A mod p variant of the André–Oort conjecture *

Bas Edixhoven & Rodolphe Richard

edix@math.leidenuniv.nl rodolphe.richard@normalesup.org

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Abstract

We state and prove a variant of the André–Oort conjecture for the product of 2 modular curves in positive characteristic, assuming GRH for quadratic fields.

1 Introduction

The André–Oort conjecture says that, for Σ any set of special points in a Shimura variety S, the irreducible components of the Zariski closure of Σ are special subvarieties. See [8] and [15] for the current state of affairs around this conjecture. In the simplest non-trivial case of this conjecture the Shimura variety S is $\mathbb{C}^2$, the product of two copies of the j-line, hence the coarse moduli space for pairs of complex elliptic curves. The irreducible special curves in $\mathbb{C}^2$ are, apart from the fibres of the two projections over CM-points, the images of the modular curves $Y_0(n)$ $(n \geq 1)$, and consist of the pairs $(j(E), j(E/\langle P\rangle))$ with E a complex elliptic curve and $P \in E$ of order n. In this case, the conjecture was proved in [1], and, conditionally on the generalised Riemann hypothesis (GRH) for quadratic fields, in [4]. In this article we state a variant in positive characteristic, and prove it under GRH for quadratic fields.

1.1 Definition For a point x in a scheme X we let $\kappa(x) = \mathcal{O}_{X,x}/m_x$ be its residue field, and we denote $\iota_x \colon \mathrm{Spec}(\kappa(x)) \to X$ the induced $\kappa(x)$-point of X. So we may view ι_x as an element of $X(\kappa(x))$, the set of $\kappa(x)$-valued points of X. For $X = \mathbb{A}^2$, we have $X(\kappa(x)) = \kappa(x)^2$.

By *CM-point in* $\mathbb{A}^2_{\mathbb{Q}}$ we mean a closed point s of the affine plane over $\mathbb{Q}$, such that both coordinates of $\iota_s \in \kappa(s)^2$ are j-invariants of CM elliptic curves.

By *CM-point in* $\mathbb{A}^2_{\mathbb{Z}}$ we mean the closure in $\mathbb{A}^2_{\mathbb{Z}}$ of a CM-point in $\mathbb{A}^2_{\mathbb{Q}}$. We view such a CM-point $\overline{\{s\}}$ as a closed subset, or as a reduced closed subscheme. For any prime number p we then denote by $\overline{\{s\}}_{\mathbb{F}_p}$ the reduced fibre over p and call it the reduction of s at p.

1.2 Theorem *Assume the generalised Riemann hypothesis for quadratic fields. Let p be a prime number. Let Σ be a set of finite closed subsets s of $\mathbb{A}^2_{\mathbb{F}_p}$ that are reductions of CM-points in $\mathbb{A}^2_{\mathbb{Z}}$. Let Z be the Zariski closure of the union of all s in Σ. Then every irreducible component*

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1.3 Remark If $K_1, \ldots, K_n$ are quadratic subfields of $\overline{\mathbb{Q}}$, then GRH holds for their compositum K if and only if it holds for each quadratic subfield of K (the zeta function of K is the product of the Riemann zeta-function and the L-functions of the quadratic subfields of K).

2 Some facts on CM elliptic curves

We will need some results on CM elliptic curves and their reduction mod p. For more detail see [4, §2], and references therein.

For E over $\overline{\mathbb{Q}}$ an elliptic curve with CM, $\mathrm{End}(E)$ is an order in an imaginary quadratic field K, hence isomorphic to $O_{K,f} = \mathbb{Z} + fO_K$, with O_K the ring of integers in K, and $f \in \mathbb{Z}_{\geq 1}$, unique, called the conductor.

For $K \subset \overline{\mathbb{Q}}$ imaginary quadratic and $f \geq 1$, we let $S_{K,f}$ be the set of isomorphism classes of (E, α), where E is an elliptictic curve over $\overline{\mathbb{Q}}$ and $\alpha\colon O_{K,f} \to \mathrm{End}(E)$ is an isomorphism, such that the action of $\mathrm{End}(E)$ on the tangent space of E at 0 induces the given embedding $K \to \overline{\mathbb{Q}}$. The group $\mathrm{Pic}(O_{K,f})$ acts on $S_{K,f}$, making it a torsor. This action commutes with the action of $G_K := \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, giving a group morphism $G_K \to \mathrm{Pic}(O_{K,f})$ through which G_K acts on $S_{K,f}$. This map is surjective, unramified outside f, and the Frobenius element at a maximal ideal m of O_K outside f is the class $[m^{-1}]$ in $\mathrm{Pic}(O_{K,f})$.

