Pink's conjecture on unlikely intersections and families of semi-abelian varieties *

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Abstract

The Poincaré torsor of a Shimura family of abelian varieties can be viewed both as a family of semi-abelian varieties and as a mixed Shimura variety. We show that the special subvarieties of the latter cannot all be described in terms of the group subschemes of the former. This provides a counter-example to the relative Manin-Mumford conjecture, but also a confirmation of Pink's conjecture on unlikely intersections in mixed Shimura varieties. The main part of the article concerns mixed Hodge structures and the uniformization of the Poincaré torsor, but other, more geometric, approaches are also discussed.

Contents

1 Introduction

In the unpublished preprint [26] Pink formulated a very influential conjecture (the equivalent Conjectures 1.1–1.3) on so-called "unlikely intersections" in mixed Shimura varieties. Here we merely recall the statement of his Conjecture 1.3:

> if Y is a Hodge generic irreducible closed subvariety of a mixed Shimura variety S, then the union of the intersections of Y with the special subvarieties of S of codimension at least $\dim(Y)+1$ is not Zariski dense in Y.

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We refer to [31] for more details on such intersections, and for their relations to the conjectures by Manin–Mumford, Mordell–Lang (which are now theorems), and André–Oort. See also [26], [25], and [19]. The André–Oort conjecture was recently proved for all $\mathcal{A}_g$ in [30].

In the last section of [26], Pink states a relative version of the Manin-Mumford conjecture for families of semi-abelian varieties, Conjecture 6.1:

> if $B \to X$ is a family of semi-abelian varieties over $\mathbb{C}$ and Y is an irreducible closed subvariety in B that is not contained in any proper closed subgroup scheme of $B \to X$, then the union of the intersections of Y with algebraic subgroups of codimension at least $\dim(Y) + 1$ of the fibres of $B \to X$ is not Zariski dense in Y.

Furthermore, Thm 6.3 of [26] claims that Conjecture 1.3 implies Conjecture 6.1. However, a counter-example to Conjecture 6.1 was given in the unpublished preprint [2], based on a relative version of a construction of Ribet ([17], [27]), leading to the notion of Ribet sections on certain semi-abelian schemes. But it was also shown in [2] that this counter-example was not in contradiction with Conjecture 1.3, and so, the error was in the proof of Theorem 6.3 (see Remark 5.4.4). The conclusion is that the context of mixed Hodge structures *is* the right one for a relative Manin-Mumford conjecture for families of *semi*-abelian varieties: indeed, the image of a Ribet section is a special subvariety that can in general *not* be interpreted as a subgroup scheme (see Remark 5.4.2). However, for families of *abelian* varieties (that is, mixed Shimura varieties of Kuga type), Theorem 6.3 is correct, see [25], Proposition 4.6, [14], Proposition 3.4, and again Remark 5.4.4 below.

The aim of this article is to provide not only a published account of this story, sharpening the results of [2], but also a self-contained description of the involved mixed Hodge structures and the corresponding mixed Shimura varieties, made as accessible as possible.

The article is structured as follows. In Section 2 we present the (counter)example, in the case of complex elliptic curves with complex multiplications, and in Section 3 (which introduces a different viewpoint) for abelian schemes. In Sections 4 and 5 we give the description of the example in the context of mixed Shimura varieties whose pure part parametrises principally polarised abelian varieties. We show that it gives evidence for Pink's Conjecture 1.3. Finally, in Section 6 we give a description of the example, in the case of elliptic curves, in terms of generalised jacobians.

1.1 Remark In each section, we construct Ribet sections under various denominations, namely t_φ in (2.1.1), r_f in Proposition 3.1, r_f^{Sh} in Thm 5.2, and t_φ^J in (6.0.2). At each step, we prove their compatibility, as well as some of their properties. The main property, leading to the searched-for counterexample, is stated in Theorem 2.4 and asserts that the Ribet section t_φ maps torsion points of the base to torsion points of their fibres. The proof (with sharper additional properties) is given in terms of r_f in Proposition 3.3, of r_f^{Sh} in Proposition 5.3 and of t_φ^J in Theorem 6.1. So, these proofs have logically unnecessary overlaps, but their settings are sufficiently distinct to justify this presentation. We should mention that yet another construction of the Ribet sections was proposed in [2], based as in [17] on the theory of 1-motives. But as shown in [8], the latter is equivalent to the construction of t_φ in Section 2.

1.2 Remark We will sometimes abbreviate "the image of a given section" by "the section". On the other hand, the image of a Ribet section will be called a Ribet variety.

2 The example with elliptic curves

The key player in the example in [2] is the Poincaré torsor $\mathcal{P}$ on a product $E \times E^\vee$, where E is a complex elliptic curve and where $E^\vee$ is its dual.

To make $\mathcal{P}$ and $E^\vee$ more explicit, we choose the isomorphism $\lambda\colon E \to E^\vee$ that sends a point P to the class of the invertible $\mathcal{O}$-module $\mathcal{O}((-P)-0) \cong \mathcal{O}(0-P)$ (this is the unique principal polarisation of E). In the notation of [23, Section 6], $\lambda = \varphi_{\mathcal{M}}$, where $\mathcal{M}$ is the invertible $\mathcal{O}$-module $\mathcal{O}(0)$ on E, and where $\varphi_{\mathcal{M}}$ sends P to the class of $(\mathrm{tr}_P^*\mathcal{M}) \otimes_{\mathcal{O}} \mathcal{M}^{-1}$, with tr_P the translation by P map on E.

The *Poincaré bundle* $\mathcal{L}$ on $E \times E$ is then

$$\mathcal{L} = \mathrm{add}^*\mathcal{M} \otimes_{\mathcal{O}} \mathrm{pr}_1^*\mathcal{M}^{-1} \otimes_{\mathcal{O}} \mathrm{pr}_2^*\mathcal{M}^{-1} \otimes_{\mathcal{O}} 0^*\mathcal{M}, \tag{2.0.1}$$

where add, pr_1, pr_2, and 0 are the addition map, the projections, and the constant map 0 from $E \times E$ to E. It is isomorphic (with the isomorphism given by the choice of a non-zero element of the fibre $\mathcal{M}(0)$ of $\mathcal{M}$ at 0, i.e., of a non-zero tangent vector of E at 0) to $\mathcal{O}(D)$, with

$$D = \mathrm{add}^{-1}0 - \mathrm{pr}_1^{-1}0 - \mathrm{pr}_2^{-1}0 \tag{2.0.2}$$

The fibre $\mathcal{L}(x,y)$ at a point (x,y) is given by:

$$\mathcal{L}(x,y) = \mathcal{M}(x+y) \otimes \mathcal{M}(x)^{-1} \otimes \mathcal{M}(y)^{-1} \otimes \mathcal{M}(0). \tag{2.0.3}$$

In particular: $\mathcal{L}(x,0) = \mathcal{M}(x) \otimes \mathcal{M}(x)^{-1} \otimes \mathcal{M}(0)^{-1} \otimes \mathcal{M}(0) = \mathbb{C}$, and similarly for $\mathcal{L}(0,y)$. Hence $\mathcal{L}$ is canonically trivial on the union of $E \times \{0\}$ and $\{0\} \times E$. But let us remark that the pullback of $\mathcal{L}$ via $\mathrm{diag}\colon E \to E \times E$ has fibre at x equal to $\mathcal{M}(2x) \otimes \mathcal{M}(x)^{-2} \otimes \mathcal{M}(0)$, hence is given by the divisor $\sum_{P \in E[2]} P - 2{\cdot}0$ which is of degree 2 and linearly equivalent to $2{\cdot}0$.

The *Poincaré torsor* $\mathcal{P}$ is then the $\mathbb{G}_{\mathrm{m}}$-torsor on $E \times E$ (trivial locally for the Zariski topology) of isomorphisms from $\mathcal{O}$ to $\mathcal{L}$:

$$\mathcal{P} = \mathrm{Isom}(\mathcal{O}, \mathcal{L}). \tag{2.0.4}$$

It is represented by a complex algebraic variety over $E \times E$, also denoted $\mathcal{P}$. Its fibre $\mathcal{P}(x,y)$ over (x,y) is the $\mathbb{C}^\times$-torsor $\mathrm{Isom}(\mathbb{C}, \mathcal{L}(x,y))$.

The theorem of the cube ([23, Section 6]) says that any invertible $\mathcal{O}$-module $\mathcal{N}$ on E^n with $n \geq 3$, whose restrictions to $\ker(\mathrm{pr}_i)$ are trivial for all i in $\{1,\ldots,n\}$, is trivial. If this is so, then, for any non-zero element s_0 of $\mathcal{N}(0,\ldots,0)$ there is a unique s in $\mathcal{N}(E^n)$ such that $s(0) = s_0$ (the reason is that $\mathcal{O}(E^n) = \mathbb{C}$).

For example, the invertible $\mathcal{O}$-module

$$\bigotimes_{I \subset \{1,2,3\}} \mathrm{add}_I^*\mathcal{M}^{(-1)^{\#I}} \quad \text{on } E \times E \times E,$$

where $\mathrm{add}_I\colon E^3 \to E$, $(x_1,x_2,x_3) \mapsto \sum_{i \in I} x_i$, is canonically trivial (canonically because its fibre at $(0,0,0)$ is $\mathcal{M}(0)^{\otimes 4} \otimes \mathcal{M}(0)^{\otimes -4} = \mathbb{C}$). Explicitly: for all points (x,y,z) of E^3 we have

$$\mathcal{M}(x+y+z) \otimes \mathcal{M}(x+y)^{-1} \otimes \mathcal{M}(x+z)^{-1} \otimes \mathcal{M}(y+z)^{-1} \otimes \mathcal{M}(x) \otimes \mathcal{M}(y) \otimes \mathcal{M}(z) \otimes \mathcal{M}(0)^{-1} = \mathbb{C}.$$

Similarly, the invertible $\mathcal{O}$-modules on E^3 with fibres

$$\mathcal{L}(x, y+z) \otimes \mathcal{L}(x,y)^{-1} \otimes \mathcal{L}(x,z)^{-1} \quad \text{and} \quad \mathcal{L}(x+y, z) \otimes \mathcal{L}(x,z)^{-1} \otimes \mathcal{L}(y,z)^{-1}$$

are canonically trivial. Therefore, for all points x, y and z of E we have:

$$\mathcal{L}(x, y+z) = \mathcal{L}(x,y) \otimes \mathcal{L}(x,z), \quad \mathcal{L}(x+y, z) = \mathcal{L}(x,z) \otimes \mathcal{L}(y,z). \tag{2.0.5}$$

This gives two composition laws on $\mathcal{P}$: for $\alpha\colon \mathbb{C} \to \mathcal{L}(x,y)$ in $\mathcal{P}(x,y)$ and $\beta\colon \mathbb{C} \to \mathcal{L}(x,z)$ in $\mathcal{P}(x,z)$ we get $\alpha \otimes \beta\colon \mathbb{C} \to \mathcal{L}(x, y+z)$ in $\mathcal{P}(x, y+z)$, and similarly with the 2nd variable fixed.

With the first variable fixed, $\mathcal{P}$ is a commutative group-variety over E, via pr_1, whose fibres are extensions of E by $\mathbb{G}_\mathrm{m}$, and similarly for pr_2; for details, see Chapter I, Section 2.5 of [22] and the Proposition of Section 2.6 there. In particular, $\mathcal{P}$ is a *bi-extension* of E and E by $\mathbb{G}_\mathrm{m}$: the two partial group laws commute with each other in the following sense. For x_1, x_2, y_1 and y_2 in E, and $p_{i,j}$ in $\mathcal{P}(x_i, y_j)$, the various ways of summing the $p_{i,j}$ leads to the same result in $\mathcal{P}(x_1 + x_2, y_1 + y_2)$. This is proved by considering the universal case $T := E^4$, $x_1 = \mathrm{pr}_1$, $x_2 = \mathrm{pr}_2$, $y_1 = \mathrm{pr}_3$ ad $y_2 = \mathrm{pr}_4$, and concluding that the trivialisations of

$$\mathcal{L}(x_1 + x_2, y_1 + y_2) \otimes \mathcal{L}(x_1, y_1)^{-1} \otimes \mathcal{L}(x_1, y_2)^{-1} \otimes \mathcal{L}(x_2, y_1)^{-1} \otimes \mathcal{L}(x_2, y_2)^{-1}$$

corresponding to the various ways of summing are equal because they are so at $(0, 0, 0, 0)$: writing it out in terms of $\mathcal{M}$ leads to the tensor product of as many $\mathcal{M}(0)$'s as $\mathcal{M}(0)^{-1}$'s.

With these preliminaries behind us, we can finally proceed to the construction of Ribet sections. Let φ be an endomorphism of E and let $\overline{\varphi} := \lambda^{-1} \circ \varphi^\vee \circ \lambda$ be the conjugate of φ. Let

$$\gamma = (\mathrm{id}, \varphi - \overline{\varphi})\colon E \to E \times E, \quad P \mapsto (P, (\varphi - \overline{\varphi})(P))$$

be the graph map attached to $\varphi - \overline{\varphi}$. The following fact was observed in [7]; see also [17] for a description in terms of 1-motives.

2.1 Proposition *The invertible $\mathcal{O}$-module $\gamma^*\mathcal{L}$ on E is canonically trivial.*

Proof As this is the crucial ingredient of the example that we present in this article, we give two proofs: one for readers who prefer a computation using divisors, and one for those who prefer universal properties. But first we note that if $\varphi = \overline{\varphi}$, then $\gamma = (\mathrm{id}, 0)$ and $\gamma^*\mathcal{L}$ is canonically trivial because, as mentioned above, $\mathcal{L}$ is canonically trivial on $E \times \{0\}$. So in the first proof below we may and do assume that $\varphi \neq \overline{\varphi}$.

A proof by divisors.

As the fibre of $\gamma^*\mathcal{L}$ at 0 is $\mathcal{L}(0, 0) = \mathbb{C}$, and $\mathcal{L}$ is isomorphic to $\mathcal{O}(D)$ with

$$D = \mathrm{add}^{-1}0 - \mathrm{pr}_1^{-1}0 - \mathrm{pr}_2^{-1}0$$

as in (2.0.2) it suffices to show that γ^*D is linearly equivalent to 0 on E. Let $\alpha := \varphi - \overline{\varphi}$. We note that

$$\mathrm{add} \circ \gamma = \mathrm{add} \circ (\mathrm{id}, \alpha) = \mathrm{id} + \alpha, \quad \mathrm{pr}_1 \circ \gamma = \mathrm{id}, \quad \text{and} \quad \mathrm{pr}_2 \circ \gamma = \alpha.$$

Hence we have the following equalities of divisors on E:

$$(\mathrm{id}, \alpha)^*D = (\mathrm{id} + \alpha)^*0 - \mathrm{id}^*0 - \alpha^*0 = \sum_{P \in \ker(\mathrm{id}+\alpha)} P - 0 - \sum_{Q \in \ker(\alpha)} Q.$$

This divisor has degree 0 because, in $\mathrm{End}(E)$, α is imaginary, so we have

$$\deg(\mathrm{id} + \alpha) = (\mathrm{id} + \alpha)(\mathrm{id} + \overline{\alpha}) = \mathrm{id} + \alpha\overline{\alpha} = 1 + \deg(\alpha).$$

Any degree zero divisor on E is linearly equivalent to $R - 0$, with R the image of the divisor under the group morphism $\mathrm{Div}^0(E) \to E$ that sends each point to itself. So in our case R is the sum of the points in $\ker(\mathrm{id} + \alpha)$, minus the sum of the points in $\ker(\alpha)$. These two kernels are finite commutative groups. For such a group, the sum of the elements is 0, except when its 2-primary part is cyclic and non-trivial, in which case it is the element of order 2. Let $a := \varphi + \overline{\varphi}$ be the trace of φ; it is in the subring $\mathbb{Z}$ of $\mathrm{End}(E)$. Then $\alpha = -a + 2\varphi$, and $\mathrm{id} + \alpha = (1 - a) + 2\varphi$. So one of these has odd degree, and the other is divisible by 2 in $\mathrm{End}(E)$, and so for none of them the 2-primary part of the kernel is cyclic and non-trivial.

A proof by universal properties.

We view $E\times E$ as an E-scheme via pr_2. Then $\mathcal{L}$ is the universal invertible $\mathcal{O}$-module of degree 0 on E with given trivialisation at 0: for every complex algebraic variety S and every invertible $\mathcal{O}$-module $\mathcal{N}$ on E_S, fibrewise of degree 0, and with a given trivialisation $\mathcal{O}_S \to 0^*\mathcal{N}$, there is a unique $f\colon S\to E$ such that the pullback of $\mathcal{L}$ via $\mathrm{id}\times f\colon E_S\to E_E$ is isomorphic to $\mathcal{N}$. Moreover, in this case there is a unique isomorphism $g\colon \mathcal{N}\to(\mathrm{id}\times f)^*\mathcal{L}$ that is compatible with the given trivialisations at 0. Of course, the analogous statements are true with pr_2 replaced by pr_1.

