Semistable reduction of modular curves associated with maximal subgroups in prime level

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Abstract

We complete the description of semistable models for modular curves associated with maximal subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ (for p any prime, $p > 5$). That is, in the new cases of non-split Cartan modular curves and exceptional subgroups, we identify the irreducible components and singularities of the reduction mod p, and the complete local rings at the singularities. We review the case of split Cartan modular curves. This description suffices for computing the group of connected components of the fibre at p of the Néron model of the Jacobian.

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Contents

$$w_p$$

$$g(X_0(p)) + 1$$
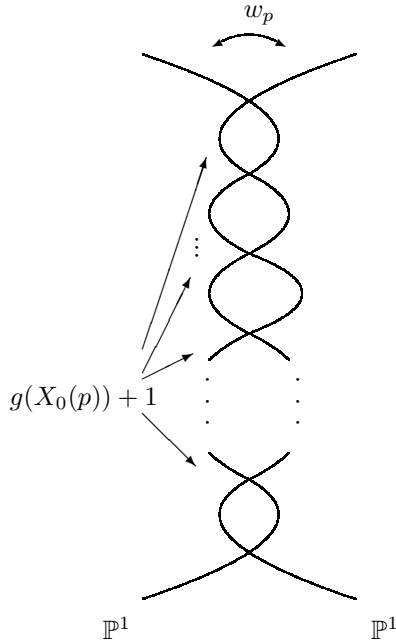
$$\mathbb{P}^1 \qquad \mathbb{P}^1$$

Figure 1: $X_0(p)_{/\overline{\mathbb{F}}_p}$

1 Introduction

Let p be a prime number. The picture given in Figure 1 of the geometric special fiber of the stable model of $X_0(p)$ over $\mathbb{Z}_p$ now looks familiar to many number-theorists[1]. It has been described in the work [5] of Deligne and Rapoport, and was actually known, in a slightly different guise, by Kronecker. Having such a model has proven crucial in many questions – not only for direct applications such as the computation of semistable Néron models of the jacobian $J_0(p)$ but also in diophantine issues, such as the determination of the non-cuspidal rational points of $X_0(p)$ in Mazur's famous works [15] and [17].

It is actually under similar motivations that we describe here a semistable model, over a suitable extension of $\mathbb{Z}_p$, of the modular curve $X_{\mathrm{ns}}^+(p)$ attached to the normaliser of a non-split Cartan subgroup in $\mathrm{GL}_2(\mathbb{F}_p)$. Recently indeed J. Balakrishnan and her coauthors managed to elaborate on the Chabauty-Kim method and prove that the modular curve $X_{\mathrm{ns}}^+(13)$ had only the expected trivial rational points (see [1]). That constituted a tour-de-force, as the latter curve had so far resisted all known methods on Earth. Their strategy needs at some point a bit of knowledge of the reduction type of the curve under study, and that knowledge was available because $X_{\mathrm{ns}}^+(13)$ is isomorphic to $X_{\mathrm{s}}^+(13)$, attached to the normaliser of a split Cartan, see [2], and for that latter curve the necessary information was already available from [7]. For $p > 13$, there is no isomorphism between split and non-split Cartan curves, so our models for $X_{\mathrm{ns}}^+(p)$ shall prove necessary for applying the quadratic Chabauty method of [1] to the latter curves.

A bit more generally, we describe stable models of modular curves associated with all maximal subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$. One classically knows (see e.g. [15]) that those subgroups (up to conjugation) are the Borel, the normalizer of split and non-split Cartan (defining the curves denoted by $X_{\mathrm{s}}^+(p)$ and $X_{\mathrm{ns}}^+(p)$ respectively) and some exceptional subgroups, which are lifts of the permutation groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$ in $\mathrm{PGL}_2(\mathbb{F}_p)$. Among those, the curve $X_{\mathrm{s}}^+(p)$, under its avatar $X_0(p^2)/w_{p^2}$, had already been treated in the article [7] by the first-named author of the present work (see also [8] and [9]). We shall recall the computations in Section 4 below. The newest part of the present study however is a complete description of fibre at p of the stable model for the non-split Cartan curve $X_{\mathrm{ns}}^+(p)$ and the "widths" of its singularities (cf. Section 3).

The case of exceptional groups is probably of lesser interest. From the diophantine point of view, for instance, Serre remarked that a simple argument on the monodromy of mod p Galois rep-

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[1]To be completely correct, when $p \leq 19$ that model is only semistable.

resentations attached to elliptic curves show that the modular curves associated with exceptional groups have no local points (with values in p-adic fields) as soon as p is large enough, cf. [16], p. 118. We however compute semistable models for those modular curves in Section 5.

Our method is first to describe stable models for the curves $\overline{\mathcal{M}}(\Gamma(p), \mathcal{P})$ associated with the full level-p structure, enhanced by some additional (finite, étale, representable) moduli problem $\mathcal{P}$ over $\mathbb{Z}_p$. This is what we do in Section 2, essentially following the unpublished [9]. We then take quotients by relevant subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$, starting with the normalizer of non-split Cartan. The fact that we added a level structure $\mathcal{P}$ allows us to keep working with a fine moduli space. Finally we assume $\mathcal{P}$ is Galois with group G, and taking the quotient by G yields a stable model for the coarse curve $X_{\mathrm{ns}}^+(p)$. We repeat that process for the split-Cartan curve and the exceptional subgroups.

We must mention that this approach is not well-suited to deal with cases of level divided by powers p^r of p, when $r \geq 2$, because of algebro-geometric reasons recalled in Remark 2.4. In that situation, fortunately, rigid-analytic methods can prove much more efficient, and allowed J. Weinstein to compute semistable models for $X(p^n)$ for arbitrary n (see [20]). It however seems that those techniques do not provide at the moment the widths of the singularities of the stable models.