For $f' \geq 1$ dividing f, the inclusion $O_{K,f} \to O_{K_{f'}}$ induces compatible surjective maps of groups $\mathrm{Pic}(O_{K,f}) \to \mathrm{Pic}(O_{K,f'})$ and of torsors $S_{K,f} \to S_{K,f'}$: (E, α) is mapped to $O_{K,f'} \otimes_{O_{K,f}} E$ with its $O_{K,f'}$-action. In terms of complex lattices: $O_{K,f'} \otimes_{O_{K,f}} \mathbb{C}/\Lambda = \mathbb{C}/O_{K,f'}\Lambda$.

For p a prime number, and f' the prime to p part of f, the map $S_{K,f} \to S_{K,f'}$ is the quotient by the inertia subgroup at any of the maximal ideals m of O_K containing p (to show this, use the adelic description of ramification in class field theory).

Elliptic curves with CM over $\overline{\mathbb{Q}}$ extend uniquely over $\overline{\mathbb{Z}}$ (the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$), and their endomorphisms as well.

For K and f as above we define $j_{K,f}$ to be the image of $j(E)\colon \mathrm{Spec}(\overline{\mathbb{Z}}) \to \mathbb{A}^1_{\mathbb{Z}}$, where E is an elliptic curve over $\overline{\mathbb{Z}}$ with $\mathrm{End}(E)$ isomorphic to $O_{K,f}$; this does not depend on the choice of E. Then $j_{K,f}$ is an irreducible closed subset of $\mathbb{A}^1_{\mathbb{Z}}$. We equip it with its reduced induced scheme structure. Then it is finite over $\mathbb{Z}$ of degree $\#\mathrm{Pic}(O_{K,f})$, and in fact $j_{K,f}(\overline{\mathbb{Z}})$ is in bijection with $S_{K,f}$ and hence is a $\mathrm{Pic}(O_{K,f})$-torsor (here we use that K has a given embedding into $\overline{\mathbb{Q}}$). For p prime, we let $j_{K,f,\mathbb{F}_p}$ be the fibre of $j_{K,f}$ over $\mathbb{F}_p$.

Let p be a prime number, and K and f as above. If p is not split in O_K then $j_{K,f,\mathbb{F}_p}$ consists of supersingular points, and $j_{K,f}$ can be highly singular above p (by lack of supersingular targets). If p is split in O_K then $j_{K,f,\mathbb{F}_p}$ consists of ordinary points, and the corresponding elliptic curves over $\overline{\mathbb{F}}_p$ have endomorphism ring isomorphic to $O_{K,f'}$, where f' is the prime to p part of f, and then $j_{K,f',\mathbb{F}_p} = (j_{K,f,\mathbb{F}_p})_{\mathrm{red}}$, and for each morphism of rings $\overline{\mathbb{Z}} \to \overline{\mathbb{F}}_p$ the map $j_{K,f'}(\overline{\mathbb{Z}}) \to j_{K,f'}(\overline{\mathbb{F}}_p)$ is a bijection and it makes $j_{K,f',\mathbb{F}_p}(\overline{\mathbb{F}}_p)$ into a $\mathrm{Pic}(O_{K,f'})$-torsor. Note that every ordinary x in $\overline{\mathbb{F}}_p$ belongs to exactly one $j_{K,f'}(\overline{\mathbb{F}}_p)$.

3 Some facts about pairs of CM elliptic curves

Let s be a CM-point in $\mathbb{A}^2_{\mathbb{Q}}$ as in Def. 1.1. Then $s(\overline{\mathbb{Q}})$ is a $G_{\mathbb{Q}}$-orbit. Let (x_1, x_2) be in $s(\overline{\mathbb{Q}})$. Then x_1 is in $j_{K_1,f_1}(\overline{\mathbb{Q}})$ for a unique imaginary quadratic subfield K_1 of $\overline{\mathbb{Q}}$, and similarly for x_2, and $G_{K_1K_2}$ acts through $\mathrm{Pic}(O_{K_1,f_1}) \times \mathrm{Pic}(O_{K_2,f_2})$, and $s(\overline{\mathbb{Q}})$ decomposes into at most 4 orbits under $G_{K_1K_2}$.