Let us turn to $\overline{\varphi}$. It is defined as $\lambda^{-1}\circ\varphi^\vee\circ\lambda$. Hence, for y in E, $\overline{\varphi}(y)$ is obtained as follows: $\lambda(y)$ is the isomorphism class of some invertible $\mathcal{O}$-module $\mathcal{N}$ of degree 0 on E, and then $\lambda(\overline{\varphi}(y))$ corresponds to $\varphi^*\mathcal{N}$. Now consider $(\varphi\times\mathrm{id})^*\mathcal{L}$ on $E\times E$; fibrewise it is of degree 0 and it has its canonical trivialisation at 0. The fact that we transported the universal invertible $\mathcal{O}$-module with trivialisation at 0 from $E\times E^\vee$ to $E\times E$ via $\mathrm{id}\times\lambda$ implies that $\overline{\varphi}$ is the unique morphism from E to E such that $(\mathrm{id}\times\overline{\varphi})^*\mathcal{L}$ is isomorphic to $(\varphi\times\mathrm{id})^*\mathcal{L}$. Hence we have a canonical isomorphism between $(\mathrm{id}\times\overline{\varphi})^*\mathcal{L}$ and $(\varphi\times\mathrm{id})^*\mathcal{L}$.

As $\mathcal{L}$ together with its trivialisations on $E\times\{0\}$ and $\{0\}\times E$ is symmetric (that is, invariant under the automorphism of $E\times E$ that sends (x,y) to (y,x)), we get a canonical isomorphism between $(\mathrm{id}\times\overline{\varphi})^*\mathcal{L}$ and $(\mathrm{id}\times\varphi)^*\mathcal{L}$.

From (2.0.5), applied with $x=\mathrm{id}_E$, $y=\varphi$ and $z=\overline{\varphi}$ we get a canonical isomorphism, on E, from $\gamma^*\mathcal{L}$ to $(\mathrm{id},\varphi)^*\mathcal{L}\otimes(\mathrm{id},-\overline{\varphi})^*\mathcal{L}$. Applying it again, but now with $x=\mathrm{id}_E$, $y=\overline{\varphi}$ and $z=-\overline{\varphi}$, we get a canonical isomorphism from $\mathcal{O}$ to $(\mathrm{id},-\overline{\varphi})^*\mathcal{L}\otimes(\mathrm{id},\overline{\varphi})^*\mathcal{L}$, giving us a canonical isomorphism from $(\mathrm{id},-\overline{\varphi})^*\mathcal{L}$ to $(\mathrm{id},\overline{\varphi})^*\mathcal{L}^{-1}$. Combining, we see that

$$\gamma^*\mathcal{L}=(\mathrm{id},\varphi)^*\mathcal{L}\otimes(\mathrm{id},-\overline{\varphi})^*\mathcal{L}=(\mathrm{id},\varphi)^*\mathcal{L}\otimes(\mathrm{id},\overline{\varphi})^*\mathcal{L}^{-1}=(\mathrm{id},\varphi)^*\mathcal{L}\otimes(\mathrm{id},\varphi)^*\mathcal{L}^{-1}=\mathcal{O}.$$

□

Now we view $\mathcal{P}$ as a group variety over E via $\mathrm{pr}_1\colon E\times E\to E$. The canonical trivialisation

(2.1.1) $$t_\varphi\colon \mathcal{O}\to\gamma^*\mathcal{L}=(\mathrm{id},\alpha)^*\mathcal{L}$$

on E gives, for every x in E, an element $t_\varphi(x)$ in $\mathrm{Isom}(\mathbb{C},\mathcal{L}(x,\alpha(x)))$, hence an element in $\mathcal{P}(x,\alpha(x))$. As such, t_φ is a section of the group variety $\mathcal{P}$ over E, which we call the *Ribet section attached to* φ.

Following [2], we will now show that if $\overline{\varphi}\neq\varphi$, then t_φ gives a counterexample to Conjecture 6.1 of [26].

2.2 Lemma *Let* $\mathbb{G}_\mathrm{m}\rightarrowtail G\twoheadrightarrow E$ *be an extension whose class in* $\mathrm{Ext}(E,\mathbb{G}_\mathrm{m})$ *is not torsion. Then the only connected algebraic subgroups of* G *are* $\{0\}$, $\mathbb{G}_\mathrm{m}$ *and* G.

Proof Let H be a connected algebraic subgroup of G. Then $\dim(H)$ is 0, 1 or 2. If it is 0 then $H=\{0\}$, and if it is 2 then $H=G$, so we assume it is 1, and that H is not equal to $\mathbb{G}_\mathrm{m}$. Then H maps surjectively to E. Then H is a finite cover of E, hence is itself an elliptic curve, and there is an $n\in\mathbb{Z}_{>0}$ and a factorisation $n\cdot\colon E\to H\to E$. This means that the extension $\mathbb{G}_\mathrm{m}\rightarrowtail G\twoheadrightarrow E$ is split after pullback via $n\cdot\colon E\to E$, hence its class is torsion. □

2.3 Lemma *If* $\varphi\neq\overline{\varphi}$, *then the union over all* $n\in\mathbb{Z}$ *of the images* $(n{\cdot}t_\varphi)(E)$ *of the sections* $n{\cdot}t_\varphi$ *is Zariski dense in* $\mathcal{P}$.

Proof Let Z be the Zariski closure of the union of the $(n{\cdot}t_\varphi)(E)$. Let x in E be of infinite order. Then $y:=\alpha(x)$ is of infinite order as well. The point $t_\varphi(x)$ of the extension $\mathcal{P}_x$ of E by $\mathbb{G}_\mathrm{m}$ has image y in E. The Zariski closure in $\mathcal{P}_x$ of $\{n{\cdot}t_\varphi(x) : n\in\mathbb{Z}\}$ is a closed subgroup

H of $\mathcal{P}_x$. The image of H in E is closed ($H \to E$ is a morphism of algebraic groups), and contains y, hence is equal to E. Hence $\dim(H)$ is 1 or 2. Assume that $\dim(H) = 1$. By Lemma 2.2 the extension class of $\mathcal{P}_x$ is torsion, but as this class is x, it is not torsion. We conclude that $\dim(H) = 2$, and $H = \mathcal{P}_x$. Hence Z contains all $\mathcal{P}_x$ with x not torsion. Then $Z = \mathcal{P}$. □

2.4 Theorem *For every torsion point x in E, $t_\varphi(x)$ is torsion in $\mathcal{P}_x$.*

Proof We will give three proofs: one in the context of abelian schemes and biextensions (Proposition 3.3), one, more elementary, using generalised jacobians of elliptic curves with a double point in Section 6, and a third proof, using the description of $t_\varphi(E)$ as a special subvariety of a mixed Shimura variety (Proposition 5.3). We refer to [2, Section 1], for the initial proof of Theorem 2.4, based on the theory of 1-motives. □

We now explain why the closed subvariety $Y := t_\varphi(E)$ in the family of semi-abelian varieties $B := \mathcal{P}$ over $X := E$ is a counter-example to [26, Conjecture 6.1] when $\varphi - \overline{\varphi} \neq 0$. First of all, Y is not contained in a proper subvariety of B that is a subgroup scheme of B over X because of Lemma 2.3.

Secondly, $d := \dim(Y) = 1$, hence according to the conjecture, the intersection of Y with the set $B^{[>1]}$ that is the union, over all x in X, of all subgroups of B_x of codimension > 1, should not be Zariski dense in Y. However, $B^{[>1]}$ is the set of points that are torsion in their fibre, and Theorem 2.4 says that the intersection is infinite.

3 The example with abelian schemes

In this section we consider abelian schemes, but even in the case of elliptic curves, this section provides a new point of view on Ribet sections and their properties. We recommend Chapter I of [22] and references therein for further details about biextensions, duality and pairings.

Let S be a scheme, A an abelian scheme over S, and $A^\vee$ its dual (Section I.1 in [12]). Let $\mathcal{L}$ be the universal line bundle on $A \times_S A^\vee$, rigidified, compatibly, at $\{0\} \times A^\vee$ and $A \times \{0\}$; it identifies A with the dual of $A^\vee$. Then $\mathcal{P} = \mathrm{Isom}_{A\times_S A^\vee}(\mathcal{O}, \mathcal{L})$ is the Poincaré $\mathbb{G}_{\mathrm{m}}$-torsor on $A \times_S A^\vee$, and as described in the previous section in the case of elliptic curves, it is a biextension of A and $A^\vee$ by $\mathbb{G}_{\mathrm{m}}$. In particular, over $A^\vee$, $\mathcal{P}$ is the universal extension of A by $\mathbb{G}_{\mathrm{m}}$, and over A, $\mathcal{P}$ is the universal extension of $A^\vee$ by $\mathbb{G}_{\mathrm{m}}$. Proposition 2.1 extends to the present situation as follows (see [7], [8], [20, Section 8.3]).

3.1 Proposition *Let S be a scheme, A an abelian scheme over S, $\mathcal{P}$ the Poincaré torsor on $A \times_S A^\vee$, $f\colon A^\vee \to A$ a morphism of group schemes, $f^\vee\colon A^\vee \to (A^\vee)^\vee = A$ its dual, and*

$$\alpha := f - f^\vee\colon A^\vee \to A\,.$$

The restriction of $\mathcal{P}$ to the graph of α has a unique section r_f

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}A^\vee} \arrow[r, hook] \arrow[dr] & \mathcal{P} \arrow[r, two heads] \arrow[d, shift left] & A_{A^\vee} = A \times_S A^\vee \arrow[dl] \\
& A^\vee \arrow[u, bend left, "r_f"] \arrow[ur, bend right, "(\alpha,\mathrm{id})"'] &
\end{tikzcd}$$

with value 1 at the origin.

Proof We start in a more general situation: let A_1 and A_2 be abelian schemes over S, $\mathcal{P}_1$ and $\mathcal{P}_2$ their Poincaré torsors, and $f\colon A_1 \to A_2$. Then the dual $f^\vee\colon A_2^\vee \to A_1^\vee$ is defined by the condition that the pullback of the universal extension

$$\mathbb{G}_{\mathrm{m}A_2^\vee} \hookrightarrow \mathcal{P}_2 \twoheadrightarrow (A_2)_{A_2^\vee} = A_2 \times_S A_2^\vee$$

by $f \times \mathrm{id}\colon A_1 \times_S A_2^\vee \to A_2 \times_S A_2^\vee$ is isomorphic to the pullback of the universal extension

$$\mathbb{G}_{\mathrm{m}A_1^\vee} \hookrightarrow \mathcal{P}_1 \twoheadrightarrow (A_1)_{A_1^\vee} = A_1 \times_S A_1^\vee$$

by $\mathrm{id} \times f^\vee\colon A_1 \times A_2^\vee \to A_1 \times A_1^\vee$. Such an isomorphism is unique, hence

$$\text{for all } T \to S,\ x \in A_1(T),\ y \in A_2^\vee(T)\colon\quad \mathcal{P}_1(x, f^\vee y) = \mathcal{P}_2(fx, y)\,.$$

Now we specialise to the situation where $A_1 = A_2^\vee$. Then $A_1 \times_S A_1^\vee = A_2^\vee \times_S A_2$, with Poincaré torsors $\mathcal{P}_1$ and $\sigma^*\mathcal{P}_2$, where $\sigma\colon A_2^\vee \times_S A_2 \to A_2 \times_S A_2^\vee$ is the coordinate switch. Then we have, for $T \to S$, $x \in A_1(T) = A_2^\vee(T)$ and $y \in A_2^\vee(T)$:

$$\mathcal{P}_2(fx, y) = \mathcal{P}_1(x, f^\vee y) = \mathcal{P}_2(f^\vee y, x)\,. \tag{3.1.1}$$

Now we restrict to the situation where $y = x$, where we have $\mathcal{P}_2(fx, x) = \mathcal{P}_2(f^\vee x, x)$. Then additivity in the first factor gives that

$$\begin{aligned} \mathcal{P}_2(\alpha x, x) &= \mathcal{P}_2((f - f^\vee)x, x) = \mathcal{P}_2(fx - f^\vee x, x) \\ &= \mathcal{P}_2(fx, x) \otimes \mathcal{P}_2(f^\vee x, x)^{-1} = \mathrm{Hom}(\mathcal{P}_2(fx, x), \mathcal{P}_2(f^\vee x, x)) = \mathbb{G}_{\mathrm{m}T}\,. \end{aligned} \tag{3.1.2}$$

Now we take $A_2 = A$, and define $r_f\colon A^\vee \to \mathcal{P}$ by letting it send x to the T-point of $\mathcal{P}(\alpha x, x)$ corresponding to the unit section of $\mathbb{G}_{\mathrm{m}T}$ via the isomorphism in (3.1.2).

By construction, $r_f(0) = 1$. This condition makes it unique, as two such sections differ by a factor in $\mathcal{O}(A^\vee)^\times = \mathcal{O}(S)^\times$, with value 1 at $0 \in A^\vee(S)$. ☐

3.2 Remark When $A \to S$ is a complex elliptic curve E, and $\lambda\colon E \to E^\vee$ is as in Section 2, and φ is in $\mathrm{End}(E)$, and $f = \varphi \circ \lambda$, then t_φ as in (2.1.1) and r_f as in Proposition 3.1 are equal (well, up to switching the factors of $E \times E$), because they are sections of the same $\mathbb{G}_{\mathrm{m}}$-torsor over E, with the same value at 0. Therefore, Proposition 3.3 below proves Theorem 2.4.

The following Proposition gives the torsion property of r_f at the torsion points of $A^\vee$: it implies that for $T \to S$ and x in $A^\vee[n](T)$ we have $n^2 r_f(x) = 1$. (See Proposition 5.3 and Theorem 6.1 for other proofs of this equality.)

3.3 Proposition *Let S, A, $\mathcal{P}$, f, α and r_f be as in Proposition 3.1. Let $n \geq 1$, let T be an S-scheme, and $x \in A^\vee[n](T)$. Then*

$$n r_f(x) = e_n(fx, x) \quad \text{in} \quad \mathcal{P}(n\alpha x, x) = \mathcal{P}(0, x) = \mathbb{G}_{\mathrm{m}}(T),$$

with $e_n\colon A[n](T) \times A^\vee[n](T) \to \mu_n(T)$ the Weil pairing (whose definition is recalled below).

Proof The base change $T \to S$ reduces to the case where $T = S$. First we describe the Weil pairing in terms of $\mathcal{P}$. Let $z \in A[n](S)$ and $y \in A^\vee[n](S)$. We have the following canonical isomorphisms between $\mathbb{G}_{\mathrm{m}}$-torsors on S,

$$\begin{tikzcd} \mathbb{G}_{\mathrm{m}S} \arrow[r, equal] \arrow[d, "e_n(z,y)"] & \mathcal{P}(z,0) \arrow[r, equal] & \mathcal{P}(z,ny) \arrow[r, equal, "+_2"] & \mathcal{P}(z,y)^{\otimes n} \arrow[d, equal, "\mathrm{id}"] \\ \mathbb{G}_{\mathrm{m}S} \arrow[r, equal] & \mathcal{P}(0,y) \arrow[r, equal] & \mathcal{P}(nz,y) \arrow[r, equal, "+_1"] & \mathcal{P}(z,y)^{\otimes n} \end{tikzcd}$$

where the superscript $+_1$ means "induced by additivity in the first coordinate", etc., and where $\mathcal{P}(z,y)^{\otimes n}$ is the contracted product of n copies of $\mathcal{P}(z,y)$. As the diagram shows, we define $e_n(z,y)$ to be the image of the section 1 of the top $\mathbb{G}_{\mathrm{m}S}$ in the bottom $\mathbb{G}_{\mathrm{m}S}$. We claim that this is the usual Weil pairing: let $\mathcal{P}_y$ be the extension of A by $\mathbb{G}_{\mathrm{m}S}$ at y, then there is a unique $\tilde{n}\colon A \to \mathcal{P}_y$ that lifts $n\cdot\colon A \to A$, and the restriction $\tilde{n}\colon A[n] \to \mu_n$ sends z to $e_n(z,y)$.