A last word about stability versus semistability: as for the model of $X_0(p)$ recalled in Figure 1, our semistable models will actually be stable, for large enough p, in many cases but not all. The curves $X_{\mathrm{s}}^+(p)$ and $X_{\mathrm{ns}}^+(p)$, for any p which is $-1 \bmod 4$, are indeed not stable, as explained in Theorems 3.5, 4.4, and Remark 3.6. In all cases however it is easy to spot what projective lines need to be contracted in order to obtain a stable model. About that issue, see Remark 4.5.

2 Stable model for full level p structure

2.1 Katz-Mazur's model $\overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))$

Our starting point will be the modular model over $\mathbb{Z}[\zeta_p]$, as given by Katz and Mazur ([11]; Chapter 13), for modular curves with full level p structure plus some additional level structure $\mathcal{P}$ with nice properties at p. Let us very briefly recall Katz-Mazur "Drinfeldian approach" to moduli problems. We will not discuss stable models for the curves $X(p)_{\mathbb{Q}}$ with no additional level structure.

We let $\mathcal{P}$ be a representable finite étale moduli problem over $(\mathrm{Ell})_{\mathbb{Z}_p}$. One can take for instance $\mathcal{P} = [\Gamma_1(N)]$ for $N \geq 4$ a prime-to-p integer. Later, when we want to get rid of $\mathcal{P}$, we will assume moreover that $\mathcal{P}$ is Galois over $(\mathrm{Ell})_{\mathbb{Z}_p}$.

There was a time when for N any positive integer, $\Gamma(N)$ denoted the kernel of the reduction morphism $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. But since [5] it became clear that it was better to attach modular curves to compact open subgroups of the finite adèle group $\mathrm{GL}_2(\mathbb{Q} \otimes \hat{\mathbb{Z}})$. So, we let $\Gamma(N)$ denote the kernel of the surjective morphism $\mathrm{GL}_2(\hat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. Following [11], if E/S is an elliptic curve over an arbitrary scheme S, we say that a group morphism $\phi \colon (\mathbb{Z}/N\mathbb{Z})^2 \to E(S)$ is a $\Gamma(N)$-structure (or "full level-N structure") if the effective Cartier divisor

$$D_N := \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^2} [\phi(a)]$$

is a group scheme which is equal to $E[N]$. The ordered pair $(P := \phi(1,0), Q := \phi(0,1))$ is then said to be a *Drinfeld basis* of $E[N]$. The set of $\Gamma(N)$-structures on E/S is denoted $[\Gamma(N)](E/S)$.

Of course when N is invertible on S, this notion of level-N structure brings nothing new to the naive usual definition. On the other hand, over a field k of positive characteristic p, a Drinfeld basis of $E[p]$ is easily seen to be a pair (P, Q) such that at least one of the two points has order p, in the usual sense, in $E(k)$, at least if E is ordinary (and the only possible $(0,0)$ if E is supersingular).

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Let us fix from now on some prime number p. If S is an $\mathbb{F}_p$-scheme, and n any non-negative integer, let $F_{E/S}^n \colon E \to E^{(p^n)}$ denote the nth-power of the relative Frobenius, and $V_{E/S}^n \colon E^{(p^n)} \to E$ the nth-power of the Verschiebung, that is, the dual isogeny to $F_{E/S}^n$.

One knows that $F_{E/S}^n$ is purely radicial, and $V_{E/S}^n$ is étale exactly when E/S is ordinary. In any case both isogenies are cyclic with order p^n, that is, after a suitable surjective finite locally free base change, there is a group morphism $\phi \colon (\mathbb{Z}/p^n\mathbb{Z}) \to E(S)$ such that their kernel is equal, as effective Cartier divisor, to $\sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})} [\phi(a)]$. An *Igusa structure of level p^n* on E/S is the datum of some point P in $E^{(p^n)}[p^n](S)$ such that the equality

$$\operatorname{Ker}(V_{E/S}^n) = \sum_{a \in (\mathbb{Z}/p^n\mathbb{Z})} [aP]$$

between effective Cartier divisors holds. The associated moduli problem on the elliptic stack $(\mathrm{Ell})_{\mathbb{F}_p}$ is denoted by $[\mathrm{Ig}(p^n)]$. Igusa proved that $[\mathrm{Ig}(p^n)]$ is relatively representable: there is a complete smooth curve $\overline{\mathcal{M}}(\mathcal{P}, [\mathrm{Ig}(p^n)])$ over $\mathbb{F}_p$ such that the complement of the cusps $\mathcal{M}(\mathcal{P}, [\mathrm{Ig}(p^n)])$ represents $(\mathcal{P}, [\mathrm{Ig}(p^n)])$. To state Katz-Mazur's central result in the simplest way, we shall actually work with so-called *exotic Igusa structures*. So - restricting ourselves to level p - we define the moduli problem:

$$(E/S/\mathbb{F}_p) \mapsto [\mathrm{ExIg}(p,1)](E/S) := \{P \in E(S), (0,P) \text{ is a Drinfeld } p-\text{basis of } E/S\}$$

and we then can check there is an (exotic) isomorphism

$$[\mathrm{Ig}(p)] \xrightarrow{\sim} [\mathrm{ExIg}(p,1)], \quad (E/S, P \in [\mathrm{Ig}(p)](E/S)) \mapsto (E^{(p)}/S, (0,P)).$$

