(H \to \mathrm{Aut}_k(Y))$. Then the restriction to $Y$ of the quotient morphism $X \to G\backslash X$ factorizes as $Y \to (H/K)\backslash Y \to Z$, where the first morphism is separable and the second is purely inseparable of degree $\#K$.*

**Proof.**

## 6 Divisors and leading terms of new forms.

**6.1 Definitions.** Let $f$ be a normalized new form of weight $k \geq 2$ on $\overline{\mathcal{M}}(\Gamma_0(p^2), \Gamma_1(N))_{\overline{\mathbf{Q}}}$ as in §3.2. We fix an embedding $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. Let $X_0^{\mathrm{st}}$ denote the model of $\overline{\mathcal{M}}(\Gamma_0(p^2), \Gamma_1(N))$ over $\mathbf{Z}[\pi]$ as constructed in §5.1; recall that $\pi^{p+1} = 1 - \zeta_p$. We fix an embedding $\mathbf{Z}[\pi] \to \overline{\mathbf{Z}}_p$. Then we can view $f$ as a rational section of the invertible sheaf $\underline{\omega}^{\otimes k}$ on $X^{\mathrm{st}}_{0,\overline{\mathbf{Z}}_p}$. As such, $f$ has a divisor. This is not completely "standard" since $\overline{\mathbf{Z}}_p$ is not noetherian, so we must explain what we mean. Let $v_p\colon \overline{\mathbf{Q}}_p^* \to \mathbf{Q}$ denote the normalized valuation. Then $v_p$ extends uniquely to the local rings of $X^{\mathrm{st}}_{0,\overline{\mathbf{Z}}_p}$ at the generic points of the irreducible components of the fibre $X^{\mathrm{st}}_{0,p}$ over $\overline{\mathbf{F}}_p$. We can now define:

$$\mathrm{div}(f) = H + a_{(2,0)}\mathrm{Ig}_{(2,0)} + a_{(0,2)}\mathrm{Ig}_{(0,2)} + a_+\mathrm{Ig}_+ + a_-\mathrm{Ig}_- + \sum_s a_s D_s \tag{6.1.1}$$

with $H$ an effective "horizontal" divisor (as in the theory of arithmetic surfaces, i.e., it is the closure of the scheme of zeros of $f$ over $\overline{\mathbf{Q}}_p$), and with the $a_* \in \mathbf{Q}$ the valuations of $f$ at the corresponding generic points. If $f$ has valuation $a_*$ along an irreducible component $C$ of $X^{\mathrm{st}}_{0,p}$ then $(p^{-a_*}f)|_C$ is a rational section of $\underline{\omega}^{\otimes k}|_C$, called the leading term of $f$ along $C$. The numbers $a_{(2,0)}$, $a_{(0,2)}$, $a_+$ and $a_-$ can be read off from the $q$-expansions of $f$. More precisely, we have formal charts at infinity:

$$\begin{array}{llll} 1\colon & (\mathrm{Tate}(q), \mu_{p^2}, 1 \mapsto \zeta_N) & \text{over} & \mathbf{Z}_p[\pi, \zeta_N][[q]] \\ 2\colon & (\mathrm{Tate}(\zeta_p^i q), \langle q^{1/p}\rangle, 1 \mapsto \zeta_N) & \text{over} & \mathbf{Z}_p[\pi, \zeta_N][[q]] \qquad (i \in \mathbf{F}_p{}^*) \\ 3\colon & (\mathrm{Tate}(q), \langle q^{1/p^2}\rangle, 1 \mapsto \zeta_N) & \text{over} & \mathbf{Z}_p[\pi, \zeta_N][[q^{1/p^2}]] \end{array} \tag{6.1.2}$$

The closed fibre of chart 1 is a chart for $\mathrm{Ig}_{(2,0)}$; chart 3 gives a chart for $\mathrm{Ig}_{(0,2)}$ and one can check that chart 2 gives a chart for $\mathrm{Ig}_+$ (resp. $\mathrm{Ig}_-$) if $-i$ is (resp. is not) a square in $\mathbf{F}_p$. Prop. 3.5.3 can now be interpreted as follows.

**6.2 Proposition.** *Let $f$ be as in §6.1, $\chi$ as in §3.2 and $a_n$, $a'_n$ and $a''_n$ as in 3.3.6.*

1. $a_{(2,0)} = v_p(\sum a_n q^n) = 0$.
2. $a_{(0,2)} = v_p(\sum a''_n q^n) = -k$.
3. *if* $\chi = \pi(\tau^m, \tau^{-m})$ *with* $0 < m < (p-1)/2$ *then* $a_+ = a_- = v_p(\sum a'_n q^n) = -m/(p-1)$,
4. *if* $\chi = \pi(\tau_2^{(p-1)m})$ *with* $0 < m < (p+1)/2$ *then* $a_+ = a_- = v_p(\sum a'_n q^n) = (1-m)/(p-1)$,
5. *if* $\chi = \pi^-(\tau^{(p-1)/2})$ *then* $a_+ = a_- = v_p(\sum a'_n q^n) = -1/2$. □

Let $a := a_+ = a_-$; then we have $-\frac{1}{2} \leq a \leq 0$. Let $b := \min_s a_s$. Our main interest in this section will be to get more information on $b$ and to determine the leading terms of $f$ along the components of $X^{\mathrm{st}}_{0,p}$ of Igusa type.