Let p be a prime. Let s be a finite closed subset of $\mathbb{A}^2_{\mathbb{F}_p}$ that is the reduction at p of a CM-point in $\mathbb{A}^2_{\mathbb{Z}}$ (see Def. 1.1). Then $s(\overline{\mathbb{F}}_p)$ is a finite subset of $\overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p$ that is stable under $G_{\mathbb{F}_p} := \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$. For each of the 2 projections, the image of $s(\overline{\mathbb{F}}_p)$ consists entirely of ordinary points or entirely of supersingular points (this follows from the facts recalled in §2). If for all (x_1, x_2) in $s(\overline{\mathbb{F}}_p)$ both x_1 and x_2 are ordinary, then the x_1 form a $\mathrm{Pic}(O_{K_1,f_1})$-orbit, and the x_2 form a $\mathrm{Pic}(O_{K_2,f_2})$-orbit, with f_1 and f_2 prime to p.

4 Images under suitable Hecke correspondences

For ℓ a prime number, T_ℓ denotes the correspondence on the j-line, over any field not of characteristic ℓ, sending an elliptic curve E over an algebraically closed field k to the sum of its $\ell+1$ quotients by the subgroups of $E(k)$ of order ℓ. Similarly, $T_\ell \times T_\ell$ is the correspondence on the j-line times itself that sends a pair of elliptic curves (E_1, E_2) to the sum of all $(E_1/C_1, E_2/C_2)$ with C_1 and C_2 subgroups of order ℓ.

4.1 Theorem *Assumptions as in Theorem 1.2, and assume that all irreducible components of Z are of dimension 1, and are not a fibre of any of the 2 projections. There are infinitely many prime numbers ℓ such that $Z \cap (T_\ell \times T_\ell)Z$ is of dimension 1.*

Proof There are only finitely many points (x_1, x_2) in $Z(\overline{\mathbb{F}}_p)$ such that x_1 or x_2 is not ordinary. Therefore we can replace Σ by its subset of s's whose images under both projections are ordinary.

At this point we combine the arguments of [5] with reduction modulo p. Let d_1 and d_2 be the degrees of the projections from Z to $\mathbb{A}^1_{\mathbb{F}_p}$.

For s in Σ and (x_1, x_2) in $s(\overline{\mathbb{F}}_p)$, let $O_{1,s}$ and $O_{2,s}$ be the endomorphism rings of the elliptic curves E_1 and E_2 over $\overline{\mathbb{F}}_p$ corresponding to x_1 and x_2.

We claim that for all but finitely many s there is a prime number ℓ such that ℓ is split in both $O_{1,s}$ and $O_{2,s}$, and $\#s(\overline{\mathbb{F}}_p) > 2d_1d_2(\ell+1)^2$, and $\ell > \log(\#s(\overline{\mathbb{F}}_p))$. This claim follows, as in the proof of [5, Lemma 7.1], from the (conditional) effective Chebotarev theorem of Lagarias and Odlyzko [9] as stated in Theorem 4 of [12], and Siegel's theorem on class numbers of imaginary quadratic fields, [14] and [10, Ch. XVI].