The following commutative diagram relates $nr_f(x)$ to $e_n(fx,x)$ and $e_n(x,f^\vee x)$: going from bottom right to upper right and then upper left is multiplication by $e_n(x,f^\vee x)$, going from bottom right to middle right and then middle left and then upper left is $nr_f(x)$, and from bottom right to upper left via bottom left is $e_n(fx,x)$.

$$\begin{tikzcd}
\mathbb{G}_{\mathrm{m}S} \arrow[r, equal, "\mathrm{id}"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{b}"] & \mathbb{G}_{\mathrm{m}S} \arrow[r, equal, "\mathrm{id}"] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{f}"] & \mathbb{G}_{\mathrm{m}S} \arrow[d, equal] \\
\mathcal{P}(0,x) \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{c}"] & (\sigma^*\mathcal{P})(0,f^\vee x) \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{g}"] & \mathcal{P}(f^\vee x,0) \arrow[d, equal] \\
\mathcal{P}(fx,x)^{\otimes n} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{d}"] & (\sigma^*\mathcal{P})(x,f^\vee x)^{\otimes n} \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{h}"] & \mathcal{P}(f^\vee x,x)^{\otimes n} \arrow[d, equal] \arrow[ll, bend right=15, "nr_f(x)"'] \\
\mathcal{P}(fx,0) \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{e}"] & (\sigma^*\mathcal{P})(x,0) \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\mathrm{i}"] & \mathcal{P}(0,x) \arrow[d, equal] \\
\mathbb{G}_{\mathrm{m}S} \arrow[r, equal, "\mathrm{id}"'] \arrow[uuuu, bend left=60, "e_n(fx,x)"] \arrow[uuuu, phantom, bend left=30, "\mathrm{a}"] & \mathbb{G}_{\mathrm{m}S} \arrow[r, equal, "\mathrm{id}"'] & \mathbb{G}_{\mathrm{m}S} \arrow[uuuu, bend right=60, "e_n(x,f^\vee x)"'] \arrow[uuuu, phantom, bend right=30, "\mathrm{j}"]
\end{tikzcd} \tag{3.3.1}$$

Here are arguments for the commutativity of all faces (a–j) in the diagram.

a This is the definition of $e_n(fx,x)$.

b–e This is because the equality signs in (3.1.1) are isomorphisms of biextensions on $A_2^\vee \times_S A_2^\vee$.

f–i These follow directly from the definition of $\sigma^*\mathcal{P}$.

j This is the definition of $e_n(x,f^\vee x)$.

Let us remark that the commutativity of this diagram shows that $f^\vee$ and f are adjoints for the e_n-pairing, and that when $f^\vee = f$, $e_n(fx,x) = 1$ for all x in $A^\vee[n](S)$, in particular, that the pairings attached to a polarisation are alternating. □

4 The Poincaré torsor as mixed Shimura variety

In this section we describe the Poincaré torsor of the universal family of principally polarised complex abelian varieties of dimension d as a mixed Shimura variety, that is, as a moduli space for mixed Hodge structures. We recommend [25, Section 2] (and also [18] and [10]) as an introduction to mixed Hodge structures and (connected) mixed Shimura varieties, but we do not assume the reader to be familiar with these notions. In fact, we hope that the example treated here also provides a good introduction, and perhaps a motivation to read more. We find that the point of view of mixed Shimura varieties gives a simple and beautiful perspective on the uniformisation of the universal Poincaré torsor. The notion of 1-motives from [11] provides an algebraic description of the mixed Hodge structures that we encounter, but we will not use this.

4.1 Pure Hodge structures

For n in $\mathbb{Z}$, a *$\mathbb{Z}$-Hodge structure of weight n* is a finitely generated $\mathbb{Z}$-module M together with a decomposition (called Hodge decomposition) of the complex vector space $M_{\mathbb{C}} := \mathbb{C} \otimes M$:

$$M_{\mathbb{C}} = \bigoplus_{\substack{p,q\in\mathbb{Z}\\ p+q=n}} M^{p,q}, \quad \text{such that for all } p,q \text{ in } \mathbb{Z} \text{ with } p+q=n\text{:} \quad M^{q,p} = \overline{M^{p,q}},$$

where $\overline{M^{p,q}}$ is the image of $M^{p,q}$ under the map $M_{\mathbb{C}} \to M_{\mathbb{C}}$ that sends $z\otimes m$ to $\overline{z}\otimes m$. A *pure $\mathbb{Z}$-Hodge structure* is a finitely generated $\mathbb{Z}$-module M, together with a direct sum decomposition

$$M/M_{\mathrm{tors}} = \bigoplus_{n\in\mathbb{Z}} M_n, \quad \text{and for each } n \text{ a Hodge structure of weight } n, \quad M_{n,\mathbb{C}} = \bigoplus_{p+q=n} M^{p,q}.$$

For $T \subset \mathbb{Z}^2$, M is said to be of type T, if, for all (p,q) not in T, $M^{p,q}$ is zero.

A morphism of pure $\mathbb{Z}$-Hodge structures from $(M,(M^{p,q})_{p,q})$ to $(N,(N^{p,q})_{p,q})$ is a morphism $f\colon M \to N$ of $\mathbb{Z}$-modules such that for all (p,q) one has $f_{\mathbb{C}}(M^{p,q}) \subset N^{p,q}$.

For M and N pure $\mathbb{Z}$-Hodge structures, $M^\vee$, $M\otimes N$ are given pure $\mathbb{Z}$-Hodge structures as follows:

$$(M^\vee)^{p,q} = (M^{-p,-q})^\vee\,, \quad (M\otimes N)^{p,q} = \bigoplus_{\substack{a+c=p\\ b+d=q}} (M^{a,b}\otimes N^{c,d})\,,$$

and this dictates the rule for $\mathrm{Hom}(M,N)$:

$$\mathrm{Hom}(M,N)^{p,q} = (M^\vee\otimes N)^{p,q} = \bigoplus_{\substack{-a+c=p\\ -b+d=q}} \mathrm{Hom}(M^{a,b},N^{c,d})\,.$$

It is convenient to define, for m in $\mathbb{Z}$, the $\mathbb{Z}$-Hodge structure $\mathbb{Z}(m)$ of weight $-2m$ as the sub-$\mathbb{Z}$-module $(2\pi i)^m\mathbb{Z}$ of $\mathbb{C}$, with $\mathbb{Z}(m)_{\mathbb{C}} = \mathbb{Z}(m)^{-m,-m}$. For M a pure $\mathbb{Z}$-Hodge structure, and m in $\mathbb{Z}$, $M(m)$ denotes $M\otimes\mathbb{Z}(m)$. The embedding $(2\pi i)^m\mathbb{Z}\subset\mathbb{C}$ gives the isomorphisms $\mathbb{Z}(m)_{\mathbb{C}} = \mathbb{C}$ and $M(m)_{\mathbb{C}} = M_{\mathbb{C}}$.

A *polarisation* on a pure $\mathbb{Z}$-Hodge structure M of weight n is a morphism of pure $\mathbb{Z}$-Hodge structures $\Psi\colon M\otimes M\to\mathbb{Z}(-n)$ such that for every (p,q) with $p+q=n$ the map

$$M^{p,q}\times M^{p,q}\to\mathbb{C}, \quad (v,w)\mapsto(-1)^p\Psi(v,\overline{w})$$

is a complex inner product (that is, for all (v,w), $\Psi(w,\overline{v}) = \overline{\Psi(v,\overline{w})}$, and, for all $v\neq 0$, $(-1)^p\Psi(v,\overline{v})>0$). The symmetry condition is equivalent to Ψ being symmetric if n is even and antisymmetric if n is odd. The symmetry and positivity conditions are equivalent to the restriction to $M_{\mathbb{R}}\times M_{\mathbb{R}}$ of the $\mathbb{C}$-bilinear map

$$M_{\mathbb{C}}\times M_{\mathbb{C}}\to\mathbb{C}\,, \quad (x,y)\mapsto(2\pi i)^n\Psi(x\otimes i{\cdot}y)$$

with i acting on $M^{p,q}$ as multiplication by $i^{-p}\overline{i}^{-q}$ being $\mathbb{R}$-valued, symmetric and positive definite.

4.2 Principally polarised abelian varieties

Let d be in $\mathbb{Z}_{\geq 1}$. Principally polarised complex abelian varieties of dimension d are conveniently described as follows. Their lattice is a free $\mathbb{Z}$-module M of rank $2d$ with a Hodge structure $M_{\mathbb{C}} = M^{-1,0}\oplus M^{0,-1}$, and the polarisation $\Psi\colon M\otimes M\to\mathbb{Z}(1) = 2\pi i\mathbb{Z}$ is antisymmetric and induces an isomorphism $M\to M^\vee(1)$. The abelian variety is then $M_{\mathbb{C}}/(M^{0,-1}+M)$. Then M

together with Ψ is isomorphic to $\mathbb{Z}^{2d}$ with $\Psi\colon \mathbb{Z}^{2d}\otimes\mathbb{Z}^{2d}\to\mathbb{Z}(1)$, $x\otimes y\mapsto 2\pi i x^t\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)y$, and such an isomorphism is unique up to composition with an element of $\mathrm{Sp}(\Psi)(\mathbb{Z})$ (the stabiliser of Ψ in $\mathrm{GL}_{2d}(\mathbb{Z})$). Let $(e_1,\ldots,e_{2d})$ be the standard basis of $\mathbb{Z}^{2d}$. The subspace $M^{0,-1}$ of $\mathbb{C}^{2d}$, on which $(v,w)\mapsto\Psi(v,\overline{w})$ is an inner product, has trivial intersection with the isotropic subspaces generated by $e_1,\ldots,e_d$ and $e_{d+1},\ldots,e_{2d}$, hence there is a unique τ in $\mathrm{GL}_d(\mathbb{C})$ such that $M^{0,-1}=\{\left(\begin{smallmatrix}\tau v\\ v\end{smallmatrix}\right): v\in\mathbb{C}^d\}$. As Ψ is a morphism of Hodge structures, $M^{0,-1}$ is isotropic for Ψ, giving $\tau^t=\tau$. The positivity of the complex inner product on $M^{0,-1}$ gives that $\mathrm{Im}(\tau)=(\tau-\overline{\tau})/2i$ is positive definite. Conversely, for every $\tau\in\mathrm{M}_d(\mathbb{C})$ with $\tau^t=\tau$ and $\mathrm{Im}(\tau)$ positive definite, τ is in $\mathrm{GL}_d(\mathbb{C})$ and $M^{0,-1}:=\{\left(\begin{smallmatrix}\tau v\\ v\end{smallmatrix}\right): v\in\mathbb{C}^d\}$ gives a Hodge structure on $\mathbb{Z}^{2d}$ such that Ψ is a principal polarisation.

We conclude: the set D_Ψ of Hodge structures of type $\{(-1,0),(0,-1)\}$ on $\mathbb{Z}^{2d}$ for which Ψ is a polarisation is in bijection with the Siegel half space $\mathbb{H}_d$ of symmetric $\tau\in\mathrm{M}_d(\mathbb{C})$ with $\mathrm{Im}(\tau)$ positive definite, via $\tau\mapsto M_\tau^{0,-1}:=\{\left(\begin{smallmatrix}\tau v\\ v\end{smallmatrix}\right): v\in\mathbb{C}^d\}$. Note that $\mathbb{H}_d$ is a convex open subset of the set of symmetric d by d complex matrices. The action of $\mathrm{Sp}(\Psi)(\mathbb{Z})$ describes the moduli of complex principally polarised abelian varieties of dimension d: the quotients by suitable congruence subgroups give fine moduli spaces, and the stacky quotient by $\mathrm{Sp}(\Psi)(\mathbb{Z})$ gives the stack of complex principally polarised abelian varieties of dimension d. Let us write more explicitly the abelian variety $A_\tau:=\mathbb{C}^{2d}/(M_\tau^{0,-1}+\mathbb{Z}^{2d})$ at τ in $\mathbb{H}_d$. The $\mathbb{C}$-linear map $\mathbb{C}^{2d}\to\mathbb{C}^d$, $\left(\begin{smallmatrix}w\\ v\end{smallmatrix}\right)\mapsto w-\tau v$ is surjective and has kernel $M^{0,-1}$. So A_τ is the cokernel of $(1_d\quad -\tau)\cdot\colon \mathbb{Z}^{2d}\to\mathbb{C}^d$, $\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right)\mapsto x-\tau y$, that is, A_τ is the quotient of $\mathbb{C}^d$ by the lattice generated by $\mathbb{Z}^d$ and the columns of τ.

For $M^{0,-1}$ in D_Ψ and g in $\mathrm{GL}_{2d}(\mathbb{R})$, $gM^{0,-1}$ is a Hodge structure of type $\{(-1,0),(0,-1)\}$ for which $g\Psi$ is a polarisation, where, for all x,y in $\mathbb{R}^{2d}$, $(g\Psi)(x\otimes y)=\Psi((g^{-1}x)\otimes(g^{-1}y))$. Hence $\mathrm{Sp}(\Psi)(\mathbb{R})$, the subgroup of $\mathrm{GL}_{2d}(\mathbb{R})$ that preserves Ψ, acts on D_Ψ.

The following argument shows that this action is transitive. Let $M^{0,-1}$ be in D_Ψ, and let $v_1,\ldots,v_d$ be an orthonormal basis for $M^{0,-1}$. Then $\mathrm{Re}(v_1),\ldots,\mathrm{Re}(v_d),\mathrm{Im}(v_1),\ldots,\mathrm{Im}(v_d)$ is an $\mathbb{R}$-basis of $\mathbb{R}^{2d}$ with respect to which $M^{0,-1}$ and Ψ do not depend on $M^{0,-1}$.

In fact a slightly bigger group acts on D_Ψ. We view Ψ as an element of the $\mathbb{R}$-vector space $(\mathbb{R}^{2d}\otimes_\mathbb{R}\mathbb{R}^{2d})^\vee\otimes_\mathbb{R}\mathbb{R}(1)$, on which the group $\mathrm{GL}_{2d}(\mathbb{R})\times\mathbb{R}^\times$ acts. An element (g,λ) acts as $(g^{-1}\otimes g^{-1})^\vee\otimes\lambda$. Then (g,λ) fixes Ψ if and only if for all $x,y\in\mathbb{R}^{2d}$, $\Psi(gx,gy)=\lambda\Psi(x,y)$. We let $\mathrm{GSp}_\Psi(\mathbb{R})$ be the group of such (g,λ), and $\mathrm{GSp}_\Psi(\mathbb{R})^+$ the subgroup of the (g,λ) with $\lambda>0$. Then $\mathrm{GSp}_\Psi(\mathbb{R})^+$ acts on D_Ψ via $M^{0,-1}\mapsto g\cdot M^{0,-1}$.

4.3 Mixed Hodge structures

A *mixed Hodge structure* on a finitely generated $\mathbb{Z}$-module M is the data of an increasing filtration $(W_nM)_{n\in\mathbb{Z}}$ (called the weight filtration) with $W_nM=M_{\mathrm{tors}}$ for n small enough and $W_nM=M$ for n large enough, with all M/W_nM torsion free, and a decreasing filtration $(F^pM_\mathbb{C})_{p\in\mathbb{Z}}$ of the $\mathbb{C}$-vector space $M_\mathbb{C}$, with $F^pM_\mathbb{C}=M_\mathbb{C}$ for small enough p and $F^pM_\mathbb{C}=0$ for large enough p, such that for each n in $\mathbb{Z}$ the filtration induced by F on $(\mathrm{Gr}_n^WM)_\mathbb{C}:=((W_nM)/(W_{n-1}M))_\mathbb{C}$ is a Hodge structure of weight n:

$$(\mathrm{Gr}_n^WM)_\mathbb{C}=\bigoplus_{p+q=n}(\mathrm{Gr}_n^WM)_\mathbb{C}^{p,q},\quad\text{with}\quad(\mathrm{Gr}_n^WM)_\mathbb{C}^{p,q}=F^p(\mathrm{Gr}_n^WM)_\mathbb{C}\cap\overline{F^q(\mathrm{Gr}_n^WM)_\mathbb{C}}.$$

As an example, let us determine all mixed Hodge structures on $M:=\mathbb{Z}\cdot e_1\oplus\mathbb{Z}\cdot e_2$, with $W_{-3}(M)=0$, $W_{-2}(M)=W_{-1}(M)=\mathbb{Z}\cdot e_1$ and $W_0(M)=M$, of type $\{(-1,-1),(0,0)\}$, that is, extensions of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$. Then $F^{-1}M_\mathbb{C}=M_\mathbb{C}$, $F^1M_\mathbb{C}=0$, and $F^0M_\mathbb{C}\cap\mathbb{C}\cdot e_1=0$ and under the quotient map $q\colon M_\mathbb{C}\to M_\mathbb{C}/W_{-1}M_\mathbb{C}=\mathbb{C}\cdot e_2$, $F^0M_\mathbb{C}$ is mapped surjectively. So $F^0M_\mathbb{C}$ is a line, of the form $L_a:=\mathbb{C}\cdot(e_2+ae_1)$ for a unique a in $\mathbb{C}$, giving a bijection from $\mathbb{C}$ to the set D_W of mixed Hodge structures of the type we consider.