**6.3 Lemma.** *Let $V$ be a complete discrete valuation ring. Let $n \geq 1$, $t \neq 0$ in the maximal ideal of $V$ and let $X := \mathrm{Spec}(V[[x,y]]/(xy - t))$. Suppose that $f \in \mathcal{O}_X(X)$, considered as a function on the subscheme defined by the ideal $(y, \pi)$, has a zero of order $k$ at $x = 0$. Then the valuation of $f$ at the generic point of the subscheme defined by $(x, \pi)$ is at most $k$ times the valuation of $t$.*

**Proof.** Let $\pi \in V$ be a uniformizer and let $\tilde{f} = \sum_{i,j\geq 0} a_{i,j}x^iy^j \in V[[x,y]]$ be a lift of $f$. Using the relation $xy = t$ we can write $\tilde{f} = a_0 + \sum_{i>0} a_ix^i + \sum_{i>0} b_iy^i$. Modulo $(y,\pi)$ we have: $\tilde{f} = \overline{a_0} + \sum_{i>0} \overline{a_i}x^i$. Hence $\overline{a_i} = 0$ for $i < k$ and $\overline{a_k} \neq 0$. On the open subscheme $D(y)$ of $X$ where $y$ is invertible the subscheme defined by $(x,\pi)$ is defined by the equation $\pi = 0$. It follows that the valuation we are after is equal to the valuation of $a_0 + \sum_{i>0} a_it^iy^{-i} + \sum_{i>0} b_iy^i$ in $\mathcal{O}_X(D(y)) = V[[y,ty^{-1}]][y^{-1}]$. The valuation of the coefficient of $y^{-k}$ is $k$ times the valuation of $t$. $\square$

**6.4 Proposition.** *We have:*

$$b \leq a + \lfloor\frac{k-1}{2}\rfloor\frac{2}{p^2-1}$$

*where for any real number $x$, $\lfloor x\rfloor$ is the largest integer less than or equal to $x$.*

**Proof.** We may suppose that $b > a$. Then $\mathrm{div}(p^{-a}f) = H - a\mathrm{Ig}_{(2,0)} - (a+k)\mathrm{Ig}_{(0,2)} + \sum_s(a_s-a)D_s$. Since $X^{\mathrm{st}}_{0,\overline{\mathbf{Z}}_p}$ is normal, $p^{-a}f$ is a section of $\underline{\omega}^{\otimes k}(-\mathrm{cusps})$ in an open neighborhood of $\mathrm{Ig}_+$. Recall that in [19] §12.8 a construction is given of a global section $A^{1/(p-1)}$ of $\underline{\omega}$ on $\overline{\mathcal{M}}(\mathrm{Ig}(p))$ which has a first order zero at every supersingular point, and no other zeros. Because $\mathrm{Ig}_+$ is isomorphic over $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ to $\overline{\mathcal{M}}(\mathrm{Ig}(p)/\{\pm1\},\Gamma_1(N))$, it follows that there is at least one $s$ where $(p^{-a}f)|_{\mathrm{Ig}_+}$ has a zero of order at most $\lfloor\frac{k-1}{2}\rfloor$. The complete local ring of $X^{\mathrm{st}}_{0,\mathbf{Z}[\pi]}$ at $x_{+,s}$ is isomorphic to $W[\pi][[x,y]]/(xy-\pi^2)$ (see §5.2). We can apply Lemma 6.3 with $V$ a finite extension of $W[\pi]$ over which $f$ is defined. $\square$

In the same way one can easily show that $b > -k$. However, we need a better lower bound for $b$. To get such a bound we will use the Kodaira-Spencer morphism.

**6.5 The Kodaira-Spencer morphism.** Recall ([19], §10.13.10, and [15], III, §9) that for any morphisms of schemes $f\colon X \to S$ and $g\colon S \to T$, with $f$ proper and smooth, one has the Kodaira-Spencer class:

$$\mathrm{KS} \in \Gamma\left(S, (\mathbf{R}^1f_*\mathrm{T}_{X/S}) \otimes_{\mathcal{O}_S} \Omega^1_{S/T}\right) \tag{6.5.1}$$

inducing a morphism of $\mathcal{O}_S$-modules:

$$\mathrm{KS}\colon \left(\mathbf{R}^1f_*\mathrm{T}_{X/S}\right)^\vee \longrightarrow \Omega^1_{S/T} \tag{6.5.2}$$

where $\mathrm{T}_{X/S} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Omega^1_{X/S},\mathcal{O}_X)$ and $(-)^\vee$ denotes the $\mathcal{O}_S$-dual. For $f'\colon X' \to S'$ obtained from $f$ by a base change $S' \to S$ over $T$, there is an obvious compatibility between the elements in 6.5.1 for $f$ and $f'$. If $\mathbf{R}^1f_*\mathrm{T}_{X/S}$ is locally free, then it is of course equivalent to give 6.5.1 or 6.5.2.