The moduli problem $(\mathcal{P}, [\Gamma(p)])$ classifies triples $(E/S, \alpha, \phi)$ for S a $\mathbb{Z}_p$-scheme, E/S an elliptic curve, $\alpha \in \mathcal{P}(E/S)$, and $\phi \in [\Gamma(p)](E/S)$. Katz-Mazur's theorems about $\Gamma(p)$-structures ([11], Theorems 3.6.0, 5.1.1 and 10.9.1) then assert that $(\mathcal{P}, [\Gamma(p)])$ is representable by a regular $\mathbb{Z}_p$-scheme $\mathcal{M}(\mathcal{P}, [\Gamma(p)])$, that has a compactification $\overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])$ which enjoys the following properties. Weil's pairing $e_p(\cdot, \cdot)$ shows that the morphism $\overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)]) \to \operatorname{Spec}(\mathbb{Z}_p)$ factorizes through $\operatorname{Spec}(\mathbb{Z}_p[\zeta_p])$, with $\mathbb{Z}_p[\zeta_p] := \mathbb{Z}_p[x]/(x^{p-1} + \cdots + x + 1)$. For all integers i in $\{1, \ldots, p-1\}$, set

$$X_i := \overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)^{\zeta_p^i - \mathrm{can}}])$$

for the sub-moduli problem over $(\mathrm{Ell})_{\mathbb{Z}_p[\zeta_p]}$ representing triples $(E/S/\mathbb{Z}_p[\zeta_p], \alpha, \phi)$ such that

$$e_p(\phi(1,0), \phi(0,1)) = \zeta_p^i.$$

The obvious morphism:

$$\coprod_i X_i \to \overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])_{\mathbb{Z}_p[\zeta_p]}$$

induces, by normalization, an isomorphism of schemes over $\mathbb{Z}_p[\zeta_p]$:

$$\coprod_i X_i \xrightarrow{\sim} \overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])_{\mathbb{Z}_p[\zeta_p]}^{\sim}, \tag{1}$$

with $\overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])_{\mathbb{Z}_p[\zeta_p]}^{\sim} \to \overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])_{\mathbb{Z}_p[\zeta_p]}$ the normalization. The triviality of pth-roots of unity in characteristic p shows that, after the base change $\mathbb{Z}_p[\zeta_p] \to \mathbb{F}_p$, the $X_{i,\mathbb{F}_p}$ are not only isomorphic to each other but actually equal. Moreover, the modular interpretation of a $\Gamma(p)$-structure $\phi \colon (\mathbb{Z}/p\mathbb{Z})^2 \to E(k)$, in the generic case of an ordinary elliptic curve E over a field k of characteristic p, amounts to choosing some line L in $(\mathbb{Z}/p\mathbb{Z})^2$ that plays the role of $\operatorname{Ker}(\phi)$, then some point P in $E(k)$ which defines the induced isomorphism $(\mathbb{Z}/p\mathbb{Z})^2/L \xrightarrow{\sim} E[p](k)$. Making that into a proof, Katz and Mazur give the following theorem.
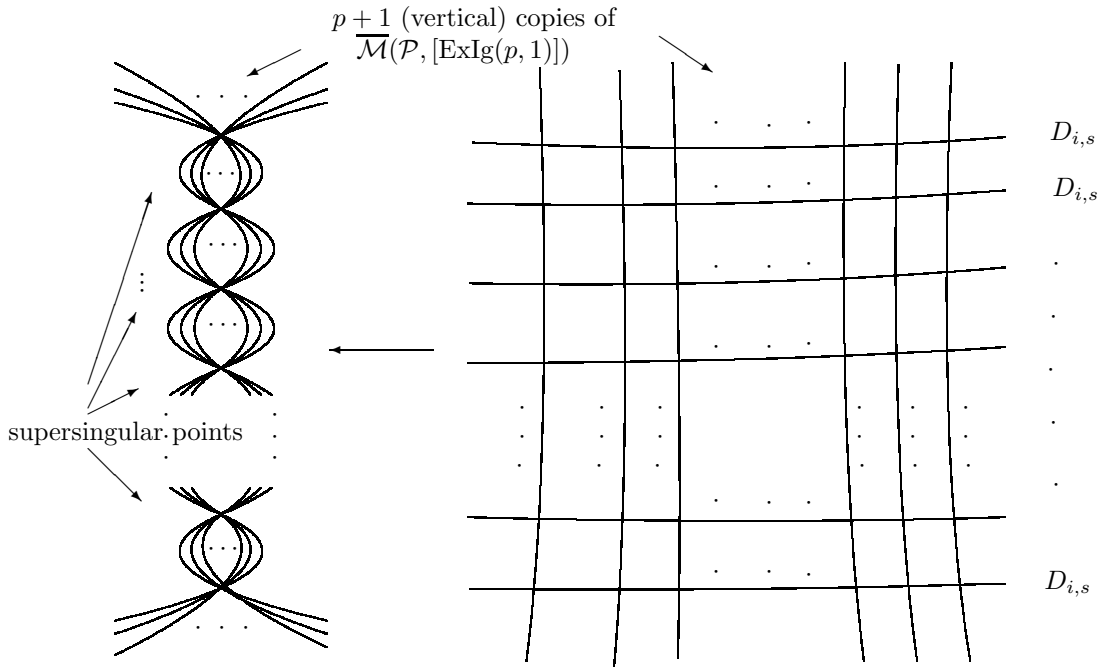
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$p+1$ (vertical) copies of $\overline{\mathcal{M}}(\mathcal{P}, [\mathrm{ExIg}(p,1)])$

supersingular points

$D_{i,s}$

$D_{i,s}$

$D_{i,s}$

Figure 2: Katz-Mazur and stable fibers of X_i

Theorem 2.1 (Katz-Mazur [11], 13.7.6). *Each curve $X_{i,\mathbb{F}_p}$ obtained from X_i over $\mathbb{Z}_p[\zeta_p]$ via $\mathbb{Z}_p[\zeta_p] \to \mathbb{F}_p$, is the disjoint union, with crossings at the supersingular points, of $p+1$ copies of the $\overline{\mathcal{M}}(\mathcal{P})$-schemes $\overline{\mathcal{M}}(\mathcal{P}, [\mathrm{ExIg}(p,1)])$ (cf. Figure 2). We label those Igusa schemes as $\mathrm{Ig}_{i,L}$ for (i,L) running through $\mathbb{F}_p^\times \times \mathbb{P}^1(\mathbb{F}_p)$.*

Remark 2.2 One would like to think of the copies of the scheme $\overline{\mathcal{M}}(\mathcal{P}, [\mathrm{ExIg}(p,1)])$ as the "components" of $X_{i,\mathbb{F}_p}$, which is morally true - note however that they may not be geometrically irreducible (being such exactly when $\overline{\mathcal{M}}(\mathcal{P})$ is). The same administrative issue will show up in our subsequent models. It of course vanishes when we eventually get rid of the auxiliary level-$\mathcal{P}$ structure, as in the coarse curves $X_{\mathrm{ns}}(p)$, $X_{\mathrm{ns}}^+(p)$, and so on below.