Now let s, (x_1, x_2) and ℓ be as in the claim above. Let $\varphi\colon \overline{\mathbb{Z}} \to \overline{\mathbb{F}}_p$ be a morphism of rings. Then there are unique embeddings of $O_{1,s}$ and $O_{2,s}$ into $\overline{\mathbb{Z}}$ that composed with φ give the actions on the tangent spaces at 0 of E_1 and E_2. Let m be a maximal ideal of index ℓ in $O_{1,s}O_{2,s} \subset \overline{\mathbb{Z}}$, and m_1 and m_2 the intersections of m with $O_{1,s}$ and $O_{2,s}$. By the facts recalled at the end of § 2, there are canonical $\tilde{x_1}$ and $\tilde{x_2}$ in $\overline{\mathbb{Z}}$ lifting E_1 and E_2 to $\tilde{E}_1$ and $\tilde{E}_2$ with $\mathrm{End}(\tilde{E}_1) = \mathrm{End}(E_1)$ and $\mathrm{End}(\tilde{E}_2) = \mathrm{End}(E_2)$. Let σ be a Frobenius element in $G_{K_1K_2}$ at m. Then $\tilde{E}_1 = [m_1]^{-1}[m_1]\tilde{E}_1$ shows that $\tilde{E}_1$ is ℓ-isogenous to $[m_1]\tilde{E}_1$ which is the conjugate of $\tilde{E}_1$ by σ^{-1}, and similarly for $\tilde{E}_2$. Then $([m_1]E_1, [m_2]E_2)$ is the reduction of $\sigma^{-1}(\tilde{E}_1, \tilde{E}_2)$, hence in $s(\overline{\mathbb{F}}_p)$. So (x_1, x_2) is in $(T_\ell \times T_\ell)([m_1]E_1, [m_2]E_2)$. So (x_1, x_2) is both in $s(\overline{\mathbb{F}}_p)$ and in $(T_\ell \times T_\ell)(s(\overline{\mathbb{F}}_p))$. We conclude that $s(\overline{\mathbb{F}}_p)$ is contained in $Z(\overline{\mathbb{F}}_p) \cap (T_\ell \times T_\ell)Z(\overline{\mathbb{F}}_p)$. Now the degrees of the projections from $(T_\ell \times T_\ell)Z$ to $\mathbb{A}^1_{\overline{\mathbb{F}}_p}$ are $(\ell+1)^2 d_1$ and $(\ell+1)^2 d_2$, so the intersection number (in $(\mathbb{P}^1 \times \mathbb{P}^1)_{\overline{\mathbb{F}}_p}$) of Z and $(T_\ell \times T_\ell)Z$ is $2d_1 d_2 (\ell+1)^2$. But the intersection contains $s(\overline{\mathbb{F}}_p)$, which has more points than this intersection number, so the intersection is not of dimension 0. $\square$

5 Goursat's lemma and Zarhin's theorem

Here we deviate from the topological approach in [4] and [5].

5.1 Theorem *Let C be an irreducible reduced closed curve in $\mathbb{A}^2_{\overline{\mathbb{F}}_p}$, not a fibre of one of the 2 projections, such that there are infinitely many prime numbers ℓ for which $(T_\ell \times T_\ell)(C)$ is reducible. Then there is an $n \in \mathbb{Z}_{>0}$ such that C is the image of an irreducible component of $Y_0(n)_{\overline{\mathbb{F}}_p}$ in $\mathbb{A}^2_{\overline{\mathbb{F}}_p}$.*

Proof Let K denote the function field of C, and let E_1 and E_2 be elliptic curves over K with j-invariants the projections π_1 and π_2, viewed as functions on C; these E_1 and E_2 are unique up to quadratic twist. We must prove that E_1 is isogeneous to a twist of E_2.

Let $K \to K^{\mathrm{sep}}$ be a separable closure and let $G := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. For $\ell \neq p$ a prime number, let $V_{\ell,1} := E_1(K^{\mathrm{sep}})[\ell]$ and $V_{\ell,2} := E_2(K^{\mathrm{sep}})[\ell]$ and let G_ℓ be the image of G in $\mathrm{GL}(V_{\ell,1}) \times \mathrm{GL}(V_{\ell,2})$, with projections $G_{\ell,1}$ and $G_{\ell,2}$. Because of the Weil pairing, G acts on $\det(V_{\ell,1})$ and $\det(V_{\ell,2})$ by the cyclotomic character $\chi_\ell \colon G \to \mathbb{F}_l^\times = \mathrm{Aut}(\mu_\ell(K^{\mathrm{sep}}))$. For all but finitely many ℓ, $G_{\ell,1}$ contains $\mathrm{SL}(V_{\ell,1})$ and similarly for E_2 (this follows, as in [2], from the fact that for n prime to p the geometric fibres of the modular curve over $\mathbb{Z}[\zeta_n, 1/n]$ parametrising elliptic curves with symplectic basis of the n-torsion are irreducible, [6, Thm. 3] and [7, Cor. 10.9.2]). Let q be the number of elements of the algebraic closure of $\mathbb{F}_p$ in K. Then, for all but finitely many ℓ, $G_{\ell,1}$ is the subgroup of elements in $\mathrm{GL}(V_{\ell,1})$ whose determinant is a power of q, and similarly for $G_{\ell,2}$. Let L be the set of prime numbers $\ell \neq 2$ for which $G_{\ell,1}$ and $G_{\ell,2}$ are as in the previous sentence, and such that $(T_\ell \times T_\ell)(C)$ is reducible. Then L is infinite.