For $E \to S$ an elliptic curve we will interpret 6.5.2, via Serre duality, as a morphism $\mathrm{KS}\colon \underline{\omega}^{\otimes 2}_{E/S} \to \Omega^1_{S/T}$. It is known ([19], Thm. 10.13.11) that for the universal elliptic curve $\mathbf{E}/\mathcal{M}$ the Kodaira-Spencer morphism is an isomorphism with a first order pole at infinity:

$$\mathrm{KS}\colon \underline{\omega}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\overline{\mathcal{M}}/\mathbf{Z}}(\mathrm{cusps}) \tag{6.5.3}$$

Equivalently, viewing KS as a section of the line bundle $\underline{\omega}^{\otimes -2} \otimes_{\mathcal{O}_{\mathcal{M}}} \Omega^1_{\mathcal{M}/\mathbf{Z}}$, we can say that the divisor of KS on $\overline{\mathcal{M}}$ is "−cusps". We want to study the divisor of KS for $X^{\mathrm{st}}_{0,\overline{\mathbf{Z}}_p} \to \mathrm{Spec}(\overline{\mathbf{Z}}_p)$, but since $\Omega^1_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}$ is not a line bundle, we will consider the composition:

$$\mathrm{KS}\colon \underline{\omega}^{\otimes 2} \longrightarrow \Omega^1_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p} \hookrightarrow \omega_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p} \tag{6.5.4}$$

where $\omega_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}$ is the dualizing sheaf ([11], I, §2.3). As in §6.1 multiplicities of components of $X^{\mathrm{st}}_{0,p}$ are measured with the normalized $p$-adic valuation.

**6.5.5 Proposition.** *1. The divisor of the Kodaira-Spencer morphism in 6.5.4 is:*

$$-\mathrm{cusps} + 2\mathrm{Ig}_{(0,2)} + \frac{1}{p+1}\sum_s D_s$$

*2. On $X^{\mathrm{st}}_{\overline{\mathbf{Z}}_p}$: $\mathrm{div}(\mathrm{KS}\colon \underline{\omega}^{\otimes 2} \to \omega_{X^{\mathrm{st}}/\overline{\mathbf{Z}}_p}) = -\mathrm{cusps} + \sum_{i,P} \mathrm{Ig}_{i,P} + \frac{p}{p+1}\sum_{i,s} D_{i,s}$.*

**Proof.** We will only prove (1); the proof of (2) is analogous. The multiplicities of the cusps and the components of Igusa type can be seen in the formal charts 6.1.2 at infinity. One knows ([19], Thm. 10.13.11) that KS for $\mathrm{Tate}(q)/\mathbf{Z}[[q]]/\mathbf{Z}$ maps $(dt/t)^{\otimes 2}$ to $dq/q$. In all three charts, $(dt/t)^{\otimes 2}$ is a local generator of $\underline{\omega}^{\otimes 2}$; local generators for $\omega_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}$ are $dq/q$, $dq/q$ and $d(q^{1/p^2})/q^{1/p^2} = p^{-2}dq/q$. It remains to compute the multiplicities of the $D_s$. To do this we use the computations of [13], §2.2.4 alluded to at the end of §5.1.

As explained in [19], Thm. 13.4.7, the complete local ring of $\mathcal{M}(\Gamma_0(p^2), \Gamma_1(N))$ at a supersingular point $s \in \mathcal{M}(\Gamma_1(N))(\overline{\mathbf{F}}_p)$ is isomorphic to $W[[x,y]]/(f_0 + pf_1)$, with $f_0 = (x^{p^2} - y)(x-y)^{p-1}(x - y^{p^2})$ and $f_1$ a unit. Moreover, the coordinates $x$ and $y$ are the "local moduli of source and target". Let $f$ be a local generator of $\underline{\omega}^{\otimes 2}$ at $s$. Then, since $\mathcal{M}(\Gamma_1(N)) \to \mathcal{M}$ is etale at $s$, $\mathrm{KS}(f) = \mathrm{unit}\cdot dx$. In [13], §2.2.4 it is shown that $u$ and $v$, defined by $y = vx$ and $x = u\pi^{p-1}$, induce non-zero rational functions on $D_s$ and that $du$ induces a non-zero rational differential form on $D_s$. It follows that the valuation of $dx$ along $D_s$ is the valuation of $\pi^{p-1}$, i.e., $1/(p+1)$. □