The situation at the supersingular points can be described as follows. Let x be a point of $X_{i,\mathbb{F}_p}$ whose underlying elliptic curve E_0 is supersingular, and let k be the residue field of x. Then x is a triple $(E_0/k, \alpha_0, \phi_0)$ with $\alpha_0 \in \mathcal{P}(E_0/k)$ and $\phi_0 \in [\Gamma(p)^{\mathrm{can}}](E_0/k)$; note that $\phi_0(1,0)$ and $\phi_0(0,1)$ are both 0, as E_0 is supersingular. Let R be the completion of the local ring of $X_{i,\mathbb{F}_p}$ at x.

By construction, R is the universal formal deformation ring of (E_0, α_0, ϕ_0) to Artin local k-algebras. That is, restricting the universal triple over $\mathcal{M}(\mathcal{P}, \Gamma(p)^{\mathrm{can}})_k$ to R gives the Cartesian diagram

$$
\begin{array}{ccc}
(E_{0,k}, \alpha_0, \phi_0) & \longrightarrow & (\mathbb{E}_R, \alpha^{\mathrm{univ}}, \phi^{\mathrm{univ}}) \\
\downarrow & \square & \downarrow \\
\mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(R)\,.
\end{array}
$$

This diagram has the property that for every Artin local k-algebra A with residue field k, every $(E/A, \alpha, \phi)$, and every Cartesian diagram

$$
\begin{array}{ccc}
(E_{0,k}, \alpha_0, \phi_0) & \longrightarrow & (E, \alpha, \phi) \\
\downarrow & \square & \downarrow \\
\mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A)
\end{array}
$$

there are unique dashed maps

$$
\begin{array}{ccccc}
(E_{0,k}, \alpha_0, \phi_0) & \longrightarrow & (E, \alpha, \phi) & \dashrightarrow & (\mathbb{E}_R, \alpha^{\mathrm{univ}}, \phi^{\mathrm{univ}}) \\
\downarrow & & \downarrow & \square & \downarrow \\
\mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(A) & \dashrightarrow & \mathrm{Spec}(R)
\end{array}
$$

that make the diagram commutative, and the right square Cartesian.

In order to get a useful description of R, let $\mathbb{E}/k[[t]]$ be a universal deformation of E_0 to Artin local k-algebras with residue field k (see Section 2.2.2 for some explicit ones). As $\mathbb{E}_R$ is a deformation of E_0 over R, we have a unique Cartesian diagram

$$
\begin{array}{ccccc}
E_{0,k} & \longrightarrow & \mathbb{E} & \dashrightarrow & \mathbb{E}_R \\
\downarrow & & \downarrow & \square & \downarrow \\
\mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k[[t]]) & \dashrightarrow & \mathrm{Spec}(R) \, .
\end{array}
$$

As $\mathcal{P}$ is étale over $(\mathrm{Ell})_{\mathbb{Z}_p}$, α_0 lifts uniquely to every deformation of E_0. Therefore, the connected component of $(\mathcal{P}, [\Gamma(p)^{\mathrm{can}}])_{\mathbb{E}/k[[t]]}$ containing x is equal to the base change of $[\Gamma(p)^{\mathrm{can}}]_{\mathbb{E}/k[[t]]}$ via $k \to k'$, and hence

$$
\mathrm{Spec}(R) = [\Gamma(p)^{\mathrm{can}}]_{\mathbb{E}/k[[t]]} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k') \, ,
$$

that is, $\mathrm{Spec}(R)$ is the $k'[[t]]$-scheme representing all $\Gamma(p)^{\mathrm{can}}$-structures on $\mathbb{E}_{k'[[t]]}/k'[[t]]$. Being this, R is a $k'[[t]]$-algebra, free of rank $\#\mathrm{SL}_2(\mathbb{F}_p)$ as $k'[[t]]$-module.

Let Z be a parameter of the formal group of $\mathbb{E}/k[[t]]$. Then, as E/k is supersingular, $\phi^{\mathrm{univ}}(1,0)$ and $\phi^{\mathrm{univ}}(0,1)$ in $\mathbb{E}(R)$ are two points of that formal group. We write

$$
x = Z(\phi(1,0)) \in R, \quad y = Z(\phi(0,1)) \in R
$$

for their respective parameters. Katz and Mazur prove in [11, §5.4] that x and y generate the maximal ideal of R, hence that R is a quotient of the formal power series ring $k'[[x,y]]$ whose x and y map to x, resp. y in R. The fact that $X_{i,\mathbb{F}_p}$ is the union of the $\mathrm{Ig}_{i,L}$ for L running through $\mathbb{P}^1(\mathbb{F}_p)$ means that the kernel of $k'[[x,y]] \to R$ is generated by the product of equations of the $\mathrm{Ig}_{i,L}$. Now the condition that ϕ^{univ} defines a point on $\mathrm{Ig}_{i,L}$ is

$$
\begin{cases}
\phi^{\mathrm{univ}}(1,0) = a \cdot \phi^{\mathrm{univ}}(0,1) & \text{if } L = \mathbb{F}_p \cdot (1,-a) \\
\phi^{\mathrm{univ}}(0,1) = 0 & \text{if } L = \mathbb{F}_p \cdot (0,1)
\end{cases}
$$

which translates on the formal group, for $\tilde{a} \in \mathbb{Z}_p$ any lift of $a \in \mathbb{F}_p$, as

$$
\begin{cases}
x = [\tilde{a}](y) = ay + \text{higher degree terms in } y & \text{if } L = \mathbb{F}_p \cdot (1,-a) \\
y = 0 & \text{if } L = \mathbb{F}_p \cdot (0,1).
\end{cases}
$$

The equation f in $k'[[x,y]]$ mod $(x,y)^{p+2}$ is therefore $y \prod_{a \in \mathbb{F}_p}(x - ay) = x^p y - x y^p$. The regularity of X_i at x, plus [11, Thm. 13.8.4] give the following.