Let $P_W(\mathbb{R})$ be the subgroup of $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}(\mathbb{R}(1)) \times \mathrm{GL}(\mathbb{R}(0))$ that fixes $\mathbb{R}(1) \to \mathbb{R}^2$, $2\pi i \mapsto e_1$, that fixes $\mathbb{R}^2 \to \mathbb{R}(0)$, $(x, y) \mapsto y$, and that fixes $\mathbb{R}(0) \otimes \mathbb{R}(0) \to \mathbb{R}(0)$, $x \otimes y \mapsto xy$. Then

$$P_W(\mathbb{R}) = \left\{ \left(\begin{pmatrix} \lambda & x \\ 0 & 1 \end{pmatrix}, \lambda, 1 \right) : \lambda \in \mathbb{R}^\times,\ x \in \mathbb{R} \right\}.$$

By definition $P_W(\mathbb{R})$ acts on D_W, and transported to $\mathbb{C}$ this action is given by $a \mapsto \lambda a + x$. This action has two orbits: $\mathbb{R}$ and $\mathbb{C} - \mathbb{R}$. We would like to have a transitive action (in order to get a "connected mixed Shimura datum" as in [25, Def. 2.1]). To get that, we allow x to be complex, that is, we let $U_W(\mathbb{C})$ be the subgroup of $\mathrm{GL}_2(\mathbb{C})$ of unipotent matrices $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$ with $x \in \mathbb{C}$, and let

$$P_W(\mathbb{R})U_W(\mathbb{C}) = \left\{ \left(\begin{pmatrix} \lambda & x \\ 0 & 1 \end{pmatrix}, \lambda, 1 \right) : \lambda \in \mathbb{R}^\times,\ x \in \mathbb{C} \right\}$$

act on D_W. The action of $P_W(\mathbb{Z})$ on $\mathbb{C}$ describes the moduli of mixed $\mathbb{Z}$-Hodge structures that are extension of $\mathbb{Z}(0)$ by $\mathbb{Z}(1)$. The coarse moduli space is the quotient $\mathbb{C} \to \mathbb{C}^\times \to \mathbb{C}$, $a \mapsto \exp(2\pi i a) \mapsto \exp(2\pi i a) + \exp(-2\pi i a)$.

4.4 The universal Poincaré torsor as moduli space of mixed Hodge structures

Let d be in $\mathbb{Z}_{\geq 1}$ and $M := \mathbb{Z}(1) \oplus \mathbb{Z}^{2d} \oplus \mathbb{Z}$, with standard basis $2\pi i e_0, e_1, \ldots, e_{2d+1}$, and with the following filtration:

$$W_{-3}M = \{0\}, \quad W_{-2}M = \mathbb{Z}{\cdot}2\pi i e_0, \quad W_{-1}M = \mathbb{Z}{\cdot}2\pi i e_0 \oplus \cdots \oplus \mathbb{Z}{\cdot}e_{2d}, \quad W_0 M = M.$$

Let D be the set of filtrations F on $M_{\mathbb{C}}$ such that (M, W, F) is a mixed $\mathbb{Z}$-Hodge structure of type $\{(-1,-1), (-1,0), (0,-1), (0,0)\}$, and such that $\Psi\colon (x, y) \mapsto 2\pi i x^t \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) y$ is, via the given bases, a polarisation on $\mathrm{Gr}^W_{-1}M$. For F in D we have $F^{-1}M_{\mathbb{C}} = M_{\mathbb{C}}$, and $F^1 M_{\mathbb{C}} = \{0\}$, so F is given by $F^0 M_{\mathbb{C}}$. We get a map from D to the set D_Ψ (see Section 4.2) by sending F^0 to $F^0(\mathrm{Gr}^W_{-1}M_{\mathbb{C}})$. Recall that we have a bijection $\mathbb{H}_d \to D_\Psi$ that sends τ to $M_\tau^{0,-1} = \left(\begin{smallmatrix} \tau \\ 1_d \end{smallmatrix}\right)\mathbb{C}^d \subset \mathbb{C}^{2d}$.

For m and n in $\mathbb{Z}_{\geq 0}$ we denote by $\mathrm{M}_{m,n}(\mathbb{C})$ the set of complex m by n matrices.

4.5 Proposition *There is a bijection*

$$\mathbb{H}_d \times \mathrm{M}_{1,d}(\mathbb{C}) \times \mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C} \longrightarrow D, \quad (\tau, u, v, w) \mapsto \begin{pmatrix} u & w \\ \tau & v \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \subset M_{\mathbb{C}} = \bigoplus_{j=0}^{2d+1} \mathbb{C}e_j\,.$$

Proof Let τ be in $\mathbb{H}_d$. The $F^0(W_{-1}(M)_{\mathbb{C}})$ in the fibre over τ are the subspaces of $W_{-1}(M)_{\mathbb{C}}$ that are mapped isomorphically to the subspace $M_\tau^{0,-1}$ of $\mathrm{Gr}^W_{-1}(M)_{\mathbb{C}}$ in the short exact sequence

$$0 \to W_{-2}(M)_{\mathbb{C}} \to W_{-1}(M)_{\mathbb{C}} \to \mathrm{Gr}^W_{-1}(M)_{\mathbb{C}} \to 0.$$

This accounts for the first d columns in the matrix above. We take these columns as the first d elements of our basis of $F^0 M_{\mathbb{C}}$.

Each $F^0(M_{\mathbb{C}})$ in D that restricts to $F^0(W_{-1}M_{\mathbb{C}})$ given by a (τ, u) has a unique $d{+}1$th basis vector $\sum a_i e_i$ ending with d zeros and then a 1. This accounts for the last column. □

Let P be the subgroupscheme of $\mathrm{GL}(M) \times \mathrm{GL}(\mathbb{Z}(1))$ that fixes W, $\mathbb{Z}(1) \to W_{-2}(M)$, $2\pi i a \mapsto 2\pi i a e_0$, $\mathbb{Z}(0) \to \mathrm{Gr}^W_0(M)$, $a \mapsto a e_{2d+1}$, and $\Psi\colon \mathrm{Gr}^W_{-1}(M) \otimes \mathrm{Gr}^W_{-1}(M) \to \mathbb{Z}(1)$. Then, for any $\mathbb{Z}$-algebra R (we will only use $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$), we have

$$P(R) = \left\{ \begin{pmatrix} \mu(g) & x & z \\ 0 & g & y \\ 0 & 0 & 1 \end{pmatrix} : \begin{array}{c} (g, \mu(g)) \in \mathrm{GSp}(\Psi)(R), \\ \\ x \in \mathrm{M}_{1,2d}(R),\ y \in \mathrm{M}_{2d,1}(R),\ z \in R \end{array} \right\}, \tag{4.5.1}$$

where the matrices are with respect to the $\mathbb{Z}$-basis $2\pi i e_0, e_1, \ldots, e_{2d+1}$ of M. We let U be the subgroupscheme of P given by

$$U(R) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : z \in R \right\}.$$

We also let P^u be the unipotent radical of P, that is,

$$P^u(R) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x \in \mathrm{M}_{1,2d}(R),\ y \in \mathrm{M}_{2d,1}(R),\ z \in R \right\},$$

also known as the Heisenberg group. Then P^u is a central extension of the vector group P^u/U by $\mathbb{G}_\mathrm{a}$. The commutator pairing on P^u/U sends $((x, y), (x', y'))$ to $xy' - x'y$.

For R a subring of $\mathbb{C}$, the matrix with respect to the $\mathbb{C}$-basis $e_0, \ldots, e_{2d+1}$ of $M_\mathbb{C}$ of the element of $P(R)$ above is

$$\begin{pmatrix} \mu(g) & 2\pi i x & 2\pi i z \\ 0 & g & y \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.5.2}$$

By definition, $P(\mathbb{R})^+U(\mathbb{C})$ acts on D. We make this explicit for elements of $P^u(\mathbb{R})U(\mathbb{C})$, with respect to the $\mathbb{C}$-basis $e_0, \ldots, e_{2d+1}$, writing $2\pi i x = (2\pi i x_1\ 2\pi i x_2)$ and $y = \binom{y_1}{y_2}$:

$$\begin{aligned} &\begin{pmatrix} 1 & 2\pi i x_1 & 2\pi i x_2 & 2\pi i z \\ 0 & 1_d & 0 & y_1 \\ 0 & 0 & 1_d & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u & w \\ \tau & v \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \\ &= \begin{pmatrix} u + 2\pi i x_1 \tau + 2\pi i x_2 & w + 2\pi i x_1 v + 2\pi i z \\ \tau & v + y_1 \\ 1_d & y_2 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \\ &= \begin{pmatrix} u + 2\pi i x_1 \tau + 2\pi i x_2 & w + 2\pi i x_1 v + 2\pi i z \\ \tau & v + y_1 \\ 1_d & y_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_d & -y_2 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1} \\ &= \begin{pmatrix} u + 2\pi i x_1 \tau + 2\pi i x_2 & w + 2\pi i x_1 v + 2\pi i z - (u + 2\pi i x_1 \tau + 2\pi i x_2) y_2 \\ \tau & v + y_1 - \tau y_2 \\ 1_d & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^{d+1}. \end{aligned} \tag{4.5.3}$$

As the action of $\mathrm{Sp}_\Psi(\mathbb{R})$ on D_Ψ is transitive, we conclude that the action of $P(\mathbb{R})^+U(\mathbb{C})$ on D

is transitive. We also write out the action of $\mathrm{GSp}_\Psi(\mathbb{R})^+$ on D:

$$
\begin{aligned}
(4.5.4)\quad \begin{pmatrix} \mu & 0 & 0 & 0\\ 0 & a & b & 0\\ 0 & c & d & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix} u & w\\ \tau & v\\ 1_d & 0\\ 0 & 1\end{pmatrix}\mathbb{C}^{d+1} &= \begin{pmatrix} \mu u & \mu w\\ a\tau+b & av\\ c\tau+d & cv\\ 0 & 1\end{pmatrix}\begin{pmatrix}(c\tau+d)^{-1} & 0\\ 0 & 1\end{pmatrix}\mathbb{C}^{d+1}\\
&= \begin{pmatrix} \mu u(c\tau+d)^{-1} & \mu w\\ (a\tau+b)(c\tau+d)^{-1} & av\\ 1_d & cv\\ 0 & 1\end{pmatrix}\begin{pmatrix}1_d & -cv\\ 0 & 1\end{pmatrix}\mathbb{C}^{d+1}\\
&= \begin{pmatrix} \mu u(c\tau+d)^{-1} & \mu w-\mu u(c\tau+d)^{-1}cv\\ (a\tau+b)(c\tau+d)^{-1} & av-(a\tau+b)(c\tau+d)^{-1}cv\\ 1_d & 0\\ 0 & 1\end{pmatrix}\mathbb{C}^{d+1}.
\end{aligned}
$$

4.6 Proposition *The quotient $P^u(\mathbb{Z})\backslash D$ is the universal Poincaré torsor over $\mathbb{H}_d$.*

Proof We prove this by showing that the universal extension of the universal abelian variety over $\mathbb{H}_d$ by $\mathbb{C}^\times$ is unformised in exactly the same way when we express everything in terms of matrices. We view $\mathrm{M}_{1,d}(\mathbb{C})$ and $\mathrm{M}_{d,1}(\mathbb{C})$ as duals via the matrix multiplication (row times column).

Let us first consider a complex torus $A=V/L$, and an extension of complex Lie groups

$$0\to\mathbb{C}^\times\to E\to A\to 0\,.$$

Passing to universal covers gives us an extension of $\mathbb{C}$-vector spaces

$$0\to\mathbb{C}\to\tilde{E}\to V\to 0\,,$$

mapping to the previous sequence by exponential maps. The kernels of these maps form an extension

$$0\to\mathbb{Z}(1)\to M\to L\to 0\,.$$

The extensions of V by $\mathbb{C}$ and of L by $\mathbb{Z}(1)$ admit splittings, unique up to $V^\vee:=\mathrm{Hom}_\mathbb{C}(V,\mathbb{C})$ and $\mathrm{Hom}_\mathbb{Z}(L,\mathbb{Z}(1))=L^\vee(1)$. It follows that all extensions of A by $\mathbb{C}^\times$ are obtained as cokernels of maps

$$(4.6.1)\quad \mathbb{Z}(1)\oplus L\to\mathbb{C}\oplus V,\quad (2\pi i n,m)\mapsto(2\pi i n-\alpha(m),m),\quad \text{with } \alpha\in\mathrm{Hom}_\mathbb{Z}(L,\mathbb{C})=L^\vee_\mathbb{C}.$$

Our reason for choosing $2\pi i n-\alpha(m)$ in the line above, and not $2\pi i n+\alpha(m)$, is to avoid a sign in the isomorphism under construction between our universal extension here and that given by $P^u(\mathbb{Z})\backslash D$; see the term $-uy_2$ in the upper right coefficient in the last matrix in (4.5.3).

More explicitly, over $L^\vee_\mathbb{C}$ we have a family of extensions, with fibre at α the cokernel above. This family is universal for extensions with given splitting of their tangent spaces at 0 and given splitting of the kernel of the exponential map. On it, we have actions of $V^\vee$ and $L^\vee(1)$, the quotient by which gives us the universal extension of A by $\mathbb{C}^\times$, with base $L^\vee_\mathbb{C}/(V^\vee+L^\vee(1))$, which is therefore the dual complex torus. The family itself is the quotient of $L^\vee_\mathbb{C}\times V\times\mathbb{C}$ by a joint action of $V^\vee$, $L^\vee(1)$, L and $\mathbb{Z}(1)$. By "joint action" we mean that the actions of the individual elements of these four groups taken in this order induce a group structure on $V^\vee\times L^\vee(1)\times L\times\mathbb{Z}(1)$ and an action by that group on $L^\vee_\mathbb{C}\times V\times\mathbb{C}$. We make this more explicit for the family over $\mathbb{H}_d$.

Let τ be in $\mathbb{H}_d$. As in Section 4.2 we have

$$A_\tau = \mathbb{C}^{2d}/(\left(\begin{smallmatrix}\tau\\ 1_d\end{smallmatrix}\right)\mathbb{C}^d + \mathbb{Z}^{2d}) = \mathbb{C}^d/((1_d \ \ -\tau)\mathbb{Z}^{2d}) = \mathrm{M}_{d,1}(\mathbb{C})/((1_d \ \ -\tau)\mathrm{M}_{2d,1}(\mathbb{Z})).$$

The universal extension of A_τ by $\mathbb{C}^\times$ is the quotient of $\mathrm{M}_{1,2d}(\mathbb{C}) \times \mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C}$ by the joint actions of $\mathrm{M}_{1,d}(\mathbb{C})$, $\mathrm{M}_{2d,1}(\mathbb{Z})$, $\mathrm{M}_{1,2d}(\mathbb{Z}(1))$, and $\mathbb{Z}(1)$. We admit that this is not the same order as a few lines above, but the rest of the proof shows that once the quotient by $\mathrm{M}_{1,d}(\mathbb{C})$ has been taken, the remaing three groups match the corresponding pieces of the Heisenberg group, and therefore the order in which we consider their actions is irrelevant.

An element l in $\mathrm{M}_{1,d}(\mathbb{C})$ acts by postcomposing the embedding of $\mathbb{Z}(1) \oplus \mathrm{M}_{2d,1}(\mathbb{Z})$ in $\mathbb{C} \oplus \mathrm{M}_{d,1}(\mathbb{C})$ as in (4.6.1) with

$$\begin{pmatrix} w \\ v \end{pmatrix} \mapsto \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix} \begin{pmatrix} w \\ v \end{pmatrix} = \begin{pmatrix} w + lv \\ v \end{pmatrix}$$

giving the embedding

$$\begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & l \\ 0 & 1_d \end{pmatrix} \begin{pmatrix} 1 & -\alpha_1 & -\alpha_2 \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 & -\alpha_1 + l & -\alpha_2 - l\tau \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} .$$

The two displayed formulas above give the actions of l on (v, w) in $\mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C}$ and on (α_1, α_2) in $\mathrm{M}_{1,2d}(\mathbb{C})$, and therefore the action on $\mathrm{M}_{1,2d}(\mathbb{C}) \times \mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C}$

$$l\colon (\alpha_1, \alpha_2, v, w) \mapsto (\alpha_1 - l, \alpha_2 + l\tau, v, w + l(v)).$$

We make a quotient map for this action as follows. For every $(\alpha_1, \alpha_2, v, w)$ there is a unique l, namely, α_1, that brings it to the subset of all $(0, \alpha_2, v, w)$. This gives us the quotient map

$$\begin{aligned} q\colon \mathrm{M}_{1,2d}(\mathbb{C}) \times \mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C} &\to \mathrm{M}_{1,d}(\mathbb{C}) \times \mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C}, \\ (\alpha_1, \alpha_2, v, w) &\mapsto (\alpha_1\tau + \alpha_2, v, w + \alpha_1 v) , \end{aligned}$$

whose target is the source at τ of the bijection in Proposition 4.5. Now we consider the other actions and push them to this quotient.