**6.6 Proposition.** *Let $f$ be as in §6.1 and $a$ and $b$ as in Prop. 6.4. Then we have:*

$$a - \frac{k}{p+1} \leq b$$

**Proof.** Let us first suppose that $k$ is even. Then $\omega(f) := \mathrm{KS}^{\otimes k/2}(f)$ is a section of $\omega^{\otimes k/2}_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}$ with divisor:

$$\mathrm{div}(\omega(f)) = H - \frac{k}{2}\mathrm{cusps} + a(\mathrm{Ig}_+ + \mathrm{Ig}_-) + \sum_s a'_s D_s \tag{6.6.1}$$

with $a'_s = a_s + \frac{k}{2(p+1)}$. We have to prove that for all $s$: $a - a'_s \leq \frac{k}{2(p+1)}$. Assume that for some $s$ we have $a - a'_s > \frac{k}{2(p+1)}$. Then $p^{-a'_s}\omega(f)|_{D_s}$ is a global section of $\omega^{\otimes k/2}_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}|_{D_s}$; according to Lemma 6.3 is has zeros of order at least $k/2$ at $x_{(2,0),s}$ and $x_{(0,2),s}$. On the other hand, $\deg_{D_s}(\omega_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}) = p+1$. One concludes that at $x_{+,s}$ or $x_{-,s}$, $p^{-a'_s}\omega(f)|_{D_s}$ has a zero of order at most $k(p-1)/4$. Lemma 6.3 now gives $a - a'_s \leq k/(2(p+1))$. This finishes the proof for $k$ even.

Let us now deal with the odd $k$. In that case we can simply repeat the proof for $k$ even but now applied to $f^{\otimes 2}$; the fact that $f^{\otimes 2}$ is not an eigenform does not matter at all. $\square$

**6.7 $q$-expansions of leading terms.** Let $f$, $a$ and $b$ be as in Prop. 6.6. We are interested in the $q$-expansions of the leading terms of $f$ along the components of $X_{0,p}^{\mathrm{st}}$ that are of Igusa type. More precisely, we want to evaluate $f$ on the formal charts 6.1.2 and take the leading terms (in the $p$-adic sense as in §6.1) of the power series obtained in that way. We have already computed those power series in §3.3 and §3.4; their $p$-adic valuations are given in Prop. 3.5.3. It seems convenient to formulate the result one gets not on the $\mathrm{Ig}_*$ but on $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ and on $\overline{\mathcal{M}}(\mathrm{Ig}(p),\Gamma_1(N))_{\overline{\mathbf{F}}_p}$. Recall from §5.2 that we have isomorphisms:

$$\phi_{(2,0)}\colon \overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p} \xrightarrow{\sim} \mathrm{Ig}_{(2,0)}, \quad \phi_{(0,2)}\colon \overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p} \xrightarrow{\sim} \mathrm{Ig}_{(0,2)} \tag{6.7.1}$$

and two morphisms:

$$\phi_{\pm}\colon \overline{\mathcal{M}}(\mathrm{Ig}(p),\Gamma_1(N))_{\overline{\mathbf{F}}_p} \longrightarrow \mathrm{Ig}_{\pm} \tag{6.7.2}$$

that are the quotient for the action of $\{1,-1\}$ on $\mathrm{Ig}(p)$. Note that $\phi_{(0,2)}$ is “exotic” ([19], §11) in the sense that it is not compatible with the maps to $\overline{\mathcal{M}}$; the other three are not exotic. Via these morphisms we can interpret the leading terms $(p^{-a_*}f)|_{\mathrm{Ig}_*}$ as modular forms on $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ or on $\overline{\mathcal{M}}(\mathrm{Ig}(p),\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ with poles (possibly) in the super singular points. Using the description of the inertia action on $\mathrm{Ig}_{\pm}$ given in §5.2 one finds that $\phi_{\pm}^*(p^{-a}f|_{\mathrm{Ig}_{\pm}})$ has eigenvalue $x^{-2(p-1)a}$ for $\langle x\rangle_p^*$, $x\in\mathbf{F}_p{}^*$. Let (again) $A^{1/(p-1)}\in\overline{\mathcal{M}}(\mathrm{Ig}(p),\underline{\omega})$ denote the $(p-1)$th root of the Hasse invariant $A$ as constructed in [19], §12.8; then $A^{1/(p-1)}$ has non-zero constant $q$-expansion at all cusps and $\langle x\rangle_p^*A^{1/(p-1)}=xA^{1/(p-1)}$ for all $x$ in $\mathbf{F}_p{}^*$. It follows that $f'_{\pm}:=(A^{1/(p-1)})^{2(p-1)a}\phi_{\pm}^*(p^{-a}f|_{\mathrm{Ig}_{\pm}})$ is a modular form on $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$, possibly with poles in the supersingular points, of weight $k+2(p-1)a$, with character $\varepsilon$, whose $q$-expansion at $(\mathrm{Tate}(q),1{\mapsto}\zeta_N)$ is up to a constant factor and some $(dt/t)$’s equal to that of $\phi_{\pm}^*(p^{-a}f|_{\mathrm{Ig}_{\pm}})$ at $(\mathrm{Tate}(q),q,1{\mapsto}\zeta_N)$. Using the results of §3 it is now straightforward (but somewhat long) to prove the following proposition.