Theorem 2.3 (Katz-Mazur) *The complete local ring of X_i at a supersingular point* x *is isomorphic to*

$$
W(k')[\zeta_p][[x,y]]/(x^p y - x y^p + g + (1 - \zeta_p)f_1) \, ,
$$

with k' the residue field at x*, g in the ideal $(x,y)^{p+2}$ and f_1 a unit of $W(k')[\zeta_p][[x,y]]$.*

2.2 The stable model

We can now describe how to compute the (semi)stable model of $\overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])$ over "the" completely ramified degree-$(p^2 - 1)$ extension of $\mathbb{Z}_p$ as in [9]. First we recall a general tool for explicitly computing semistable model of curves in tame situations. Let S be the spectrum of some discrete valuation ring, whose generic and closed point we denote by η and s respectively. Let $\mathcal{C} \to S$ be a curve, that is, a S-scheme purely of relative dimension 1. Assume $\mathcal{C}$ is proper and flat over S, that $\mathcal{C}$ is regular, and $\mathcal{C}_\eta := C \to S_\eta$ is smooth. After sufficiently many blow-ups in closed singular points of $\mathcal{C}$ we can assume $\mathcal{C}_s$ is a Cartier divisor on $\mathcal{C}$ with normal crossings. Write n for the least common multiple of the multiplicities of irreducible components of $\mathcal{C}$ and set $T := S[\pi_0^{1/n}]$, for π_0 some uniformizer on S. Let $\widetilde{\mathcal{C}_T}$ be the normalization of the base change $\mathcal{C} \times_S T \to T$. Then, assuming n is prime to p, one knows that $\widetilde{\mathcal{C}_{T/T}}$ is a semistable curve.

Remark 2.4 • From our semistable model it is not hard to obtain a stable one via appropriate contractions.

- In the case of modular curves, the hypothesis that n be prime to p is *not* satisfied when the level is divisible by p^2. For those more difficult cases rigid analytic methods are more succesful, as shown in the work of Weinstein ([20]; see also references in the Introduction of loc. cit.).

We apply the above to compute a semistable model for $\overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])$. Actually, for p not too small, that model will happen to be even *stable*. Starting from the regular curve $\overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])_{\widetilde{\mathbb{Z}_p[\zeta_p]}}$ over $\mathbb{Z}_p[\zeta_p]$, equal to $\coprod_i X_i$ by (1), we sum-up the algorithm we follow:

(a) blow-up singular points in the closed fiber until having normal crossings;

(b) provided the l.c.m. n of multiplicities of components is prime to p (which will be the case for us), base-change to "the" purely ramified-at-p extension of $\mathbb{Q}_p$ of degree n and

(c) normalize; we denote the result by $\overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])^{\mathrm{st}}$.

It is clear from that construction and Theorem 2.1 that the special fiber of our semistable model $\overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])^{\mathrm{st}}$ over $\mathbb{Z}[(1 - \zeta_p)^{1/p+1}]$ will have two types of irreducible components: the "vertical ones", obtain by simple base change from the components of Katz-Mazur model, and the "horizontal ones", which contract to supersingular points in that model. The former vertical components, which are copies of the $\overline{\mathcal{M}}(\mathcal{P}, [\mathrm{ExIg}(p, 1)])$, will be called *Igusa parts*. The latter horizontal ones will be referred to as *Drinfeld components* and computed in next section.

2.2.1 Drinfeld components

We know from Theorem 2.3 that the complete local ring of X_i at some singular point s is $W(k)[\zeta_p][[x, y]]/(f)$, for k the residue field of s, and $f = x^p y - x y^p + g + (1 - \zeta_p)f_1$, with g in the ideal $(x, y)^{p+2}$ and f_1 a unit of $W(k)[\zeta_p][[x, y]]$. The completion along the exceptional divisor of the blow up of X_i in s is therefore covered by two affine open $\mathrm{Spf}(A_1)$ and $\mathrm{Spf}(A_2)$, with $A_1 = W(k)[\zeta_p][v][[x]]/f(x, vx)$ and $A_2 = W(k)[\zeta_p][u][[y]]/f(ux, y)$ (see [7], 1.3.1). So here

$$A_1 = W(k)[\zeta_p][v][[x]]/(x^{p+1}v(1 - v^{p-1}) + g + (1 - \zeta_p)f_1(x, xv))$$

which shows that the exceptional divisor in $\mathrm{Spf}(A_1)$ has multiplicity $p + 1$, and same with A_2. So we extend the base ring $W(k)[\zeta_p]$ to $W(k)[\pi]$ with

$$\pi := (1 - \zeta_p)^{1/(p+1)} \tag{2}$$

so that, writing $g(x, vx) = x^{p+2}h$,

$$A_1 \otimes_{W(k)[\zeta_p]} W(k)[\pi] = W(k)[\pi][v][[x]]/(x^{p+1}v(1 - v^{p-1}) + x^{p+2}h + \pi^{p+1}f_1(x, xv))$$

and, to normalize it, we blow up at (x, π). This means we set $\pi = xw$ and

$$A_1 \tilde{\otimes}_{W(k)[\zeta_p]} W(k)[\pi] = W(k)[\pi][v][[x]][w]/(\pi - xv, (v - v^p) + xh + w^{p+1} f_1(x, xv)).$$

The corresponding affine part of the exceptional divisor $D_{i,s}$ above s is given by $x = 0$, so that $D_{i,s}$ has an affine model with equation

$$aw^{p+1} = v^p - v \tag{3}$$

for $a \in k^*$ the image of $f_1(0, 0)$. One could possibly determine that a but we will content ourselves in that paper with geometric models so we will henceforth assume $a = -1$. Putting $\alpha = 1/w$ and $\beta = v/w$ gives the other model

$$\alpha^p \beta - \alpha \beta^p = 1. \tag{4}$$

Note the singularities of our model have width 1.