At the point $(\alpha_1 \ \alpha_2)$ in $\mathrm{M}_{1,2d}(\mathbb{C})$ the embedding of $\mathbb{Z}(1) \oplus \mathrm{M}_{2d,1}(\mathbb{Z})$ in $\mathbb{C} \oplus \mathrm{M}_{d,1}(\mathbb{C})$ is

$$\begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -\alpha_1 & -\alpha_2 \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} ,$$

and therefore $(2\pi i n, \left(\begin{smallmatrix} m_1 \\ m_2 \end{smallmatrix}\right))$ in $\mathbb{Z}(1) \times \mathrm{M}_{2d,1}(\mathbb{Z})$ acts on $\mathrm{M}_{1,2d}(\mathbb{C}) \times \mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C}$ by the translations

$$(2\pi i n, \left(\begin{smallmatrix} m_1 \\ m_2 \end{smallmatrix}\right))\colon (\alpha_1, \alpha_2, v, w) \mapsto (\alpha_1, \alpha_2, v + m_1 - \tau m_2, w + 2\pi i n - \alpha_1 m_1 - \alpha_2 m_2) .$$

It follows that $2\pi i n$ and $\left(\begin{smallmatrix} m_1 \\ m_2 \end{smallmatrix}\right)$ act on $\mathrm{M}_{1,d}(\mathbb{C}) \times \mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C}$ by

$$(4.6.2) \quad 2\pi i n\colon (u, v, w) \mapsto (u, v, w + 2\pi i n), \quad \left(\begin{smallmatrix} m_1 \\ m_2 \end{smallmatrix}\right)\colon (u, v, w) \mapsto (u, v + m_1 - \tau m_2, w - u m_2) .$$

An element $2\pi i(n_1 \ n_2)$ in $\mathrm{M}_{1,2d}(\mathbb{Z}(1))$ acts by precomposing the embedding of $\mathbb{Z}(1) \oplus \mathrm{M}_{2d,1}(\mathbb{Z})$ in $\mathbb{C} \oplus \mathrm{M}_{d,1}(\mathbb{C})$ with

$$\begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -2\pi i n_1 & -2\pi i n_2 \\ 0 & 1_d & 0 \\ 0 & 0 & 1_d \end{pmatrix} \cdot \begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 2\pi i(n - n_1 m_1 - n_2 m_2) \\ m_1 \\ m_2 \end{pmatrix} .$$

where we have introduced a factor -1 because we want a left action. This gives the embedding

$$\begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -\alpha_1 & -\alpha_2 \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 1 & -2\pi i n_1 & -2\pi i n_2 \\ 0 & 1_d & 0 \\ 0 & 0 & 1_d \end{pmatrix} \cdot \begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -\alpha_1 - 2\pi i n_1 & -\alpha_2 - 2\pi i n_2 \\ 0 & 1_d & -\tau \end{pmatrix} \cdot \begin{pmatrix} 2\pi i n \\ m_1 \\ m_2 \end{pmatrix} .$$

So the identity on $\mathbb{C} \oplus \mathrm{M}_{d,1}(\mathbb{C})$ and the inverse of the action of $2\pi i(n_1\ n_2)$ on $\mathbb{Z}(1) \oplus \mathrm{M}_{2d,1}(\mathbb{Z})$ induce an isomorphism from the extension at (α_1, α_2) to the extension at $(\alpha_1 + 2\pi i n_1, \alpha_2 + 2\pi i n_2)$. Therefore the action of $2\pi i(n_1\ n_2)$ in $\mathrm{M}_{1,2d}(\mathbb{Z}(1))$ on $\mathrm{M}_{1,2d}(\mathbb{C}) \times \mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C}$ is by the translations

$$2\pi i(n_1\ n_2)\colon (\alpha_1, \alpha_2, v, w) \mapsto (\alpha_1 + 2\pi i n_1, \alpha_2 + 2\pi i n_2, v, w).$$

Pushing this to the quotient gives

$$2\pi i(n_1\ n_2)\colon (u, v, w) \mapsto (u + 2\pi i n_1 \tau + 2\pi i n_2, v, w + 2\pi i n_1 v)\,. \tag{4.6.3}$$

By inspection, one sees that the bijection in Proposition 4.5 is equivariant for the actions on its source by $\mathrm{M}_{2d,1}(\mathbb{Z})$, $\mathrm{M}_{1,2d}(\mathbb{Z}(1))$, and $\mathbb{Z}(1)$ given in (4.6.2) and (4.6.3) and the action on its target by $P^u(\mathbb{Z})$ given in (4.5.3), where $2\pi i n$ in $\mathbb{Z}(1)$, $\binom{m_1}{m_2}$ in $\mathrm{M}_{2d,1}(\mathbb{Z})$ and $2\pi i(n_1\ n_2)$ in $\mathrm{M}_{1,2d}(\mathbb{Z}(1))$ respectively correspond to

$$\begin{pmatrix} 1 & 0 & 2\pi i n \\ 0 & 1_{2d} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1_d & 0 & m_1 \\ 0 & 0 & 1_d & m_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2\pi i n_1 & 2\pi i n_2 & 0 \\ 0 & 1_d & 0 & 0 \\ 0 & 0 & 1_d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.6.4}$$

This finishes our identification of $P^u(\mathbb{Z})\backslash D$ with the universal Poincaré torsor over $\mathbb{H}_d$. $\square$

4.7 Duality and the Poincaré torsor

Proposition 4.6 and equations (4.5.3) give us an explicit description of the Poincaré torsor over $\mathbb{H}_d$. Let τ be in $\mathbb{H}_d$. Then we have (as in Section 4.2) $A_\tau = \mathrm{M}_{d,1}(\mathbb{C})/(1_d\ \ -\tau)\cdot\mathrm{M}_{2d,1}(\mathbb{Z})$ (see the 2nd column of the last matrix in (4.5.3), and $B_\tau = \mathrm{M}_{1,d}(\mathbb{C})/\mathrm{M}_{1,2d}(\mathbb{Z}(1))\cdot\binom{\tau}{1_d}$ (consider the first row), and the Poincaré torsor $\mathcal{P}_\tau$ on $A_\tau \times B_\tau$ that is the universal extension of A_τ by $\mathbb{C}^\times$ and of B_τ by $\mathbb{C}^\times$, giving isomorphisms $B_\tau = \mathrm{Ext}^1(A_\tau, \mathbb{C}^\times) = A_\tau^\vee$ and $A_\tau = \mathrm{Ext}^1(B_\tau, \mathbb{C}^\times) = B_\tau^\vee$.

Let now $f\colon B_\tau \to A_\tau$ be a morphism of abelian varieties. Then f is given by a complex linear map

$$\mathrm{M}_{1,d}(\mathbb{C}) \longrightarrow \mathrm{M}_{d,1}(\mathbb{C}), \quad u \mapsto f_{\mathbb{C}}\cdot u^t, \quad \text{with } f_{\mathbb{C}} \text{ in } \mathrm{M}_d(\mathbb{C})\,,$$

and a $\mathbb{Z}$-linear map

$$\mathrm{M}_{1,2d}(\mathbb{Z}(1)) \longrightarrow \mathrm{M}_{2d,1}(\mathbb{Z}), \quad 2\pi i(n_1 \quad n_2) \mapsto f_{\mathbb{Z}}\cdot \begin{pmatrix} n_1^t \\ n_2^t \end{pmatrix}, \quad \text{with} \quad f_{\mathbb{Z}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{M}_{2d}(\mathbb{Z})\,.$$

The fact that these form a commutative diagram

$$\begin{array}{ccc} \mathrm{M}_{1,2d}(\mathbb{Z}(1)) & \xrightarrow{\cdot\binom{\tau}{1_d}} & \mathrm{M}_{1,d}(\mathbb{C}) \\ \big\downarrow{\scriptstyle f_{\mathbb{Z}}} & & \big\downarrow{\scriptstyle f_{\mathbb{C}}} \\ \mathrm{M}_{2d,1}(\mathbb{Z}) & \xrightarrow[(1_d\ -\tau)\cdot]{} & \mathrm{M}_{d,1}(\mathbb{C}) \end{array}$$

gives us

$$2\pi i f_{\mathbb{C}}\tau^t = \alpha - \tau\gamma, \quad \text{and} \quad 2\pi i f_{\mathbb{C}} = \beta - \tau\delta\,. \tag{4.7.1}$$

The morphism $f\colon B_\tau \to A_\tau$ gives us the dual $f^\vee\colon B_\tau \to A_\tau$. We want to know what $(f^\vee)_{\mathbb{C}}$ and $(f^\vee)_{\mathbb{Z}}$ are. The following proposition answers this question.

4.8 Proposition *In the situation above,* $(f^\vee)_{\mathbb{Z}} = -(f_{\mathbb{Z}})^t$, *and* $(f^\vee)_{\mathbb{C}} = \frac{1}{2\pi i}(-\gamma^t + \tau\delta^t)$.

Proof Let $b \in B_\tau$. Then $f^\vee(b)$ is the unique $a \in A_\tau$ such that there is a morphism of extensions

$$\begin{tikzcd}
\mathbb{C}^\times \arrow[r, hook] & \mathcal{P}_{\tau,b} \arrow[r, two heads] & A_\tau \\
\mathbb{C}^\times \arrow[r, hook] \arrow[u, equal] & \mathcal{P}_{\tau,a} \arrow[r, two heads] \arrow[u] & B_\tau \arrow[u, "f"']
\end{tikzcd}\,.$$

Let $u \in \mathrm{M}_{1,d}(\mathbb{C})$ be an element that maps to b. Then we are looking for a v in $\mathrm{M}_{d,1}(\mathbb{C})$ (mapping to a), b_1 and b_2 in $\mathrm{M}_{d,1}(\mathbb{Z})$, and y in $\mathrm{M}_{d,1}(\mathbb{C})$ such that the diagram

$$\begin{tikzcd}
(2\pi i n, 2\pi i(n_1\ n_2)) \arrow[r, mapsto] & \left(2\pi i(n + (n_1\ n_2)\cdot\left(\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}\right)), f_{\mathbb{Z}}\cdot\left(\begin{smallmatrix} n_1^t \\ n_2^t \end{smallmatrix}\right)\right) \\
\mathbb{Z}(1) \oplus \mathrm{M}_{1,2d}(\mathbb{Z}(1)) \arrow[r] \arrow[d, "\cdot\left(\begin{smallmatrix} 1 & 0 \\ v & \tau \\ 0 & 1_d \end{smallmatrix}\right)"'] & \mathbb{Z}(1) \oplus \mathrm{M}_{2d,1}(\mathbb{Z}) \arrow[d, "\left(\begin{smallmatrix} 1 & 0 & -u \\ 0 & 1_d & -\tau \end{smallmatrix}\right)\cdot"] \\
\mathbb{C} \oplus \mathrm{M}_{1,d}(\mathbb{C}) \arrow[r] & \mathbb{C} \oplus \mathrm{M}_{d,1}(\mathbb{C}) \\
(z, x) \arrow[r, mapsto] & (z + x\cdot y, f_{\mathbb{C}}\cdot x^t)
\end{tikzcd}$$

is commutative. This commutativity is equivalent to:

$$\forall n_1, n_2 \in \mathrm{M}_{1,d}(\mathbb{Z})\colon 2\pi i(n_1\cdot(v + \tau\cdot y) + n_2\cdot y) = 2\pi i(n_1\cdot b_1 + n_2\cdot b_2) - u\cdot(\gamma\cdot n_1^t + \delta\cdot n_2^t)\,,$$

which in turn is equivalent to:

$$2\pi i(v + \tau\cdot y) = 2\pi i b_1 - \gamma^t\cdot u^t \quad \text{and} \quad 2\pi i y = 2\pi i b_2 - \delta^t\cdot u^t\,.$$

We solve this by taking

$$b_1 = 0, \quad b_2 = 0, \quad y = -(2\pi i)^{-1}\delta^t\cdot u^t, \quad v = (2\pi i)^{-1}(-\gamma^t\cdot u^t + \tau\cdot\delta^t\cdot u^t)\,.$$

We conclude that $f^\vee\colon B_\tau \to A_\tau$ is given by

$$\mathrm{M}_{1,d}(\mathbb{C}) \to \mathrm{M}_{d,1}(\mathbb{C}), \quad u \mapsto (f^\vee)_{\mathbb{C}}\cdot u^t, \quad \text{with} \quad (f^\vee)_{\mathbb{C}} = (2\pi i)^{-1}(-\gamma^t + \tau\cdot\delta^t)\,.$$

The fact that $(f^\vee)_{\mathbb{Z}}$ is as claimed follows from the commutativity of the diagram

$$\begin{tikzcd}
2\pi i (n_1 \ n_2) \arrow[r, mapsto] \arrow[ddd, mapsto, bend right] & -\left(\begin{smallmatrix} \alpha^t & \gamma^t \\ \beta^t & \delta^t \end{smallmatrix}\right)\cdot\left(\begin{smallmatrix} n_1^t \\ n_2^t \end{smallmatrix}\right) \arrow[ddd, mapsto, bend left] \\
\mathrm{M}_{1,2d}(\mathbb{Z}(1)) \arrow[r] \arrow[d, "\cdot\left(\begin{smallmatrix} \tau \\ 1_d \end{smallmatrix}\right)"'] & \mathrm{M}_{2d,1}(\mathbb{Z}) \arrow[d, "(1_d \ -\tau)\cdot"] \\
\mathrm{M}_{1,d}(\mathbb{C}) \arrow[r] & \mathrm{M}_{d,1}(\mathbb{C}) \\
2\pi i (n_1 \cdot \tau + n_2) \arrow[r, mapsto] & (-\gamma^t + \tau\cdot\delta^t)\cdot(\tau^t\cdot n_1^t + n_2^t) \,.
\end{tikzcd}$$

To establish this commutativity one uses (4.7.1). ☐

To finish this section, we include the polarisation

$$\Psi\colon \mathrm{M}_{2d,1}(\mathbb{Z}) \otimes \mathrm{M}_{2d,1}(\mathbb{Z}) \longrightarrow \mathbb{Z}(1), \quad x \otimes y \mapsto 2\pi i\, x^t \cdot \left(\begin{smallmatrix} 0 & -1_d \\ 1_d & 0 \end{smallmatrix}\right)\cdot y$$

in the present discussion (up to here we haven't used it, and the results above are valid for τ in $\mathrm{M}_d(\mathbb{C})$ whose imaginary part is invertible). Fixing the second variable in Ψ gives us the isomorphism

$$\Psi_1\colon \mathrm{M}_{2d,1}(\mathbb{Z}) \longrightarrow \mathrm{M}_{2d,1}(\mathbb{Z})^\vee(1), \quad y \mapsto (x \mapsto \Psi(x \otimes y))$$

of $\mathbb{Z}$-Hodge structures (at τ in $\mathbb{H}_d$), and therefore an isomorphism of complex tori

$$\lambda_\tau\colon A_\tau = \mathrm{M}_{2d,1}(\mathbb{C})/(M_\tau^{0,-1} + \mathrm{M}_{2d,1}(\mathbb{Z})) \longrightarrow \mathrm{M}_{2d,1}(\mathbb{C})^\vee/((M^\vee)_\tau^{0,-1} + \mathrm{M}_{2d,1}(\mathbb{Z})^\vee(1)) = B_\tau \,,$$

where the identification with B_τ is via universal extensions as in the proof of Proposition 4.6.

4.9 Proposition *With the notation above, the $\mathbb{C}$-linear and $\mathbb{Z}$-linear maps corresponding to λ_τ are*

$$\mathrm{M}_{d,1}(\mathbb{C}) \to \mathrm{M}_{1,d}(\mathbb{C}) \,, \quad v \mapsto 2\pi i\, v^t$$

and

$$\mathrm{M}_{2d,1}(\mathbb{Z}) \to \mathrm{M}_{1,2d}(\mathbb{Z})(1) \,, \quad y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) \mapsto 2\pi i\, y^t \cdot \left(\begin{smallmatrix} 0 & 1_d \\ -1_d & 0 \end{smallmatrix}\right) = 2\pi i\,(-y_2^t \ \ y_1^t) \,.$$

Proof For $(\lambda_\tau)_{\mathbb{Z}}$, this follows directly from the proof of Proposition 4.6. For $(\lambda_\tau)_{\mathbb{C}}$, it follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{M}_{2d,1}(\mathbb{Z}) \arrow[r] \arrow[d, "(1_d \ -\tau)\cdot"'] & \mathrm{M}_{1,2d}(\mathbb{Z})(1) \arrow[d, "\cdot\left(\begin{smallmatrix} \tau \\ 1_d \end{smallmatrix}\right)"] & \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) \arrow[r, mapsto] & 2\pi i\,(-y_2^t \ \ y_1^t) \\
\mathrm{M}_{d,1}(\mathbb{C}) \arrow[r] & \mathrm{M}_{1,d}(\mathbb{C}) & v \arrow[r, mapsto] & 2\pi i\, v^t \,.
\end{tikzcd}$$

Here one uses that $\tau^t = \tau$. ☐

It is reassuring to see, using Proposition 4.8, that $\lambda_\tau^\vee = \lambda_\tau$, as $(\lambda_\tau)_{\mathbb{Z}} = \left(\begin{smallmatrix} 0 & 1_d \\ -1_d & 0 \end{smallmatrix}\right)$ is antisymmetric.