**6.7.3 Proposition.** *The four forms $\phi_{(2,0)}^*(f|_{\mathrm{Ig}_{(2,0)}})$, $\phi_{(0,2)}^*(p^kf|_{\mathrm{Ig}_{(0,2)}})$ and $f'_{\pm}$ are eigenforms for all $T_l^*$ (including $l{=}p$) with character $\varepsilon$ and weights $k$, $p^2k$ and $k+2(p-1)a$ respectively. The first two have eigenvalues $\overline{a_l}$, where $\overline{a_l}$ is the image of $a_l$ as in 3.3.5 under $\overline{\mathbf{Z}}_p\to\overline{\mathbf{F}}_p$. The eigenvalues of $f'_{\pm}$ depend on the character $\chi$ associated to $f$ as in §3.2. For $\chi=\pi(\tau^m,\tau^{-m})$ with $0<m<(p-1)/2$ one has $T_l^*f'_{\pm}=l^{-m}\overline{a_l}f'_{\pm}$ $(l\neq p)$ and $T_p^*f'_{\pm}=\overline{\lambda'_p}f'_{\pm}$ ($\lambda'_p$ as in 3.4.4). For $\chi=\pi(\tau_2^{(p-1)m})$, with $0<m<(p+1)/2$, one has $T_l^*f'_{\pm}=l^{1-m}\overline{a_l}f'_{\pm}$ $(l\neq p)$ and $T_p^*f'_{\pm}=0$. For $\chi=\pi^-(\tau^{(p-1)/2})$ one has $T_l^*f'_{\pm}=l^{(p-1)/2}\overline{a_l}f'_{\pm}$ $(l\neq p)$ and $T_p^*f'_{\pm}=\overline{b_p}f'_{\pm}$ ($b_p$ as in Prop. 3.4.13).* $\square$

**6.8 Forms of weight 2 defined over unramified extensions.** From now on we suppose that $f$ is a normalized new form on $\overline{\mathcal{M}}(\Gamma_0(p^2),\Gamma_1(N))_{\overline{\mathbf{Q}}}$ of weight 2, character $\varepsilon$ and that $f$ is defined over an unramified subextension of $\mathbf{Q}_p\to\overline{\mathbf{Q}}_p$ (recall that we have a fixed $\overline{\mathbf{Q}}\to\overline{\mathbf{Q}}_p$). In this case we have $a_{(2,0)}{\in}\mathbf{Z}$, $a_{(0,2)}{\in}\mathbf{Z}$, $a{\in}(p-1)^{-1}\mathbf{Z}$ and $a_s{\in}(p+1)^{-1}\mathbf{Z}$ for all $s$ (cf. 6.1.1). Let $\omega(f):=\mathrm{KS}(f)\in\mathrm{H}^0(X_{0,\overline{\mathbf{Z}}_p}^{\mathrm{st}},\omega_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p})\otimes\mathbf{Q}$ (note that since $f$ is a cusp form, $\mathrm{KS}(f)$ has no pole at

$\infty$); then $\operatorname{div}(\omega(f))$ is given in 6.6.1. Let $b'$ be the minimum of the $a'_s$, hence $b' = b + 1/(p{+}1)$. According to Prop.'s 6.4 and 6.6 we have:

$$a - \frac{1}{p+1} \;\leq\; b' \;\leq\; a + \frac{1}{p+1} \tag{6.8.1}$$

Recall that we already know $a$ in terms of $\chi$ (cf. Prop. 6.2), hence in terms of the $\rho_\lambda|_I$ with $\lambda \not| p$ (cf. Prop. 4.8). Now we want to find the exact value of $b'$ in terms of the representation $\overline{\rho}$ associated to $f$. By definition, $\overline{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is the unique (up to isomorphism) semi-simple continuous representation unramified outside $pN$ such that $\operatorname{trace}(\overline{\rho}(\mathrm{Frob}_l)) = \overline{a_l}$ and $\det(\overline{\rho}(\mathrm{Frob}_l)) = \varepsilon(l)l^{k-1}$ for all $l \not| pN$.

Recall that the filtration $w(g)$ of a non-zero mod $p$ modular form $g \in \mathrm{H}^0(\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}, \underline{\omega}^{\otimes k})$ of weight $k$ is defined to be the smallest integer $k'$ for which there exists a form $g' \in \mathrm{H}^0(\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}, \underline{\omega}^{\otimes k'})$ which, at some cusp, has $q$-expansion equal to that of $g$ (see for example [14], §3 and references given there). Equivalently, $(p{-}1)^{-1}(k - w(g))$ is the minimum of the orders of the zeros of $g$ at the super singular points. We extend this definition to forms $g$ that are allowed to have poles at the super singular points (the form $g'$ is of course not allowed to have poles).

**6.8.2 Proposition.** *Under the assumptions made at the beginning of §6.8 we have:*

1. $w(\phi^*_{(2,0)}(f|_{\mathrm{Ig}_{(2,0)}})) = 2 - (p^2{-}1)b$
2. $w(f'_\pm) = 2 + (p^2{-}1)(a{-}b) = p{+}1 + (p^2{-}1)(a{-}b') \leq 2p$
3. *there exist super singular* $s{\in}\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ *with* $a_s = b$ *and* $x_{(2,0),s}$, $x_{(0,2),s}$, $x_{+,s}$ *and* $x_{-,s}$ *not in the support of* $H$.