Keeping track of our parameters, we register that

$$\alpha = w^{-1} = \pi^{-1} x \quad \text{and} \quad \beta = w^{-1} v = \pi^{-1} vx = \pi^{-1} y. \tag{5}$$

2.2.2 Points with exceptional automorphisms

In order to compute stable models for level structures defining non-rigid moduli problems, that is, to compute stable models for coarse moduli spaces, we shall consider quotients of the above stable models $\overline{\mathcal{M}}(\mathcal{P}, [\Gamma(p)])^{\mathrm{st}}$ by relevant subgroups $H \subseteq \mathrm{GL}_2(\mathbb{F}_p)$, such as $H = \Gamma_{\mathrm{s}}(p)$, $\Gamma_{\mathrm{ns}}(p)$, $\Gamma_{\mathrm{s}}^+(p)$ or $\Gamma_{\mathrm{ns}}^+(p)$, the split or non-split Cartan subgroup, or their respective normalizers. Then to get rid of the rigidifying level structure $\mathcal{P}$, we shall assume it is representable, finite étale over $(\mathrm{Ell})_{\mathbb{Z}_p}$, and Galois of group G; finally we take the quotient of our $\overline{\mathcal{M}}(\mathcal{P}, [H])^{\mathrm{st}}$ by the action of G. To describe the local situation above singular points of $\overline{\mathcal{M}}(\mathcal{P}, H^{\zeta_p - \mathrm{can}})_{\mathbb{Z}_p[\zeta_p]}$ with extra automorphisms, we however need to describe the action of those automorphisms on the relevant deformation rings.

So let E_0 be a supersingular elliptic curve over $k := \mathbb{F}_{p^2}$, such that $\mathrm{Aut}_k(E_0)$ is cyclic of order 4 $(j = 1728)$ or 6 $(j = 0)$. Let x be a point of $\mathcal{M}(\mathcal{P}, \Gamma(p)^{\mathrm{can}})_k$ whose underlying elliptic curve is E_0, and let k' be its residue field. Then x is a triple $(E_{0,k'}, \alpha_0, \phi_0)$ with α_0 in $\mathcal{P}(E_{0,k'}/k')$ and ϕ_0 the unique (trivial) element of $[\Gamma(p)^{\mathrm{can}}](E_{0,k'}/k')$. Let R be the completion of the local ring of $\mathcal{M}(\mathcal{P}, \Gamma(p)^{\mathrm{can}})_k$ at x. In order to get a useful description of R, we first give a universal deformation of E_0 to Artin local k-algebras with residue field k.

If $j = 1728$, one can check (cf. [7], 1.3.2) that the elliptic curve $\mathbb{E}$ over $k[[t]]$ given by the Weierstrass equation

$$Y^2 = X^3 - X + t \tag{6}$$

is universal. (Indeed, it is well-known that one can choose for E_0 an equation of shape $Y^2 = X^3 - X$. Any deformation of E_0 to an Artinian local k-algebra A with residue field k and maximal ideal m can then be given an equation $Y^2 = X^3 + aX + b$, with $a + 1$ and b in m. (Recall that $p > 3$.) Now one can write $a = -c^4$ for $c \in A$ congruent to 1 mod m. Replacing the variables X and Y by $c^{-2}X$ and $c^{-3}Y$ respectively gives the desired model for $\mathbb{E}$.) The action of a generator i of $\mu_4(k) = \mathrm{Aut}_k(E_0)$ (via action on tangent space at 0) is given by:

$$[i] \colon X \mapsto -X, \quad Y \mapsto iY, \quad t \mapsto -t.$$

In the case $j = 0$ one similarly sees that a model for $\mathbb{E}$ over $k[[t]]$ is given by the Weierstrass equation

$$Y^2 = X^3 + tX - 1 \tag{7}$$

with automorphism action given by:

$$[\zeta] \colon X \mapsto \zeta^{-2}X, \quad Y \mapsto -Y, \quad t \mapsto \zeta^2 t$$

for ζ some generator of $\mu_6(k) = \mathrm{Aut}_k(E_0)$ (again, identification via the action on the tangent space at 0).

As $\mathcal{P}$ is étale over $(\mathrm{Ell})_{\mathbb{Z}_p}$, α_0 lifts uniquely to every deformation of E_0. Therefore,

$$\mathrm{Spec}(R) = [\Gamma(p)^{\mathrm{can}}]_{\mathbb{E}/k[[t]]} \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k')\,.$$

To arrive at the description of R by Katz-Mazur in Theorem 2.3, we choose X/Y as parameter Z of the formal group of $\mathbb{E}/k[[t]]$, with X and Y the functions of the Weierstrass model above. Our description of the action of $[i]$ and $[\zeta]$ shows that $[i]\colon Z \mapsto iZ$, and $[\zeta]\colon Z \mapsto \zeta Z$. Therefore as $\phi(1,0)$ and $\phi(0,1)$ constitute our Drinfeld basis of $\mathbb{E}[p]$, we have $x = Z(\phi(1,0))$ and $y = Z(\phi(0,1)) \in R$ for their respective parameters as in Theorem 2.3 so that $[i]$ maps them to ix and iy respectively, and $[\zeta]$ maps them to ζx and ζy. It means the parameters α and β of equations (5) are mapped to $i\alpha$ and $i\beta$ respectively, and similarly to $\zeta\alpha$ and $\zeta\beta$. (One immediately checks that equation (4) is of course preserved because $p+1$ is divisible by the order of the automorphism.)