Let ℓ be in L. Let $N_{\ell,1} := \ker(G_\ell \to G_{\ell,2})$ and $N_{\ell,2} := \ker(G_\ell \to G_{\ell,1})$. Then $N_{\ell,i}$ is a normal subgroup of $G_{\ell,i} \cap \mathrm{SL}(V_{\ell,i})$, and G_ℓ is the inverse image of the graph of an isomorphism $G_{\ell,1}/N_{\ell,1} \to G_{\ell,2}/N_{\ell,2}$. The only normal subgroups of $\mathrm{SL}_2(\mathbb{F}_\ell)$ are the trivial subgroups and the center $\{\pm 1\}$, with different number of elements. As $\#G_{\ell,1} = \#G_{\ell,2}$, we have $\#N_{\ell,1} = \#N_{\ell,2}$, and so there are 3 cases.

If $N_{\ell,1} = \mathrm{SL}(V_{\ell,1})$, then G_ℓ contains $\mathrm{SL}(V_{\ell,1}) \times \mathrm{SL}(V_{\ell,2})$, contradicting the reducibility of $(T_\ell \times T_\ell)(C)$. Hence $N_{\ell,1}$ is $\{\pm 1\}$ or $\{1\}$, and G_ℓ gives us an isomorphism $\varphi_\ell \colon G_{\ell,1}/\{\pm 1\} \to G_{\ell,2}/\{\pm 1\}$. As all automorphisms of $\mathrm{SL}_2(\mathbb{F}_\ell)/\{\pm 1\}$ are induced by $\mathrm{GL}_2(\mathbb{F}_\ell)$ ([11], or [16, §3.3.4]), there is an isomorphism $\gamma \colon V_{\ell,1} \to V_{\ell,2}$ of $\mathbb{F}_\ell$-vector spaces (not necessarily G-equivariant) that induces the restriction φ_ℓ from $\mathrm{SL}(V_{\ell,1})/\{\pm 1\}$ to $\mathrm{SL}(V_{\ell,2})/\{\pm 1\}$. Let α_ℓ be the automorphism of $G_{\ell,1}/\{\pm 1\}$ obtained as the composition of first φ_ℓ and then $G_{\ell,2}/\{\pm 1\} \to G_{\ell,1}/\{\pm 1\}$, $g \mapsto \gamma^{-1} g \gamma$. Consider the short exact sequence

$$\{1\} \to \mathrm{SL}(V_{\ell,1})/\{\pm 1\} \to G_{\ell,1}/\{\pm 1\} \to \langle q \rangle \to \{1\}\,.$$

Then α_ℓ induces the identity on $\mathrm{SL}(V_{\ell,1})/\{\pm 1\}$ and on $\langle q\rangle$. Lemma 5.3 gives us that α_ℓ is the identity. Hence φ_ℓ is the morphism $G_{\ell,1}/\{\pm 1\} \to G_{\ell,2}/\{\pm 1\}$, $g \mapsto \gamma g \gamma^{-1}$. If $N_{\ell,1} = \{1\}$ then G_ℓ is $\Gamma_{\ell,\gamma} := \{(g, \gamma g \gamma^{-1}) : g \in G_{\ell,1}\}$, and if $N_{\ell,1} = \{\pm 1\}$ then G_ℓ is $\Gamma^{\pm}_{\ell,\gamma} := \{(g, \pm\gamma g \gamma^{-1}) : g \in G_{\ell,1}\}$. This means that $\gamma\colon V_{\ell,1}/\{\pm 1\} \to V_{\ell,2}/\{\pm 1\}$ is G-equivariant. Even better, writing, for g in G, $\gamma(gv) = \varepsilon_\ell(g) g(\gamma(v))$ with $\varepsilon_\ell(g) \in \{\pm 1\}$, this $\varepsilon_\ell\colon G \to \{\pm 1\} \subset \mathbb{F}_l^\times$ is a character, and γ is an isomorphism from $V_{\ell,1}$ to the twist of $V_{\ell,2}$ by ε_ℓ.