**Proof.** Let us write $a = -\alpha/(p{-}1)$; then $0 \leq \alpha \leq (p{-}1)/2$. Recall that $b{\in}(p{+}1)^{-1}\mathbf{Z}$ and that $b' = b + 1/(p{+}1)$. According to 6.8.1 there are at most three possible values for $b'$. Let us first deal with the case $0 < \alpha < (p{-}1)/2$. Then there are exactly two possibilities: $b' = -(\alpha{+}1)/(p{+}1)$ and $b' = -\alpha/(p{+}1)$; they are equivalent to $b' < a$ and $b' > a$, respectively. If $b' < a$ and $a_s = b$ then $p^{-b'}\omega(f)|_{D_s}$ is a non-zero global section of $\omega_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}|_{D_s}$ on which inertia acts via $I \to \mathbf{F}_{p^2}{}^*$ followed by $\mathbf{F}_{p^2}{}^* \to \overline{\mathbf{F}}_p{}^*$: $u \mapsto u^{(p-1)(\alpha+1)}$ (see §4.6 for conventions about inertia). The degree of $\omega_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}|_{D_s}$ is $p{+}1$. From Table 5.2.3 it follows that $p^{-b'}\omega(f)|_{D_s}$ has zeros at $x_{(2,0),s}$ and $x_{(0,2),s}$ of order $\alpha{+}1$ and zeros at $x_{+,s}$ and $x_{-,s}$ of order $(p{-}1)/2 - \alpha$.

The complete local ring at $x_{+,s}$ of $X_0^{\mathrm{st}}$ is isomorphic to $W[\pi][[x,y]]/(xy{-}\pi^2)$ (see §5.2). It follows from the construction as in §4.1 of $X_0^{\mathrm{st}}$ that the section $p^{-b'}\omega(f)$ over this ring is already defined over $W[\pi^2]$, i.e., to study its divisor we may view $p^{-b'}\omega(f)$ as an element of the regular local ring $W[\pi^2][[x,y]]/(xy - \pi^2)$. Since the $p$-adic valuation of $p^{-b'}\omega(f)$ along $\mathrm{Ig}_+$ equals $a{-}b'$ and $v_p(\pi^2) = (p^2{-}1)/2$ we have:

$$(H \cdot D_s)_{x_{+,s}} + \frac{p^2-1}{2}(a - b') = \frac{p-1}{2} - \alpha$$

where $(H \cdot D_s)_{x_{+,s}}$ denotes the local intersection number. It follows that $(H \cdot D_s)_{x_{+,s}} = 0$. Likewise: $(H \cdot D_s)_{x_{-,s}} = 0$. This shows that $x_{+,s}$ and $x_{-,s}$ are not in the support of $H$. Using this information one can verify that $w(f'_\pm) = 2 + (p^2{-}1)(a{-}b)$.

To prove that $x_{(2,0),s}$ is not in the support of $H$ we do a similar computation. Now we view $p^{-b'}\omega(f)$ as an element of $W[\pi^{p-1}][[x,y]]/(xy-\pi^{p-1})$. The fact that $-b'/v_p(\pi^{(p-1)})$ is the order of vanishing of $p^{-b'}\omega(f)|_{D_s}$ at $x_{(2,0),s}$ gives the result. One can now verify that $w(\phi^*_{(2,0)}f|_{\mathrm{Ig}_{(2,0)}}) = 2-(p^2-1)b$.

Suppose now that $b' > a$, i.e., that $b' = -\alpha/(p+1)$. Then $p^{-a}\omega(f)|_{\mathrm{Ig}_+}$ is a global section of $\omega_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}|_{\mathrm{Ig}_+}$ on which inertia acts via $I \to \mathbf{F}_{p^2}{}^*$ followed by $\mathbf{F}_{p^2}{}^* \to \overline{\mathbf{F}}_p{}^*$: $u \mapsto u^{(p+1)\alpha}$. It follows that $\phi^*_+(p^{-a}\omega(f)|_{\mathrm{Ig}_+})$ is a global section of $\Omega^1(\text{s.s.})$ on $\overline{\mathcal{M}}(\Gamma_1(N),\mathrm{Ig}(p))$ on which $\langle x\rangle^*$ has eigenvalue $x^{2\alpha}$ for all $x\in\mathbf{F}_p{}^*$; therefore it has zeros of order $2\alpha$ modulo $p-1$ at the supersingular points. A computation shows that $\deg_{\overline{\mathcal{M}}(\Gamma_1(N),\mathrm{Ig}(p))}\Omega^1(\text{s.s.}) = (p+1)\deg_{\overline{\mathcal{M}}(\Gamma_1(N))}(\text{s.s.}) - (p-1)\deg_{\overline{\mathcal{M}}(\Gamma_1(N))}(\text{cusps})$. This implies that there exist super singular $s$ where $\phi^*_+(p^{-a}\omega(f)|_{\mathrm{Ig}_+})$ has a zero of order exactly $2\alpha$, or equivalently, where $p^{-a}\omega(f)|_{\mathrm{Ig}_+}$ has a zero of order exactly $\alpha$. Fix such a $s$. Since $(p^2-1)(b'-a)/2 = \alpha$, a computation as above shows that $a_s = b$ and that $x_{+,s}$ is not in the support of $H$. Likewise: $x_{-,s}$ is not in the support of $H$. It follows that $p^{-b'}\omega(f)|_{D_s}$ has poles of order exactly $\alpha$ at $x_{\pm,s}$. The action of inertia shows that $p^{-b'}\omega(f)|_{D_s}$ has zeros of order exactly $\alpha$ modulo $p+1$ at $x_{(2,0),s}$ and $x_{(0,2),s}$. Moreover, the orders of vanishing at $x_{(2,0),s}$ and $x_{(0,2),s}$ are equal. Since $\deg_{D_s}\omega_{X_0^{\mathrm{st}}} = p+1$ we see that $p^{-b'}\omega(f)|_{D_s}$ has zeros of order exactly $\alpha$ at $x_{(2,0),s}$ and $x_{(0,2),s}$. It follows that $x_{(2,0),s}$ and $x_{(0,2),s}$ are not in the support of $H$ and that $w(f'_\pm)$ and $w(\phi^*_{(2,0)}(f|_{\mathrm{Ig}_{(2,0)}}))$ are as claimed.