2.2.3 The action of $\mathrm{GL}_2(\mathbb{F}_p)$

The action of $\mathrm{GL}_2(\mathbb{F}_p)$ on $\mathcal{M}(\mathcal{P}, \Gamma(p))$ from the right has the obvious modular interpretation:

$$r(g)\colon (E/S, \alpha, \phi) \mapsto (E/S, \alpha, \phi \circ g).$$

By construction, that extends uniquely to an action on our semistable model $\overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))^{\mathrm{st}}$, and we want to describe this on the special fiber. As $e_p(\phi \circ g(1,0), \phi \circ g(0,1)) = e_p(\phi(1,0), \phi(0,1))^{\det(g)}$, the action of g on $\coprod_i X_i \otimes_{\mathbb{Z}[\zeta_p]} \overline{\mathbb{F}}_p$ is

$$\big((E/S/\overline{\mathbb{F}}_p, \alpha, \phi), i\big) \mapsto \big((E/S/\overline{\mathbb{F}}_p, \alpha, \phi \circ g), i \det(g)\big)\,.$$

The action of $\mathrm{GL}_2(\mathbb{F}_p)$ on the $\mathrm{Ig}_{i,P}$ goes therefore as follows. Each g induces an isomorphism

$$r(g)\colon \mathrm{Ig}_{i,P} \xrightarrow{\sim} \mathrm{Ig}_{i\det(g), g^{-1}P}$$

so that the stabilizer of $\mathrm{Ig}_{i,P}$ is the Borel subgroup B_P of $\mathrm{SL}_2(\mathbb{F}_p)$ that fixes the line P. As for the Drinfeld components, g induces an isomorphism

$$r(g)\colon D_{i,s} \xrightarrow{\sim} D_{i\det(g),s}$$

and the stabilizer of $D_{i,s}$ is $\mathrm{SL}_2(\mathbb{F}_p)$. Recalling the notation we we have introduced before Theorem 2.3 we denote by Z a parameter of the formal group of the universal deformation $\mathbb{E}/\overline{\mathbb{F}}_p[[t]]$, so that our universal p-torsion basis have parameters $x = Z(\phi(1,0))$ and $y = Z(\phi(0,1))$. Writing $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{F}_p)$, we see that g acts from the left on $W[\zeta_p][[x,y]]/(f)$ by:

$$
\begin{aligned}
r(g)^{\#}x &= r(g)^{\#}Z(\phi(1,0)) = Z(\phi \circ g(1,0)) = Z(\phi(a,c)) \\
&\equiv aZ(\phi(1,0)) + bZ(\phi(0,1)) \mod (x,y)^2 \equiv ax + cy \mod (x,y)^2 \qquad (8) \\
r(g)^{\#}y &\equiv bx + dy \mod (x,y)^2. \qquad (9)
\end{aligned}
$$

It therefore follows from (5) that g acts on our model (4) by

$$r(g)^{\#}\alpha = a\alpha + c\beta \quad \text{and} \quad r(g)^{\#}\beta = b\alpha + d\beta \; ; \qquad (10)$$

one readily checks that equations (4) are preserved by the action of $\mathrm{SL}_2(\mathbb{F}_p)$.

2.2.4 Galois action

Let $G_{\mathbb{Q}}$ be the absolute Galois group of $\mathbb{Q}$, and G_p its decomposition group at a maximal ideal of $\overline{\mathbb{Z}}$ over p, which we identify with the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$. If $\mathbb{Q}_p^{\mathrm{ur}}$ and $\mathbb{Q}_p^{\mathrm{t}}$ are the

usual notations for the maximal unramified and tame extension of $\mathbb{Q}_p$ respectively, the sequence of inclusions $\mathbb{Q}_p \subset \mathbb{Q}_p^{\mathrm{ur}} \subset \mathbb{Q}_p^{\mathrm{t}} \subset \overline{\mathbb{Q}}_p$ induces the sequence of Galois subgroups

$$I_p \lhd I \lhd G_p \lhd G_\mathbb{Q}$$

where correspondingly I is the inertia subgroup, and I_p its wild inertia subgroup. The tame inertia group $I_t := I/I_p \simeq \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{t}}/\mathbb{Q}_p^{\mathrm{ur}})$ can be identified with $\varprojlim_{p\nmid n} \mu_n(\overline{\mathbb{F}}_p)$ (where μ_n stands for the nth-roots of unity) by

$$\sigma \mapsto (\sigma(p^{1/n})/p^{1/n}))_{p\nmid n} \tag{11}$$

(so that the transition morphisms $\mu_{nm} \to \mu_n$ are given by $\zeta_{nm} \mapsto \zeta_{nm}^m$), and that is still isomorphic to $\varprojlim \mathbb{F}_{p^n}^*$ (in which transition morphisms are now given by the norm): this is Serre's theory of "caractères fondamentaux", cf. [19], paragraph 1.3. Any σ in $G_\mathbb{Q}$ induces an automorphism

$$\gamma(\sigma) := \mathrm{id} \times \mathrm{Spec}(\sigma) \quad \text{of} \quad \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p)) \times_\mathbb{Q} \mathrm{Spec}(\overline{\mathbb{Q}}) = \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p)) \times_{\mathrm{Spec}(\mathbb{Q}(\pi))} \mathrm{Spec}(\overline{\mathbb{Q}}),$$

with $\pi = (1-\zeta_p)^{\frac{1}{p+1}}$ as in (2), and $\gamma(\sigma)$ extends uniquely to an automorphism of $\overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))^{\mathrm{st}} \times_{\mathrm{Spec}(\mathbb{Z}[\pi])}$ $\mathrm{Spec}(\overline{\mathbb{Z}})$ that we still denote by $\gamma(\sigma)$. It follows that any σ in $G_{\mathbb{Q}_p}$ induces an automorphism of the special fiber $\overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))_p^{\mathrm{st}} := \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))^{\mathrm{st}} \times_{\mathrm{Spec}(\mathbb{Z}[\pi])} \overline{\mathbb{F}}_p$, and if σ actually belongs to I then $\gamma(\sigma)$ is an $\overline{\mathbb{F}}_p$-automorphism.