Let $U \subset C$ be the open subscheme where C is regular and where E_1 and E_2 have good reduction. Then for all ℓ in L, and all closed x in U, ε_ℓ is unramified at x. As U is a smooth curve over a finite field, there are only finitely many characters $\varepsilon\colon G \to \{\pm 1\}$ unramified on U, if $p \neq 2$ (this uses Kummer theory). For $p = 2$, one has to be more careful; we argue as follows. There are infinitely many characters $\varepsilon\colon G \to \{\pm 1\}$ unramified on U, but only finitely many with bounded conductor on the projective smooth curve $\overline{C}$ with function field K. Let $K' \subset K^{\mathrm{sep}}$ be the extension cut out by $V_{3,1} \times V_{3,2}$, and let $\overline{C}' \to \overline{C}$ be the corresponding cover. Then both E_1 and E_2 have semistable reduction over $\overline{C}'$ by [3, Cor. 5.18]. The Galois criterion for semi-stability in [13, Exp. IX, Prop. 3.5] tells us that all ε_ℓ become unramified on $\overline{C}'$. This shows that also for $p = 2$ there are only finitely many distinct ε_ℓ. The conclusion is that, for general p, there are only finitely many distinct ε_ℓ, and therefore we can assume (by shrinking L to an infinite subset) that they are all equal to some ε. Then we replace E_2 by its twist by ε, and then ε_ℓ are trivial.

Now Zarhin's result [17, Cor. 2.7] tells us that there is a non-zero morphism $\alpha\colon E_1 \to E_2$. □

5.2 Remark Up to sign, there is a unique isogeny $\alpha\colon E_1 \to E_2$ of minimal degree n. Then C is an irreducible component of the image of $Y_0(n)_{\mathbb{F}_p}$. We write $n = p^k m$ with m prime to p. Then C is the image of the image of $Y_0(m)_{\mathbb{F}_p}$ by the p^k-Frobenius map on the first or on the second coordinate, and C is also an irreducible component of the images of all $Y_0(p^{2i}n)$ with $i \in \mathbb{Z}_{\geq 0}$.

5.3 Lemma *Let G be a group, N a normal subgroup of G and Q the quotient. Let α be an automorphism of G inducing the identity on N and on Q, and suppose that G acts trivially by conjugation on the center of N, and that there is no non-trivial morphism from Q to the center of N. Then α is the identity on G.*

Proof We write, for all $g \in G$:

$$\alpha(g) = g\beta(g), \quad \text{with } \beta \text{ a map (of sets!) from } G \text{ to itself.}$$

As α induces the identity on Q, β takes values in N. As α is the identity on N, we have $\beta(n) = 1$ for all $n \in N$. For all g_1 and g_2 in G we have:

$$g_1 g_2 \beta(g_1 g_2) = \alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2) = g_1\beta(g_1) g_2 \beta(g_2)\,,$$

and therefore

$$\beta(g_1 g_2) = g_2^{-1}\beta(g_1) g_2 \beta(g_2)\,.$$

For g_1 in N, this gives that for all g_2 in G, $\beta(g_1g_2) = \beta(g_2)$. Hence β factors through $\overline{\beta}\colon Q \to N$: $\beta(g) = \overline{\beta}(\overline{g})$. Now, for g_1 in G and g_2 in N, we have

$$\overline{\beta}(\overline{g_1}) = \overline{\beta}(\overline{g_1 g_2}) = g_2^{-1}\overline{\beta}(\overline{g_1})g_2\,,$$

hence $\overline{\beta}$ takes values in the center of N. Now let g_1 and g_2 be in G. As $g_2^{-1}\beta(g_1)g_2 = \beta(g_1)$, $\overline{\beta}$ is a morphism of groups from Q to the center of N and therefore trivial. □

6 Proof of the main theorem

We are now ready to prove Theorem 1.2.

If $Z = \mathbb{A}^2_{\mathbb{F}_p}$ or is finite, then Z has no irreducible components of dimension 1. Now assume that Z has dimension 1. We write $Z = V \cup H \cup F \cup Z'$ with V the union in Z of fibers of the 1st projection pr_1, H the union in Z of fibers of pr_2, and F the set of isolated points in Z, and Z' the union of the remaining irreducible components of Z. Let B_1 be the image of $V \cup F$ under pr_1, and B_2 the image of $H \cup F$ under pr_2.