It remains to study the cases $a = 0$ and $a = -1/2$. Suppose first that $a = 0$. Then $\chi = \pi(\tau_2^{(p-1)})$, hence $T_p^* f = 0$ by Prop. 3.3.4 and Lemma 3.5.1(4). By Prop. 6.7.3 $\phi^*_{(2,0)}(f|_{\mathrm{Ig}_{(2,0)}})$ and $f'_+$ are eigenforms for all $T_l^*$ (including $l = p$) and have the same eigenvalues, so their $q$-expansions are equal up to a scalar factor. Consequently: $w(f'_\pm) = w(f)$. According to 6.8.1 $b' \in \{1/(p+1), 0, -1/(p+1)\}$. If $b' = 1/(p+1)$ then $b = 0$, $f|_{\mathrm{Ig}_{(2,0)}}$ has no poles at the super singular points and since it is a cusp form of weight two it cannot have zeros at all super singular $s$ hence $w(f) = 2$. If $b' = -1/(p+1)$ and $a_s = b$ then $p^{1/(p+1)}\omega(f)|_{D_s}$ is a non-zero global section of $\omega_{X_0^{\mathrm{st}}/\overline{\mathbf{Z}}_p}|_{D_s}$ on which inertia acts by $u \mapsto u^{p-1}$; hence it has zeros of order exactly 1 and $(p-1)/2$ at $x_{(2,0),s}$ and $x_{\pm,s}$, respectively; computations as above prove the required statements.

Suppose now that $b' = 0$. Then $\omega(f)|_{\mathrm{Ig}_{(2,0)}}$, $\omega(f)|_{\mathrm{Ig}_{(0,2)}}$ and $\omega(f)|_{\mathrm{Ig}_\pm}$ are global sections of $\Omega^1(\text{s.s.})$ with $q$-expansions equal up to scalar factors. If all residues of $\omega(f)|_{\mathrm{Ig}_{(2,0)}}$ at the super singular points are zero then the same is true on the other components of Igusa type and the $\omega(f)|_{D_s}$ are global sections of $\Omega^1_{D_s}$ for all $s$ on which inertia acts trivially; but no such non-zero differential forms exist. Hence not all residues are zero. But then $g := A\phi^*_{(2,0)}f|_{\mathrm{Ig}_{(2,0)}}$ is a cusp form on $\overline{\mathcal{M}}(\Gamma_1(N))_{\overline{\mathbf{F}}_p}$ of weight $p+1$, $w(g) = p+1$ and $T_p^* g = 0$; by [14], Prop. 3.3 this is impossible. It follows that $b' = 0$ is impossible.

Finally, suppose that $a = -1/2$. According to 6.8.1 at most three values for $b'$ are possible. If $b' = -1/2-1/(p+1)$ and $a_s = b$ then $p^{-b'}\omega(f)|_{D_s}$ is a non-zero global section of $\Omega^1_{D_s}$ on which inertia acts by $u \mapsto u^{-(p-1)(p+3)/2}$. But no such differential forms exist, hence $b' = -1/2$ or $b' = -1/2+1/(p+1)$. In these two cases computations as above prove the required statements. □

**6.8.3 Corollary.** *Let $k(\overline{\rho}) \in \mathbf{Z}$ be the "Serre weight" associated to $\overline{\rho}$ as in [14], Def. 4.3. Suppose that $\overline{\rho}$ is irreduible. If $k(\overline{\rho}) > p+1$ then $b = (2-k(\overline{\rho}))/(p^2-1)$. If $k(\overline{\rho}) = 2$ and $\overline{\rho}$ is of* *level one (see [14], §2.4) then $b = -2/(p+1)$. If $k(\overline{\rho}) = 2$ and $\overline{\rho}$ is of level two then $b = 0$. The case $k(\overline{\rho}) = p+1$ does not occur.*