5 Ribet varieties are special subvarieties

We recall that in Section 3 we had an abelian scheme $A \to S$ and a morphism $f\colon A^\vee \to A$, and $\alpha := f - f^\vee\colon A^\vee \to A$, hence $\alpha^\vee = -\alpha$, and a section r_f of the Poincaré torsor over the graph of α. Now we describe this in the present context, over $\mathbb{C}$, in the principally polarised case.

Let $M := \mathbb{Z}(1) \oplus \mathbb{Z}^{2d} \oplus \mathbb{Z}$, W, D, and P be as in Section 4.4, and recall the notation B_τ from the beginning of Section 4.7. Let τ_0 be in $\mathbb{H}_d$, $f\colon B_{\tau_0} \to A_{\tau_0}$ a morphism, and $\alpha := f - f^\vee\colon B_{\tau_0} \to A_{\tau_0}$. Then α gives (and is given by) the $\mathbb{Z}$-linear map

(5.0.1) $$\mathrm{Gr}^W_{-1}(M)^\vee(1) = \mathrm{M}_{1,2d}(\mathbb{Z}(1)) \longrightarrow \mathrm{M}_{2d,1}(\mathbb{Z}) = \mathrm{Gr}^W_{-1}(M)\,, \quad 2\pi i(n_1 \quad n_2) \mapsto \alpha_\mathbb{Z}\cdot\begin{pmatrix} n_1^t \\ n_2^t \end{pmatrix},$$

with $\alpha_\mathbb{Z} \in \mathrm{M}_{2d}(\mathbb{Z})$. By Proposition 4.8,

$$\alpha_\mathbb{Z} = f_\mathbb{Z} - (f^\vee)_\mathbb{Z} = f_\mathbb{Z} + (f_\mathbb{Z})^t\,.$$

Hence $\alpha_\mathbb{Z}$ is symmetric and the quadratic form

$$\mathrm{M}_{1,2d}(\mathbb{Z}) \longrightarrow \mathbb{Z}, \quad x \mapsto \frac{1}{2}\,x\cdot\alpha_\mathbb{Z}\cdot x^t = x\cdot f_\mathbb{Z}\cdot x^t$$

is $\mathbb{Z}$-valued. Just for completeness, we include that the endomorphism $\beta := \alpha \circ \lambda_{\tau_0}$ of A_{τ_0} is anti-symmetric for the Rosati involution:

$$\lambda_{\tau_0}^{-1} \circ \beta^\vee \circ \lambda_{\tau_0} = \lambda_{\tau_0}^{-1} \circ (\alpha \circ \lambda_{\tau_0})^\vee \circ \lambda_{\tau_0} = \lambda_{\tau_0}^{-1} \circ \lambda_{\tau_0}^\vee \circ \alpha^\vee \circ \lambda_{\tau_0} = -\alpha \circ \lambda_{\tau_0} = -\beta\,.$$

Now, everything is in place to introduce the connected mixed Shimura subvariety of the universal Poincaré-torsor $P^u(\mathbb{Z})\backslash D$ over $\mathbb{H}_d$ (with its $\mathrm{GSp}(\Psi)(\mathbb{Z})$-action) that is dictated by the map in (5.0.1) being a morphism of Hodge structures. Concretely, we let P_α and G_α be the connected components of identity of the stabilisers of (5.0.1) in P and in $\mathrm{GSp}(\Psi)$. As the action of P on $\mathrm{Gr}^W_{-1}(M)$ factors through $\mathrm{GSp}(\Psi)$, P_α is the inverse image in P of G_α, and the unipotent radical P_α^u of P_α is equal to P^u, hence contains U. In D and $\mathbb{H}_d$, we consider the orbits

(5.0.2) $$D_\alpha := P_\alpha(\mathbb{R})^+U(\mathbb{C})\cdot\widetilde{\tau_0} \subset D \quad \text{and} \quad \mathbb{H}_{d,\alpha} := G_\alpha(\mathbb{R})^+\cdot\tau_0 \subset \mathbb{H}_d\,,$$

where $\widetilde{\tau_0}$ is the element of D that corresponds to $(\tau_0, 0, 0, 0)$ under the bijection of Proposition 4.5. More intrinsically: $\widetilde{\tau_0}$ is the mixed Hodge structure on M in which the weight filtration is split over $\mathbb{Z}$ by the given $\mathbb{Z}$-basis (hence it is pure), and which induces that given by τ_0 on $\mathrm{Gr}^W_{-1}M$. Here, it does not matter which lift of τ_0 we take, but it will matter further on when we describe the Ribet section in D_α.

Deligne's group theoretical description of Shimura varieties shows that $\mathbb{H}_{d,\alpha}$ is the connected component containing τ_0 of the set of $\tau \in \mathbb{H}_d$ where (5.0.1) is a morphism of Hodge structures (equivalently: where it induces a morphism $\alpha\colon B_\tau \to A_\tau$). Let us explain in a few lines how this works; for details, see [21, Section 2.4] and [9, Section 1.1.12]. Pure Hodge structures on an $\mathbb{R}$-vector space correspond to $\mathbb{R}$-algebraic actions of $\mathbb{C}^\times$. For G a connected linear algebraic group over $\mathbb{R}$, the set of $\mathbb{R}$-morphisms $\mathrm{Hom}(\mathbb{C}^\times, G(\mathbb{R}))$ is the set of $\mathbb{R}$-points of a smooth $\mathbb{R}$-scheme, which is the disjoint union of G-orbits (for G acting by composition with inner automorphisms). The $G(\mathbb{R})^+$-orbits in $\mathrm{Hom}(\mathbb{C}^\times, G(\mathbb{R}))$ are the connected components for the Archimedean topology. References in [29] (SGA 3): Exp. IX, Cor. 3.3, and Exp. XI, Cor. 4.2.

The pairs $(P_\mathbb{Q}, D)$, $(G_{\alpha,\mathbb{Q}}, \mathbb{H}_{d,\alpha})$ and $(P_{\alpha,\mathbb{Q}}, D_\alpha)$ are connected mixed Shimura data as in [25, Def. 2.1], and we have the diagram of morphisms of connected mixed Shimura data

(5.0.3) $$\begin{tikzcd} (P_{\alpha,\mathbb{Q}}, D_\alpha) \arrow[r, hook] \arrow[d, two heads] & (P_\mathbb{Q}, D) \arrow[d, two heads] \\ (G_{\alpha,\mathbb{Q}}, \mathbb{H}_{d,\alpha}) \arrow[r, hook] & (\mathrm{GSp}(\Psi)_\mathbb{Q}, \mathbb{H}_d)\,. \end{tikzcd}$$

The careful reader will have noticed that we must show that D is a $P(\mathbb{R})^+U(\mathbb{C})$-orbit in $\mathrm{Hom}(\mathbb{C}^\times, P(\mathbb{C}))$ and D_α is a $P_\alpha(\mathbb{R})^+U(\mathbb{C})$-orbit in $\mathrm{Hom}(\mathbb{C}^\times, P_\alpha(\mathbb{C}))$. For the fact that the natural maps from these orbits to D and D_α are isomorphisms we refer to Propositions 1.18 and 1.16(c) in [24] (the surjectivity is clear because source and target are orbits for the same group, for the injectivity one has to show that the stabilisers are the same).

5.1 Proposition *The quotient $P^u_\alpha(\mathbb{Z})\backslash D_\alpha$ is the universal Poincaré torsor over $\mathbb{H}_{d,\alpha}$. The quotient of D_α by $P^u_\alpha(\mathbb{Z})U(\mathbb{C})$ is the universal family of $A_\tau \times B_\tau$'s over $\mathbb{H}_{d,\alpha}$. The quotient of D_α by $P^u_\alpha(\mathbb{Z})\mathrm{M}_{1,2d}(\mathbb{R})U(\mathbb{C})$ is the universal family of A_τ's over $\mathbb{H}_{d,\alpha}$, and the quotient of D_α by $P^u_\alpha(\mathbb{Z})\mathrm{M}_{2d,1}(\mathbb{R})U(\mathbb{C})$ is the universal family of B_τ's over $\mathbb{H}_{d,\alpha}$.*

Proof One easily deduces this from Proposition 4.6 and parts of its proof. □

Now we proceed directly to the Ribet section, by revealing the tensor that defines it, namely, the map (encoded by a matrix $\widetilde{\alpha_\mathbb{Z}}$)

(5.1.1)
$$\begin{array}{ccccccc} M^\vee(1) & = & \mathbb{Z}\oplus \mathrm{M}_{1,2d}(\mathbb{Z}(1))\oplus\mathbb{Z}(1) & \longrightarrow & \mathbb{Z}(1)\oplus\mathrm{M}_{2d,1}(\mathbb{Z})\oplus\mathbb{Z} & = & M \\ x & = & (k_1 \quad 2\pi i n \quad 2\pi i k_2) & \longmapsto & \begin{pmatrix} -2\pi i k_2 \\ \alpha_\mathbb{Z}\cdot n^t \\ -k_1\end{pmatrix} & = & \widetilde{\alpha_\mathbb{Z}}\cdot x^t\,, \end{array}$$

where

(5.1.2)
$$\widetilde{\alpha_\mathbb{Z}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & \alpha_\mathbb{Z} & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{in } \mathrm{M}_{2d+2}(\mathbb{Z})\,.$$

This tensor was already described in [27], see also [3, Lemme 6]. We let $P_{\tilde\alpha}$ be the stabiliser in P of this map (5.1.1), as a group scheme over $\mathbb{Z}$. Then, for any $\mathbb{Z}$-algebra R and for any p in $P(R)$ we have $p\in P_{\tilde\alpha}(R)$ if and only if $p\cdot\widetilde{\alpha_\mathbb{Z}} = \mu(p)\widetilde{\alpha_\mathbb{Z}}\cdot p^{-1,t}$ in $\mathrm{M}_{2d+2}(R)$, which is equivalent to $p\cdot\widetilde{\alpha_\mathbb{Z}}\cdot p^t = \mu(p)\widetilde{\alpha_\mathbb{Z}}$. A direct computation then shows, for any $\mathbb{Z}$-algebra R in which multiplication by 2 is injective:

(5.1.3)
$$P_{\tilde\alpha}(R) = \left\{ \begin{pmatrix} \mu(g) & x & \mu(g)^{-1}xf_\mathbb{Z}x^t \\ 0 & g & \mu(g)^{-1}g\alpha_\mathbb{Z}x^t \\ 0 & 0 & 1 \end{pmatrix} : (g,\mu(g))\in G_\alpha(R),\ x\in\mathrm{M}_{1,2d}(R) \right\},$$

where the matrices are with respect to the $\mathbb{Z}$-basis $2\pi i e_0, e_1, \ldots, e_{2d+1}$ of M. We note that for R on which multiplication by 2 is injective, $P_{\tilde\alpha}(R)$ is the semi-direct product

(5.1.4)
$$P_{\tilde\alpha}(R) = \mathrm{M}_{1,2d}(R)\rtimes G_\alpha(R) = \left\{\begin{pmatrix} 1 & x & xf_\mathbb{Z}x^t \\ 0 & 1_{2d} & \alpha_\mathbb{Z}x^t \\ 0 & 0 & 1\end{pmatrix}\right\}\cdot\left\{\begin{pmatrix} \mu(g) & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1\end{pmatrix}\right\}.$$

where x ranges over $\mathrm{M}_{1,2d}(R)$ and g over $G_\alpha(R)$. In particular, the unipotent radical (over $\mathbb{Z}[1/2]$) of $P_{\tilde\alpha}$ is a vector group scheme, and the weight -2 part of its Lie algebra is zero. We define

(5.1.5)
$$D_{\tilde\alpha} := P_{\tilde\alpha}(\mathbb{R})^+\cdot\widetilde{\tau_0}\subset D_\alpha\subset D\,.$$

Then we have the following diagram of connected mixed Shimura data

(5.1.6)
$$\begin{tikzcd} (P_{\tilde\alpha,\mathbb{Q}}, D_{\tilde\alpha}) \arrow[r, hook] \arrow[dr, two heads] & (P_{\alpha,\mathbb{Q}}, D_\alpha) \arrow[d, two heads] \\ & (G_{\alpha,\mathbb{Q}}, \mathbb{H}_{d,\alpha})\,. \end{tikzcd}$$

5.2 Theorem *The quotient $P^u_{\tilde\alpha}(\mathbb{Z})\backslash D_{\tilde\alpha}$ is the image of a section r^{Sh}_f in $P^u_\alpha(\mathbb{Z})\backslash D_\alpha$ (the universal Poincaré torsor over $\mathbb{H}_{d,\alpha}$, see Proposition 5.1) over the family of B_τ with τ ranging over $\mathbb{H}_{d,\alpha}$. In particular, the image of r^{Sh}_f is a special subvariety. This section r^{Sh}_f is equal, in this setting, to the section r_f of Proposition 3.1.*

Proof It is sufficient to verify the claim at each $\tau \in \mathbb{H}_{d,\alpha}$. So let τ be such. The description in (4.5.3) of the action of $P^u_\alpha(\mathbb{R})U(\mathbb{C}) = P^u(\mathbb{R})U(\mathbb{C})$ on D shows that it is free and transitive on the fibre $D_{\alpha,\tau}$ of $D_\alpha \to \mathbb{H}_{d,\alpha}$ at τ. This gives us the bijection

$$(5.2.1) \qquad P^u_\alpha(\mathbb{R})U(\mathbb{C}) \xrightarrow{\ \simeq\ } D_{\alpha,\tau} \qquad p \mapsto p{\cdot}\widetilde\tau\,,$$

where $\widetilde\tau$ is the element of D that corresponds to $(\tau,0,0,0)$ under the bijection of Proposition 4.5. For g in $G_\alpha(\mathbb{R})^+$ with $g{\cdot}\tau_0 = \tau$, we have $g \in P_{\tilde\alpha}(\mathbb{R})^+$ via (5.1.4), and $\widetilde\tau = g{\cdot}\widetilde{\tau_0}$ (use (4.5.4)), hence $D_{\tilde\alpha,\tau} = P^u_{\tilde\alpha}(\mathbb{R}){\cdot}\widetilde\tau$. Via the bijection (5.2.1), the inclusion $D_{\tilde\alpha,\tau} \subset D_{\alpha,\tau}$ corresponds to the inclusion $P^u_{\tilde\alpha}(\mathbb{R}) \subset P^u_\alpha(\mathbb{R})U(\mathbb{C})$, and the $P^u_{\tilde\alpha}(\mathbb{Z})$-action on $D_{\tilde\alpha}$ corresponds to the action by left-multiplication on $P^u_{\tilde\alpha}(\mathbb{R})$. By (5.1.4), $P^u_{\tilde\alpha}(\mathbb{R}) = \mathrm{M}_{1,2d}(\mathbb{R})$, and (5.2.1) identifies this with $\mathrm{M}_{1,d}(\mathbb{C})$, sending (x_1,x_2) to $2\pi i(x_1\tau + x_2)$. Hence $P^u_{\tilde\alpha}(\mathbb{Z})\backslash D_{\tilde\alpha,\tau} = B_\tau$. Proposition 5.1 together with the description (5.1.4) of $P^u_{\tilde\alpha}$ show that $P^u_{\tilde\alpha}(\mathbb{Z})\backslash D_{\tilde\alpha}$ is the image of a section r^{Sh}_f of the Poincaré torsor over the graph of $\alpha\colon B_\tau \to A_\tau$ (equivalently, over B_τ). This section differs from r_f by multiplication by a global regular function on B_τ, hence by a constant factor in $\mathbb{C}^\times$. As both sections have value 1 at $0 \in B_\tau$, they are equal. $\square$