Let s be in Σ such that $\mathrm{pr}_1(s)$ meets B_1. Then either $\mathrm{pr}_1(s)(\overline{\mathbb{F}}_p)$ consists of supersingular points, or it consists of ordinary points with the same endomorphism ring as an ordinary point in B_1. Hence for such a $\mathrm{pr}_1(s)$ there are only finitely many possibilities. Similarly for the $\mathrm{pr}_2(s)$. It follows that the s in Σ with $\mathrm{pr}_1(s)$ disjoint from B_1 and $\mathrm{pr}_2(s)$ disjoint from B_2 are contained in Z'. Let Σ' be the set of these s. The s in $\Sigma - \Sigma'$ lie on a finite union of fibres of pr_1 and pr_2, and the intersection of this union with Z' is finite. Therefore the union of the s in Σ' is dense in Z'. We replace Z by Z', and Σ by Σ'. Then all irreducible components of Z are of dimension 1 and are not a fibre of pr_1 or pr_2. Let d_i (i in $\{1,2\}$) be the degree of pr_i restricted to Z.

There are only finitely many points (x_1, x_2) in $Z(\overline{\mathbb{F}}_p)$ such that x_1 or x_2 is not ordinary. Therefore we can replace Σ by its subset of s's whose image under both projections is ordinary.

Theorem 4.1 gives us an infinite set L of primes ℓ such that $Z \cap (T_\ell \times T_\ell)Z$ is of dimension 1. Let $(Z_i)_{i\in I}$ be the set of irreducible components of Z. Then for each ℓ in L there are i and j in I such that Z_i is in $(T_\ell \times T_\ell)Z_j$. If moreover $\ell > 12d_1$ then $(T_\ell \times T_\ell)Z_j$ is reducible, because if not, then $(T_\ell \times T_\ell)Z_j$ equals Z_i (as closed subsets of $\mathbb{A}^2_{\mathbb{F}_p}$), but for any ordinary (x, y) in $Z_j(\overline{\mathbb{F}}_p)$, $T_\ell(y)$ consists of at least $(\ell+1)/12 > d_1$ distinct points.

There is a $j_0 \in I$ such that for infinitely many $\ell \in L$, $(T_\ell \times T_\ell)Z_{j_0}$ is reducible. Theorem 5.1 then tells us that there is an $n \geq 1$ such that Z_{j_0} is the image in $\mathbb{A}^2_{\mathbb{F}_p}$ of an irreducible component of $Y_0(n)_{\mathbb{F}_p}$. We let $T(n)$ be the reduced closed subscheme of $\mathbb{A}^2_{\mathbb{Z}}$ whose geometric points correspond to pairs (E_1, E_2) of elliptic curves that admit a morphism $\varphi\colon E_1 \to E_2$ of degree n. Let J be the set of $j \in I$ such that Z_j is an irreducible component of $T(n)_{\mathbb{F}_p}$, let $Z(n)$ be their union, and and let Z' be the union of the Z_i with $i \notin J$.

We claim that any s in Σ that meets $T(n)_{\mathbb{F}_p}$ is contained in $T(n)_{\mathbb{F}_p}$. So let $(j(E_1), j(E_2))$ be in $s(\overline{\mathbb{F}}_p)$, and $\varphi\colon E_1 \to E_2$ of degree n. Let $\overline{\mathbb{Z}} \to \overline{\mathbb{F}}_p$ be a morphism of rings, and $\tilde{\varphi}\colon \tilde{E}_1 \to \tilde{E}_2$ be the canonical lift over $\overline{\mathbb{Z}}$. Then $\tilde{\varphi}$ is of degree n, and so are all its conjugates by $G_{\mathbb{Q}}$, and so $s(\overline{\mathbb{F}}_p)$, consisting of all reductions of these conjugates, lies in $T(n)(\overline{\mathbb{F}}_p)$.

As $T(n)_{\mathbb{F}_p} \cap Z'$ is finite, the set Σ' of s in Σ that do not meet $T(n)_{\mathbb{F}_p}$ is dense in Z' and our proof is finished by induction on the number of irreducible components of Z.