**Proof.** Let us first remark that the difference between $k(\overline{\rho})$ and Serre's original definition in [25], §1–2 of the "weight" of $\overline{\rho}$ is irrelevant in our situation; see [14], Rem. 4.4 and note that we have $p > 2$ and $k(\overline{\rho}) \equiv 2(p-1)$. Theorem 4.5 of [14] shows that there exists a mod $p$ cuspidal eigenform $g$ of weight $k(\overline{\rho})$, level $N$ and character $\varepsilon$ such that $\phi^*_{(2,0)}f|_{\mathrm{Ig}_{(2,0)}}$ and $g$ have the same eigenvalues for all $T^*_l$ ($l \neq p$). If $k(\overline{\rho}) > p+1$ then $T^*_p g = 0$ by [16], Prop. 4.12, hence $\phi^*_{(2,0)}f|_{\mathrm{Ig}_{(2,0)}}$ and $g$ have the same $q$-expansions at a fixed cusp, so $k(\overline{\rho}) = w(g) = w(\phi^*_{(2,0)}f|_{\mathrm{Ig}_{(2,0)}})$. If $k(\overline{\rho}) = 2$ and $\overline{\rho}$ is of level two then $T^*_p g = 0$ by [14], Thms. 2.5 and 2.6, hence $w(\phi^*_{(2,0)}f|_{\mathrm{Ig}_{(2,0)}}) = 2$, hence $b = 0$. If $k(\overline{\rho}) = 2$ and $\overline{\rho}$ is of level one then $T^*_p g \neq 0$ hence $w(\phi^*_{(2,0)}f|_{\mathrm{Ig}_{(2,0)}}) = w(\theta^{p-1}g = 2p$. (see [14], §3). If $k(\overline{\rho}) = p+1$ then $T^*_p g \neq 0$ and $w(\phi^*_{(2,0)}f|_{\mathrm{Ig}_{(2,0)}}) = w(\theta^{p-1}g = p^2+p$ hence $-b = 1 + 1/(p+1)$ which is impossible. □

**6.8.4 Remark.** 1. The representation $\overline{\rho}$ an only be reducible if $\overline{\rho}|_I \sim \chi_p^\alpha \oplus \chi_p^\beta$, where $\chi_p\colon G_{\mathbf{Q}} \to \mathbf{F}_p{}^*$ is the $p$-cyclotomic character and $(\alpha, \beta)$ equals $(0, 1)$ or $((p-1)/2, (p+1)/2)$.

2. We have already seen in Prop. 4.8 that $a$ is determined by $\rho_\lambda|_I$. Cor. 6.8.3 shows that $b$ is determined by $\overline{\rho}|_I$, at least if $\overline{\rho}$ is irreducible. Eqn. 6.8.1 gives a strong restriction on the possible combinations $(\rho_\lambda|_I, \overline{\rho}|_I)$.

# 7 Modular parametrizations.

**7.1 Definitions.** For $M$ a positive integer, let $X_0(M)$ denote coarse moduli scheme over $\mathbf{Z}$ associated to the stack $\overline{\mathcal{M}}(\Gamma_0(M))$. Let $J_0(M)$ be the Néron model over $\mathbf{Z}$ of the jacobian $J_0(M)_{\mathbf{Q}}$ of $X_0(M)$. An elliptic curve $E$ over $\mathbf{Q}$ is said to be modular if it is an isogeny factor of some $J_0(M)_{\mathbf{Q}}$; the smallest $M$ for which this happens is called the level of $E$. The Shimura-Taniyama conjecture states that every elliptic curve $E$ over $\mathbf{Q}$ is modular, and that the level of $E$ equals the conductor of (the system of $l$-adic representations of) $E$. One knows that for modular elliptic curves the level equals the conductor ([5], [20] and [9]). A modular elliptic curve $E$ of level $M$ is called strong if there exists a closed immersion $E \hookrightarrow J_0(M)_{\mathbf{Q}}$. It follows from the multiplicity one principle that such an immersion is unique up to sign.

Let $E$ be a strong modular elliptic curve of level $M$. The corresponding strong modular parametrization $\phi\colon X_0(M)_{\mathbf{Q}} \to E$ is obtained as follows:

$$\begin{array}{ccc} J_0(M)_{\mathbf{Q}} & \to & E \\ \uparrow & \nearrow_{\phi} & \\ X_0(M)_{\mathbf{Q}} & & \end{array} \tag{7.1.1}$$

where $J_0(M)_{\mathbf{Q}} \to E$ is the dual of one of the two closed immersions $E \hookrightarrow J_0(M)_{\mathbf{Q}}$ and where $X_0(M)_{\mathbf{Q}} \to J_0(M)_{\mathbf{Q}}$ is the standard immersion sending the cusp $\infty$ to 0. Let $\mathcal{E}$ be the Néron model over $\mathbf{Z}$ of $E$. Then the $\mathbf{Z}$-module $\Gamma(\mathcal{E}, \Omega^1)$ is free of rank one; let $\omega$ be one of the two generators. We get the differential form $\phi^*\omega$ on $X_0(M)\mathbf{Q}$. There is also another differential form on $X_0(M)_{\mathbf{Q}}$ related to $E$: the normalized newform $\sum a_n q^n dq/q$ corresponding to $E$. The multiplicity one principle gives:

$$\phi^*\omega = c \sum_{n\geq 1} a_n q^n \frac{dq}{q} \tag{7.1.2}$$

for some $c \in \mathbf{Q}^*$. This number $c$ is called the Manin constant of $E$ (see [21], [22], [23] and [4]).

# 8 Manin constants.

## 9 More on parametrizations.

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