5.3 Proposition *Let τ be in $\mathbb{H}_{d,\alpha}$. On the left hand side of the bijection in (5.2.1), B_τ is $\mathrm{M}_{1,2d}(\mathbb{R})/\mathrm{M}_{1,2d}(\mathbb{Z})$. For x in $\mathrm{M}_{1,2d}(\mathbb{R})$ and $\overline{x}$ its image in B_τ, the extension $E_{\tau,\overline{x}}$ of A_τ by $\mathbb{C}^\times$ corresponding to $\overline{x}$ is, as real Lie group, $(\mathbb{C}/2\pi i\mathbb{Z}) \times (\mathrm{M}_{2d,1}(\mathbb{R})/\mathrm{M}_{2d,1}(\mathbb{Z}))$, and $r_f(\overline{x})$ is given by $(2\pi i x f_{\mathbb{Z}} x^t, \alpha_{\mathbb{Z}} x^t)$. If $\overline{x}$ is of order n in B_τ, then $r_f(\overline{x})$ in $E_{\tau,\overline{x}}$ is killed by n^2.*

Proof Consider (5.2.1) and (4.5.3). Let $x = (x_1,x_2) \in \mathrm{M}_{1,2d}(\mathbb{R})$. This gives the element

$$p_x := \begin{pmatrix} 1 & 2\pi i x_1 & 2\pi i x_2 & 0\\ 0 & 1_d & 0 & 0\\ 0 & 0 & 1_d & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \in P^u_\alpha(\mathbb{R})\,, \quad \text{and} \quad p_x{\cdot}\tilde\tau = \begin{pmatrix} 2\pi i(x_1\tau + x_2) & 0\\ \tau & 0\\ 1_d & 0\\ 0 & 1\end{pmatrix}\mathbb{C}^{d+1} \in D_{\alpha,\tau}\,.$$

This shows that on the left hand side of (5.2.1), B_τ is $\mathrm{M}_{1,2d}(\mathbb{R})/\mathrm{M}_{1,2d}(\mathbb{Z})$. To describe $E_{\tau,\overline{x}}$, let, for z in $\mathbb{C}$ and $\left(\begin{smallmatrix} y_1\\ y_2\end{smallmatrix}\right)$ in $\mathrm{M}_{2d,1}(\mathbb{R})$,

$$p_{z,y} := \begin{pmatrix} 1 & 0 & 0 & 2\pi i z\\ 0 & 1_d & 0 & y_1\\ 0 & 0 & 1_d & y_2\\ 0 & 0 & 0 & 1\end{pmatrix}, \text{ then } p_{z,y}{\cdot}p_x{\cdot}\tilde\tau = \begin{pmatrix} 2\pi i(x_1\tau + x_2) & 2\pi i(z - (x_1\tau + x_2)y_2)\\ \tau & y_1 - \tau y_2\\ 1_d & 0\\ 0 & 1\end{pmatrix}\mathbb{C}^{d+1}\,.$$

Now observe that $2\pi i(z - (x_1\tau + x_2)y_2)$ and $y_1 - \tau y_2$ are $\mathbb{R}$-linear in z, y_1 and y_2, and that $2\pi i(x_1\tau + x_2)$ does not depend on z, y_1 and y_2. Hence the $\mathbb{R}$-vector space structure on $\{2\pi i(x_1\tau + x_2)\} \times \mathrm{M}_{d,1}(\mathbb{C}) \times \mathbb{C}$ in $D_{\alpha,\tau}$ corresponds to the $\mathbb{R}$-vector space structure on $\mathrm{M}_{2d,1}(\mathbb{R}) \times \mathbb{C}$ on the left, and therefore the same holds for the group structures. The left-action by the $p_{z,y}$ with $z \in \mathbb{Z}$ and $y \in \mathrm{M}_{2d,1}(\mathbb{Z})$ on these 2 real vector spaces then gives the description of $E_{\tau,\overline{x}}$. The description of $P^u_{\tilde\alpha}$ in (5.1.4) proves the last two claims in the proposition. $\square$

5.4 Remark Assume that α is an isogeny.

1. The tensor $\widetilde{\alpha}$ in (5.1.1) that defines the Ribet variety as an irreducible component of its Hodge locus is a selfduality of mixed $\mathbb{Q}$-Hodge structures. It is interesting to see that on the underlying $\mathbb{Z}$-module M it is a symmetric $\mathbb{Z}(1)$-valued pairing. Algebraically this can be described as a self-duality of 1-motives with $\mathbb{Q}$-coefficients, see [27] and also [3].

2. Let $\Gamma_\alpha(3)$ be the kernel of $G_\alpha(\mathbb{Z}) \to G_\alpha(\mathbb{F}_3)$. Then $\Gamma_\alpha(3)$ acts on the whole situation of Theorem 5.2, freely on the base $\mathbb{H}_{d,\alpha}$. The quotient $\Gamma_\alpha(3)\backslash D_\alpha$ is then the Poincaré torsor $\mathcal{P}$ for the abelian scheme $A := \Gamma_\alpha(3)\backslash(P^u_\alpha(\mathbb{Z})\mathrm{M}_{1,2d}(\mathbb{R})U(\mathbb{C})\backslash D_\alpha)$ over the pure Shimura variety $S := \Gamma_\alpha(3)\backslash\mathbb{H}_{d,\alpha}$, with the image of the Ribet section r_f as a special subvariety of a family of semi-abelian varieties. As a generalisation of Lemma 2.3, we will now prove that this special subvariety is not a torsion translate of a family of algebraic subgroups. Let τ be in $\mathbb{H}_{d,\alpha}$ and $x = (x_1, \ldots, x_{2d})$ be in $\mathrm{M}_{1,2d}(\mathbb{R})$ such that $x_1, \ldots, x_{2d}, x\alpha_{\mathbb{Z}}x^t$ in $\mathbb{R}$ are $\mathbb{Q}$-linearly independent. Then the coordinates of $\alpha_{\mathbb{Z}}{\cdot}x^t$ and $x\alpha_{\mathbb{Z}}x^t$ are $\mathbb{Q}$-linearly independent. By Proposition 5.3, the subgroup of $E_{\tau,\overline{x}}$ generated by $r_f(\overline{x})$ is dense, for the Archimedean topology, in $(i\mathbb{R}/2\pi i\mathbb{Z}) \times (\mathrm{M}_{2d,1}(\mathbb{R})/\mathrm{M}_{2d,1}(\mathbb{Z}))$. This shows that the union of the images of the nr_f, with $n \in \mathbb{Z}$, is dense, for the Archimedean topology, in a circle bundle of real codimension 1 in $\mathcal{P}$. The fibres of this circle bundle are the maximal compact subgroups of the corresponding complex analytic semi-abelian varieties.

3. The example just given (the image of r_f) now supports Pink's Conjecture 1.3 of [26]: indeed, it is a subvariety Y of $\mathcal{P}$ containing a Zariski dense set of special points (i.e. special subvarieties of maximal codimension in $\mathcal{P}$), and it is itself a special subvariety of $\mathcal{P}$. For further verifications in this context of [26], Conjecture 1.3, see [4] and [5].

4. Let us now clarify what is wrong in the proof of Theorem 6.3 of [26]. The error is in the statement "Since the special subvarieties of A that dominate S are precisely the translates of semiabelian subschemes by torsion points,..."; we have just seen that this is not true. Similarly, note the sentence "Conversely, for any special subvariety $T \subset A$, every irreducible component of $T \cap A_s$ is a translate of a semiabelian subvariety of A_s by a torsion point." in the proof of Theorem 5.7 of [26].

   The essential difference between the case of Kuga varieties (Shimura families of abelian varieties over pure Shimura varieties), where the statement is correct ([25], Proposition 4.6), and the case of Shimura families of tori over Kuga varieties is as follows. In the first case the morphism of mixed Shimura varieties $A \to S$ is induced by a morphism of Shimura data $(P, D_P) \to (G, D_G)$ with G reductive, and $P \to G$ surjective, split, with kernel V a $\mathbb{Q}$-vector space. Then the special subvarieties Z of A that surject to S are given by morphisms of sub-Shimura data (Q, D_Q) of (P, D_P), with $Q \to G$ is surjective. Then Q is an extension of G by $Q \cap V$, a sub-$\mathbb{Q}$-vector space of V. This extension is split because $\mathrm{H}^2(G, Q\cap V) = 0$, and the splitting is unique op to conjugation by $Q\cap V$ because $\mathrm{H}^1(G, Q \cap V) = 0$. So indeed such special subvarieties come from subfamilies $B \to S$ of $A \to S$ and Hecke correspondences that account for translations by torsion points. In the second case, say $T \to A$, these arguments no longer apply because the group P in the Shimura datum for A (such as P_α/U as above) is not necessarily reductive (and indeed the extension P_α of P_α/U by U is not split).

6 The example with elliptic curves, using generalised jacobians

In this section we give a description of the example in Section 2 in terms of the generalised jacobian of a family of singular curves. Our reason to include it is that this description is more elementary than the one using the Poincaré bundle, and that it is more explicit in terms of divisors, rational functions, Weil pairing, and is a nice application of Weil reciprocity.

We return to the situation as in Section 2, except that now we let k be an arbitrary algebraically closed field. Let E be an elliptic curve over k. Here we will view $E \times E$ as a family of elliptic curves over E via the 2nd projection $\mathrm{pr}_2\colon E_E = E \times E \to E$, $(x, y) \mapsto y$.

In our construction, we will remove a finite number of points of the base curve E, and denote the complement by U. This U will be shrunk a few times.

The diagonal morphism $\Delta\colon E \to E_E$, $x \mapsto (x, x)$, is a section, and the group law of E_E over E gives us a second section 2Δ, $x \mapsto (2x, x)$. The sections Δ and 2Δ are disjoint over the open subset $U := E - \{0\}$.

We let $C \to U$ be the singular curve over U obtained by identifying the disjoint sections 2Δ and Δ. As a set, it is the quotient of E_U by the equivalence relation generated by $(2x, x) \sim (x, x)$ with x ranging over U. The topology on C is the finest one for which the quotient map $\mathrm{quot}\colon E_U \to C$ is continuous: a subset V of C is open if and only if $\mathrm{quot}^{-1}V$ is open in E_U. The regular functions on an open set V of C are the regular functions f on $\mathrm{quot}^{-1}V$ such that $f(2x, x) = f(x, x)$ whenever $\mathrm{quot}(x, x)$ is in V. It is proved in Theorem 5.4 of [13] that this topological space with sheaf of rings is indeed an algebraic variety over k. In the category of varieties over k, $\mathrm{quot}\colon E_U \to C$ is the co-equaliser of the pair of morphisms $(2\Delta, \Delta)$ from U to E_U:

$$U \overset{2\Delta}{\underset{\Delta}{\rightrightarrows}} E_U \xrightarrow{\mathrm{quot}} C \quad .$$

The curve $C \to U$ is a family of singular curves, each with an ordinary double point; it is semi-stable of genus 2 (see [6, 9.2/6]). Its normalisation is $\mathrm{quot}\colon E_U \to C$. Its generalized jacobian

$$G := \mathrm{Pic}^0_{C/U}$$

is described in [6], 8.1/4, 8.2/7, 9.2, 9.4/1, and in more direct terms in this specific situation in [15]. As $C \to U$ has a section (for example $\overline{\Delta} := \mathrm{quot} \circ \Delta$), we have, for every $T \to U$, that $G(T) = \mathrm{Pic}^0(C_T/T)/\mathrm{Pic}(T)$, where $\mathrm{Pic}^0(C_T/T)$ is the group of isomorphism classes of invertible $\mathcal{O}$-modules on C_T that have degree zero on the fibres of $C_T \to T$. The group $\mathrm{Pic}(T)$ is contained as direct summand in $\mathrm{Pic}^0(C_T/T)$ via pullback by the projection $C_T \to T$ and a chosen section. In particular, a divisor D on C that is finite over U, disjoint from $\overline{\Delta}(U)$ and of degree zero after restriction to the fibres of $C \to U$ gives the invertible $\mathcal{O}_C$-module $\mathcal{O}_C(D)$ that has degree zero on the fibres and therefore gives an element denoted $[D]$ in $G(U)$. An alternative and very useful description, given in detail in [15], of $\mathrm{Pic}(C_T)$ is the set of isomorphism classes of $(\mathcal{L}, \sigma)$, with $\mathcal{L}$ an invertible $\mathcal{O}$-module on E_T and $\sigma\colon (2\Delta)^*\mathcal{L} \to \Delta^*\mathcal{L}$ an isomorphism of $\mathcal{O}$-modules on T, where an isomorphism from $(\mathcal{L}, \sigma)$ to $(\mathcal{L}', \sigma')$ is an isomorphism $f\colon \mathcal{L} \to \mathcal{L}'$ such that $(\Delta^* f) \circ \sigma = \sigma' \circ (2\Delta)^* f$.

For x in U, the fibre G_x is, as abelian group, the group $\mathrm{Pic}^0(C_x)$. In terms of divisors this is the quotient of the group $\mathrm{Div}^0(C_x)$ of degree zero divisors with support outside $\{\overline{\Delta}(x)\}$ by the subgroup of principal divisors $\mathrm{div}(f)$ for nonzero rational functions f in $k(C_x)^\times$ that are regular and invertible at $\overline{\Delta}(x)$. As $C_x - \{\overline{\Delta}(x)\}$ is the same as $E - \{2x, x\}$, $\mathrm{Div}^0(C_x)$ is the group of degree zero divisors on E with support outside $\{2x, x\}$. An element f of $k(C_x)^\times$ that is regular at $\overline{\Delta}(x)$ is an element of $k(E)^\times$ that is regular at $2x$ and x and satisfies $f(2x) = f(x)$. This gives us a useful description of G_x.

The normalisation map $\mathrm{quot}\colon E_U \to C$ induces a morphism of group schemes over U

$$\pi\colon G = \mathrm{Pic}^0_{C/U} \to \mathrm{Pic}^0_{E_U/U} = E_U,$$

and identifies G with the extension of E by $\mathbb{G}_m$ given by the section $\Delta \in E_U(U)$. For x in U and $D \in \mathrm{Div}^0(C_x)$, the class $[D]$ in G_x lies in the kernel $k^\times$ of π_x if and only if there exists

$f \in k(E)^\times$ such that $D = \mathrm{div}(f)$ on E, and it is then a torsion point in $k^\times$ if and only if the quotient $f(2x)/f(x) \in k^\times$, which does not depend on the choice of f, is a root of unity.

We recall that for u in $\mathrm{End}(E)$, the pullback map u^* on $\mathrm{Div}(E)$ induces $u^\vee$ in $\mathrm{End}(E^\vee)$, the dual of u, and then $\overline{u} := \lambda^{-1}u^\vee\lambda$ in $\mathrm{End}(E)$ is called the Rosati-dual of u, where λ is the standard polarization as in Section 2. The map $\mathrm{End}(E) \to \mathrm{End}(E)$, $u \mapsto \overline{u}$ is a anti-morphism of rings, in fact an involution. It is characterised by the property that in $\mathrm{End}(E)$ we have $\overline{u}u = \deg(u) = \deg(\overline{u})$ and $u + \overline{u} \in \mathbb{Z}$. Also, the pushforward map u_* on $\mathrm{Div}(E)$ induces an element still denoted u_* in $\mathrm{End}(E^\vee)$ such that $\lambda u = u_*\lambda$ in $\mathrm{Hom}(E, E^\vee)$, and $u_*u^* = \deg(u)$ in $\mathrm{End}(E^\vee)$. Hence u_* and u^* are each other's Rosati duals. For f a nonzero rational function on E and $u \neq 0$ we have $u^*\mathrm{div}(f) = \mathrm{div}(f \circ u)$, and $u_*\mathrm{div}(f) = \mathrm{div}(\mathrm{Norm}_u(f))$, where $\mathrm{Norm}_u \colon k(E)^\times \to k(E)^\times$ is the norm map along u.

We will use Weil reciprocity: for f and g nonzero rational functions on E such that $\mathrm{div}(f)$ and $\mathrm{div}(g)$ have disjoint supports, one has $f(\mathrm{div}(g)) = g(\mathrm{div}(f))$, where for $D = \sum_P D(P)\cdot P$ a divisor on E one defines $f(D) = \prod_P f(P)^{D(P)}$, cf. [28], III, Proposition 7.

We will also use the Weil pairing. For n a positive integer and P and Q in $E[n]$ the element $e_n(P, Q)$ in $\mu_n(k)$ is defined as follows. Let D_P and D_Q in $\mathrm{Div}^0(E)$ be disjoint divisors representing $\lambda(P)$ and $\lambda(Q)$. Let f and g be in $k(E)^\times$ such that $nD_P = \mathrm{div}(f)$ and $nD_Q = \mathrm{div}(g)$. Then $e_n(P, Q) = f(D_Q)/g(D_P)$. For n invertible in k this pairing e_n is a perfect alternating pairing, see [16], Chapter 12, Remark 3.7.