6.1 Remark We think that Theorem 1.2 remains true if $E \subset \overline{\mathbb{Q}}$ is a finite extension of $\mathbb{Q}$ and we work with $\mathbb{A}^2_{\overline{\mathbb{F}}_p}$ and consider reductions of G_E-orbits of CM-points in $\mathbb{A}^2(\overline{\mathbb{Z}})$. However, the case $E = \mathbb{Q}$ has a special feature: up to fibres of the projections, the Z are invariant under switching the coordinates. This comes from the dihedral nature of the Galois action. As soon as E contains an imaginary quadratic field, there are Σ such that Z consists of one irreducible component of $Y_0(p)_{\overline{\mathbb{F}}_p}$.

References

[1] Y. André. *Finitude des couples d'invariants modulaires singuliers sur une courbe algébrique plane non modulaire.* J. Reine Angew. Math. 505 (1998), 203–208.

[2] A.C. Cojocaru, C. Hall. *Uniform results for Serre's theorem for elliptic curves.* Int. Math. Res. Not. 2005, no. 50, 3065–3080. `https://doi.org/10.1155/IMRN.2005.3065`

[3] M. Deschamps. *Réduction semi-stable.* Séminaire sur les pinceaux de courbes de genre au moins deux. Astérisque No. 86 (1981), 1–34.

[4] B. Edixhoven. *Special points on the product of two modular curves.* Compositio Math. 114 (1998), no. 3, 315–328. `https://doi.org/10.1023/A:1000539721162`

[5] B. Edixhoven. *Special points on products of modular curves.* Duke Math. J. 126 (2005), no. 2, 325–348. `https://doi.org/10.1215/S0012-7094-04-12624-7`

[6] J-I. Igusa. *Fibre systems of Jacobian varieties. III. Fibre systems of elliptic curves.* Amer. J. Math. 81 1959 453–476. `https://doi.org/10.2307/2372751`

[7] N. M. Katz, B. Mazur, *Arithmetic moduli of elliptic curves.* Annals of Mathematics Studies, 108. Princeton University Press, Princeton, NJ, 1985.

[8] B. Klingler, E. Ullmo and A. Yafaev, *Bi-algebraic geometry and the André–Oort conjecture*, Algebraic geometry: Salt Lake City 2015, 319–359, Proc. Sympos. Pure Math., 97.2, Amer. Math. Soc., Providence, RI, 2018.
`https://webusers.imj-prg.fr/~bruno.klingler/papiers/Survey2.pdf`

[9] J. C. Lagarias, and A. M. Odlyzko, *Effective versions of the Chebotarev density theorem.* Algebraic number fields: L-functions and Galois properties (Proc. Sympos., Univ. Durham, Durham, 1975), 409–464.

[10] S. Lang, *Algebraic number theory.* Second edition. Graduate Texts in Mathematics, 110. Springer-Verlag, New York, 1994. `https://link.springer.com/book/10.1007%2F978-1-4612-0853-2`

[11] O. Schreier and B. L. van der Waerden, *Die Automorphismen der projektiven Gruppen.* Abh. Math. Sem. Univ. Hamburg 6 (1928), 303–322. `https://doi.org/10.1007/BF02940620`

[12] J-P. Serre, *Quelques applications du théorème de densité de Chebotarev.* Inst. Hautes Études Sci. Publ. Math. No. 54 (1981), 323–401. http://archive.numdam.org/article/PMIHES_1981__54__123_0.pdf

[13] SGA 7.1. *Groupes de monodromie en géométrie algébrique. I. Séminaire de Géométrie Algébrique du Bois–Marie 1967–1969 (SGA 7 I).* Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D.S. Rim. Lecture Notes in Mathematics, Vol. 288. Springer–Verlag, Berlin–New York, 1972.

[14] C. L. Siegel, *Über die Classenzahl quadratischer Zahlkörper.* Acta Arithmetica 1.1 (1935): 83–86. http://eudml.org/doc/205054.

[15] J. Tsimerman, *The André–Oort conjecture for* $\mathcal{A}_g$. Ann. of Math. (2) 187 (2018), no. 2, 379–390.

[16] R. A. Wilson, *The finite simple groups.* Graduate Texts in Mathematics, 251. Springer-Verlag London, Ltd., London, 2009. https://doi.org/10.1007/978-1-84800-988-2

[17] Y. Zarhin, *Abelian varieties over fields of finite characteristic.* Cent. Eur. J. Math. 12 (2014), no. 5, 659–674. https://doi.org/10.2478/s11533-013-0370-1