We assume that φ is an endomorphism of E such that $\alpha := \varphi - \overline{\varphi} \neq 0$. We set

$$D_\varphi := \varphi_*\big((\Delta) - (2\Delta)\big) - \varphi^*\big((\Delta) - (2\Delta)\big) \quad \text{in } \mathrm{Div}^0(E_U). \tag{6.0.1}$$

Note that $(\Delta) - (2\Delta)$ is linearly equivalent to $(0) - (\Delta)$, and that, under $\lambda \colon E \to E^\vee$, Δ in $E(E)$ is mapped to $[(0) - (\Delta)]$. We want the support of D_φ to be disjoint from Δ and 2Δ, and this becomes true by removing from U the kernels of $2(\varphi - 1)$, of $2\varphi - 1$ and of $\varphi - 2$ (as $\overline{\varphi} \neq \varphi$, only a finite set is removed). We can now also view D_φ as element of $\mathrm{Div}^0(C)$, and we set:

$$t^J_\varphi := [D_\varphi] \quad \text{in } G(U). \tag{6.0.2}$$

Combining Parts 2 and 4 of the following theorem provides a new proof in the elliptic case of Proposition 3.3, while Part 3 sharpens Theorem 2.4.

6.1 Theorem *1. The image $\pi(t^J_\varphi)$ of t^J_φ equals $(\alpha, \mathrm{id}_U) \colon U \to E \times U = E_U$.*

2. *Let n be a positive integer and x in E with $nx = 0$. Then $n^2 t^J_\varphi(x) = 0$ in G_x, and $nt^J_\varphi(x) = e_n(\varphi(x), x)$.*

3. *Let n be a positive odd integer that is prime to $\deg(\alpha)$, invertible in k, and that divides none among $\deg(2(\varphi-1))$, $\deg(2\varphi-1)$ and $\deg(\varphi-2)$. Then there is an $x \in U$ of order n, such that the order of $t^J_\varphi(x)$ is equal to n^2.*

4. *The extension G of E_U by $\mathbb{G}_{\mathrm{m}U}$ is uniquely isomorphic to the restriction to E_U of the Poincaré torsor $\mathcal{P}$ as in Section 2 (up to a switch of the factors of $E \times E$), and under this isomorphism, t^J_φ equals the Ribet section t_φ.*

Proof We prove part 1. The image $\pi(t^J_\varphi)$ in $E_U(U)$ of t^J_φ is the class of the divisor D_φ on E_U, hence we have, denoting by $\simeq$ linear equivalence on $\mathrm{Div}^0(E_U)$:

$$\begin{aligned} D_\varphi &\simeq \varphi_*\big((\Delta) - (2\Delta)\big) - \overline{\varphi}_*\big((\Delta) - (2\Delta)\big) \\ &= \big((\varphi(\Delta)) - (2\varphi(\Delta))\big) - \big((\overline{\varphi}(\Delta)) - (2\overline{\varphi}(\Delta))\big) \\ &\simeq \big((0) - (\varphi(\Delta))\big) - \big((0) - (\overline{\varphi}(\Delta))\big) \\ &\simeq \big((0) - ((\varphi - \overline{\varphi})(\Delta))\big) = \big((0) - (\alpha(\Delta))\big). \end{aligned}$$

Under the principal polarisation $\lambda\colon E \to E^\vee$, $x \mapsto [(0)-(x)]$, this corresponds to $\alpha(\Delta)$ in $E(U)$. This proof of part 1 is finished.

We prove part 2. So, let n be a positive integer, and let $x \in U$ be a point such that $nx = 0$ in E. As $nx = 0$, we have $n\pi t^J_\varphi(x) = n\alpha(x) = 0$ in E. This means that $nD_{\varphi,x}$ is a principal divisor on E. Let $f \in k(E)^\times$ be such that $\mathrm{div}(f) = n(x) - n(2x)$ in $\mathrm{Div}(E)$. Then we have, on E:

$$\begin{aligned}\mathrm{div}(f\circ\varphi) &= \varphi^*\mathrm{div}(f) = \varphi^*\left(n(x) - n(2x)\right),\\ \mathrm{div}(\mathrm{Norm}_\varphi(f)) &= \varphi_*\mathrm{div}(f) = \varphi_*\left(n(x) - n(2x)\right).\end{aligned}$$

We define:

$$g_\varphi := \mathrm{Norm}_\varphi(f)/(f\circ\varphi) \quad \text{in } k(E)^\times.$$

Then we have:

$$nD_{\varphi,x} = \mathrm{div}(\mathrm{Norm}_\varphi(f)) - \mathrm{div}(f\circ\varphi) = \mathrm{div}(g_\varphi) \quad \text{on } E.$$

This means that $nt^J_\varphi(x)$ in G_x is the element $g_\varphi(x)/g_\varphi(2x)$ of $k^\times$. By the construction of U, the divisor of f has support disjoint from that of g_φ and of $\varphi^*\mathrm{div}(f)$ and $\varphi_*\mathrm{div}(f)$, and Weil reciprocity gives us:

$$\begin{aligned}\left(\frac{g_\varphi(x)}{g_\varphi(2x)}\right)^n &= g_\varphi(\mathrm{div}(f)) = f(\mathrm{div}(g_\varphi)) = f(\mathrm{div}(\mathrm{Norm}_\varphi(f)) - \mathrm{div}(f\circ\varphi))\\ &= \frac{f(\mathrm{div}(\mathrm{Norm}_\varphi(f)))}{f(\mathrm{div}(f\circ\varphi))} = \frac{f(\varphi_*\mathrm{div}(f))}{(f\circ\varphi)(\mathrm{div}(f))} = \frac{f(\varphi_*\mathrm{div}(f))}{f(\varphi_*\mathrm{div}(f))} = 1.\end{aligned}$$

So, $n^2t^J_\varphi(x)$ is indeed equal to 0 in G_x. Let us also prove that $nt^J_\varphi(x) = e_n(\varphi(x), x)$. We have

$$\lambda(x) = [(x) - (2x)] \text{ in } E^\vee, \quad n((x) - (2x)) = \mathrm{div}(f) \text{ in } \mathrm{Div}(E),$$

and

$$\lambda(\varphi(x)) = [\varphi_*(x) - \varphi_*(2x)] \text{ in } E^\vee, \quad n(\varphi_*(x) - \varphi_*(2x)) = \mathrm{div}(\mathrm{Norm}_\varphi(f)) \text{ in } \mathrm{Div}(E).$$

So, by the description above of the Weil pairing,

$$\begin{aligned}e_n(\varphi(x), x) &= \frac{(\mathrm{Norm}_\varphi(f))((x) - (2x))}{f(\varphi_*((x) - (2x)))} = g_\varphi((x) - (2x))\\ &= \frac{g_\varphi(x)}{g_\varphi(2x)} = nt^J_\varphi \quad \text{in } k^\times.\end{aligned}$$

We prove part 3. Let n be a positive odd integer that is prime to $\deg(\alpha)$, invertible in k, and that divides none among $\deg(2(\varphi-1))$, $\deg(2\varphi-1)$ and $\deg(\varphi-2)$. To prove that there is a x in U such that the order of x is n and the order of $t^J_\varphi(x)$ is n^2, it is sufficient to show that there is an x in U of order n such that $e_n(\varphi(x), x)$ is of order n. As n does not divide $\deg(2(\varphi-1))$, $\deg(2\varphi-1)$, and $\deg(\varphi-2)$, each x in E of order n is in U.

Let now p be a prime number dividing n. Then p is odd, and p is invertible in k, hence $E[p]$ is of dimension two as $\mathbb{F}_p$-vector space, with the symmetric bilinear form

$$E[p] \times E[p] \longrightarrow k^\times, \quad (x, y) \mapsto e_p(\alpha(x), y).$$

As p does not divide $\deg(\alpha)$, this form is perfect. Therefore, there is an x_p in $E[p]$ such that $e_p(\alpha(x_p), x_p)$ is of order p. As $e_p(\alpha(x_p), x_p) = e_p(\varphi(x_p), x_p)^2$, $e_p(\varphi(x_p), x_p)$ is then also of order p. Let n_p be the exponent of p in the factorisation of n, and $x'_p \in E$ such that $x_p = p^{n_p-1}x'_p$, then x'_p is in $E[n]$, and the order of $e_n(\varphi(x'_p), x'_p)$ is p^{n_p}.

Taking for x the sum of the x'_p for p dividing n gives an x as desired. We have now finished the proof of part 3.

We prove part 4. The two families of extensions of E by $\mathbb{G}_{\mathrm{m}}$ are fibrewise isomorphic by construction, hence there is a unique isomorphism of extensions between them as $\mathrm{Hom}(E, \mathbb{G}_{\mathrm{m}})$ is trivial. The sections t_φ and t_φ^J lie above the graph of $\alpha\colon E \to E$. We will show that t_φ^J extends from U to E, and that $t_\varphi(0) = t_\varphi^J(0)$. Then there is a unique $c \in k^\times$ such that $t_\varphi^J = ct_\varphi$, and the c equals 1 because of the values at 0.

We show that t_φ^J extends from U to E by viewing as explained above, for $T \to U$, $\mathrm{Pic}(C_T)$ as the group of isomorphism classes of $(\mathcal{L}, \sigma)$, with $\mathcal{L}$ an invertible $\mathcal{O}$-module on E_T and $\sigma\colon \Delta^*\mathcal{L} \to (2\Delta)^*\mathcal{L}$ an isomorphism of $\mathcal{O}$-modules on T. This description extends as such to all $T \to E$, hence gives us an extension over all of E of the extension G of E_U by $\mathbb{G}_{\mathrm{m}U}$. Now we show that t_φ^J extends over E. It suffices to take $T = E$, and show that the divisor $\Delta^*(D_\varphi) - (2\Delta)^*(D_\varphi)$ on E is principal, and that the restriction $D_{\varphi,0}$ of D_φ to $E \times \{0\}$ is principal.

Definition (6.0.1) shows that $D_{\varphi,0} = 0$ as divisor. We claim that also $\Delta^*(D_\varphi) - (2\Delta)^*(D_\varphi)$ is zero, as divisor. We give the computation. Let R be any k-algebra. Then

$$\begin{aligned}
(\Delta^*(\varphi_*(\Delta)))(R) &= \{x \in E(R) : \varphi(x) = x\} \\
(\Delta^*(\varphi_*(2\Delta)))(R) &= \{x \in E(R) : 2\varphi(x) = x\} \\
(\Delta^*(\varphi^*(\Delta)))(R) &= \{x \in E(R) : \varphi(x) = x\} \\
(\Delta^*(\varphi^*(2\Delta)))(R) &= \{x \in E(R) : \varphi(x) = 2x\}\,.
\end{aligned}$$

and

$$\begin{aligned}
((2\Delta)^*(\varphi_*(\Delta)))(R) &= \{x \in E(R) : \varphi(x) = 2x\} \\
((2\Delta)^*(\varphi_*(2\Delta)))(R) &= \{x \in E(R) : 2\varphi(x) = 2x\} \\
((2\Delta)^*(\varphi^*(\Delta)))(R) &= \{x \in E(R) : 2\varphi(x) = x\} \\
((2\Delta)^*(\varphi^*(2\Delta)))(R) &= \{x \in E(R) : 2\varphi(x) = 2x\}\,.
\end{aligned}$$

A little bit of bookkeeping shows that the balance is zero. □

References

[1] Y. André, *Mumford-Tate groups and the theorem of the fixed part,* Compositio Math. 82 (1992), 1–24.

[2] D. Bertrand, *Special points and Poincaré bi-extensions,* with an Appendix by Bas Edixhoven. `https://arxiv.org/abs/1104.5178`

[3] D. Bertrand, *Extensions panachées autoduales,* J. K-theory, 11, 2013, 393–411.

[4] D. Bertrand, D. Masser, A. Pillay, U. Zannier, *Relative Manin-Mumford for semi-abelian surfaces,* Proc. Edin. MS 59, 2016, 837–875.
`https://doi.org/10.1017/S0013091515000486`

[5] D. Bertrand, H. Schmidt, *Unlikely intersections in semi-abelian surfaces.*
`https://arxiv.org/abs/1803.04835`

[6] S. Bosch, W. Lütkebohmert, M. Raynaud: *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 21. Springer-Verlag, Berlin, 1990.

[7] L. Breen, *Biextensions alternées,* Compositio Math. 63 (1987), 99–122.

[8] A. Chambert-Loir, *Géométrie d'Arakelov et hauteurs canoniques sur des variétés semi-abéliennes,* Math. Ann. 314, 1999, 381–401.

[9] P. Deligne, *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques.* Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 247–289, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[10] P. Deligne, *Théorie de Hodge. II.* Inst. Hautes Etudes Sci. Publ. Math. No. 40 (1971), 5–57.

[11] P. Deligne, *Théorie de Hodge III,* Inst. Hautes Etudes Sci. Publ. Math. No. 44 (1974), 5–77.

[12] G. Faltings and Ching-Li Chai, *Degeneration of abelian varieties.* With an appendix by David Mumford. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 22. Springer-Verlag, Berlin, 1990. xii+316 pp. ISBN: 3-540-52015-5.

[13] D. Ferrand, *Conducteur, descente et pincement*, Bull. Soc. Math. France 131 (2003), no. 4, 553–585.

[14] Z. Gao, *A special point problem of André-Pink-Zannier in the universal family of Abelian varieties.* Ann. Sc. Norm. Super. Pisa Cl. Sci. 17 (2017), 231–266.

[15] S. Howe: Higher genus counterexamples to relative Manin-Mumford; ALGANT Master Thesis, July 2012. Available at
`http://http://www.math.leidenuniv.nl/scripties/HoweMaster.pdf`

[16] D. Husemoller: *Elliptic curves.* With an appendix by Ruth Lawrence. Graduate Texts in Mathematics, 111. Springer-Verlag, New York, 1987.

[17] O. Jacquinot, K. Ribet, *Deficient points on extensions of abelian varieties by $\mathbb{G}_m$,*; J. Number Th., 25, 1987, 133–151.

[18] B. Klingler, *Hodge loci and atypical intersections: conjectures,*
`https://arxiv.org/abs/1711.09387`

[19] B. Klingler, E. Ullmo and A. Yafaev, *Bi-algebraic geometry and the André–Oort conjecture*, Algebraic geometry: Salt Lake City 2015, 319–359, Proc. Sympos. Pure Math., 97.2, Amer. Math. Soc., Providence, RI, 2018.
`https://webusers.imj-prg.fr/~bruno.klingler/papiers/Survey2.pdf`

[20] M.A. Lopuhaä, *Pink's conjecture on semi-abelian varieties,* MSc. thesis, September 2014, Leiden.
`https://www.universiteitleiden.nl/binaries/content/assets/science/mi/scripties/lopuh`

[21] B. Moonen, *Linearity properties of Shimura varieties. I.* J. Algebraic Geom. 7 (1998), no. 3, 539–567.

[22] L. Moret-Bailly, *Pinceaux de variétés abéliennes.* Astérisque No. 129 (1985).

[23] D. Mumford, *Abelian varieties.* With appendices by C.P. Ramanujam and Yuri Manin. Corrected reprint of the second (1974) edition. Tata Institute of Fundamental Research Studies in Mathematics, 5. Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008. xii+263 pp. ISBN: 978-81-85931-86-9; 81-85931-86-0.

[24] R. Pink, *Arithmetical compactification of mixed Shimura varieties.* Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989. Bonner Mathematische Schriften, 209. Universität Bonn, Mathematisches Institut, Bonn, 1990.
`https://people.math.ethz.ch/~pink/dissertation.html`

[25] R. Pink, *A combination of the conjectures by Mordell-Lang and André-Oort*, In "Geometric methods in algebra and number theory," 251–282, Progr. Math., 235, Birkhäuser Boston, Boston, MA, 2005.

[26] R. Pink, *A common generalization of the conjectures of André–Oort, Manin-Mumford, and Mordell-Lang*, preprint, April 2005,
`https://people.math.ethz.ch/~pink/ftp/AOMMML.pdf`.

[27] K. Ribet, *Cohomological realization of a family of one-motives,* J. Number Th. 25, 1987, 152–161.

[28] J-P. Serre, *Groupes algébriques et corps de classes*, Publications de l'institut de mathématique de l'université de Nancago, VII. Hermann, Paris 1959.

[29] SGA 3.II. Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux. Séminaire de Géométrie Algébrique du Bois Marie 1962/64. Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152 Springer-Verlag, Berlin-New York 1970.

[30] J. Tsimerman, *The André-Oort conjecture for $\mathcal{A}_g$,* Ann. of Math. (2) 187 (2018), 379–390.

[31] U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry,* with appendixes by David Masser. Annals of Mathematics Studies, 181. Princeton University Press, Princeton, NJ, 2012.