The extension $\mathbb{Z} \to \mathbb{Z}[\pi]$ having degree p^2-1 and being totally ramified at p, the inertia action just defined factorizes through an antihomomorphism $\overline{\gamma} \colon \mathbb{F}_{p^2}^* \to \mathrm{Aut}_{\overline{\mathbb{F}}_p}(\overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))_p^{\mathrm{st}})$. The action of I on $\mu_p(\mathbb{F}_p)$ factorizes through $I \to \mathbb{F}_{p^2}^* \xrightarrow{N} \mathbb{F}_p^*$, for N the norm map $x \mapsto x^{p+1}$. This means that the action of inertia on the left-hand side $\coprod_{i\in\mathbb{F}_p^*} X_i$ of (1) has the modular interpretation:

$$(E/S \xrightarrow{f} \mathrm{Spec}(\mathbb{Z}[\zeta_p]), \phi)_i \xrightarrow{\overline{\gamma}(u)} (E/S \xrightarrow{f'} \mathrm{Spec}(\mathbb{Z}[\zeta_p]), \phi)_{iu^{-p-1}}$$

for $f' = \mathrm{Spec}(u^{p+1}) \circ f$. It follows that $u \in \mathbb{F}_{p^2}^*$ induces isomorphisms:

$$\overline{\gamma}(u) \colon \begin{cases} \mathrm{Ig}_{i,P} & \xrightarrow{\sim} & \mathrm{Ig}_{iu^{-p-1},P} \\ D_{i,P} & \xrightarrow{\sim} & D_{iu^{-p-1},P} \,. \end{cases} \tag{12}$$

The stabilizer in $\mathbb{F}_{p^2}^*$ of both components is the kernel $\mu_{p+1}(\mathbb{F}_{p^2}) = (\mathbb{F}_{p^2}^*)^{p-1}$ of the norm map. Because the $\mathrm{Ig}_{i,P}$ are already components of the special fiber of some $\mathbb{Z}[\zeta_p]$-scheme, $\mu_{p+1}(\mathbb{F}_{p^2})$ acts trivially on each of them. As for the $D_{i,s}$, we see from (11) that $\overline{\gamma}(u)(\pi) = u\pi \mod \pi^2$, so that (5) implies that $u \in \mu_{p+1}(\mathbb{F}_{p^2})$ induces the automorphism

$$\overline{\gamma}(u)^\# \colon \alpha \mapsto u^{-1}\alpha \quad \text{and} \quad \beta \mapsto u^{-1}\beta \tag{13}$$

on the model (4) of $D_{i,s}$.

3 Non-split Cartan structures

3.1 Stable model for $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{ns}}(p))$

We compute the stable model for modular curves associated with a non-split Cartan group $\Gamma_{\mathrm{ns}}(p) \subseteq \mathrm{GL}_2(\mathbb{F}_p)$ (but *not* its normalizer), endowed with some additional level structure $\mathcal{P}$.

Theorem 3.1 *Let $p > 3$ be a prime, and let $[\Gamma_{\mathrm{ns}}(p)]$ be the moduli problem over $\mathbb{Z}[1/p]$ associated with $\Gamma_{\mathrm{ns}}(p)$. Let $\mathcal{P}$ be a representable Galois moduli problem, which is finite étale over $(\mathrm{Ell})_{/\mathbb{Z}_p}$ (take for instance $\mathcal{P} = [\Gamma(N)]$ for some $N \geq 3$ not divisible by p). Let $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{ns}}(p)) = \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))/\Gamma_{\mathrm{ns}}(p)$ be the associated compactified fine moduli space. Let W be a totally ramified extension of $\mathbb{Z}_p$ of degree $p^2 - 1$ (one can take for instance $W = \mathbb{Z}_p[(1 - \zeta_p)^{1/(p+1)}])$.*

Then $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{ns}}(p))$ has a semistable model over W whose special fiber is made of two vertical Igusa parts, which are linked by horizontal Drinfeld components above each supersingular points of $\overline{\mathcal{M}}(\mathcal{P})$ via the projection $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{ns}}(p)) \to \overline{\mathcal{M}}(\mathcal{P})$.

Both vertical parts, call them $\mathrm{Ig}(p, \mathcal{P})_1$ and $\mathrm{Ig}(p, \mathcal{P})_d$ (for $d \in \mathbb{F}_p^*$ a non-square), are isomorphic to the enhanced Igusa curve $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$.

If $s_{\mathcal{P}}$ is the number of supersingular points of $\overline{\mathcal{M}}(\mathcal{P})(\overline{\mathbb{F}}_p)$, the $s_{\mathcal{P}}$ horizontal (Drinfeld) components are all copies of some hyperelliptic smooth curve D for which an affine model is given by

$$U^2 = V^{p+1} + A_{\mathrm{ns}} \tag{14}$$

for some A_{ns} in $\overline{\mathbb{F}}_p^*$.

The singular points have local equations $W[[x, y]]/(xy - \pi^2)$, for π a uniformizer of W (e.g. $\pi = (1 - \zeta_p)^{1/(p+1)}$).

Remark 3.2 Recall, as in Remark 2.2, that we would have liked to call "vertical components" our "vertical parts" $\mathrm{Ig}(p, \mathcal{P})_1$ and $\mathrm{Ig}(p, \mathcal{P})_d$ above, but were formally prevented from doing so because $\overline{\mathcal{M}}(\mathcal{P})$ may not be irreducible itself.

Note also that our model is not regular, but easy to make so by performing a few blow ups.

Finally, the constant terms A_{ns} in equations (14) could obviously been taken as 1, as we here are only interested in geometric models; the same holds for similar terms in the forthcoming parallel statements about split Cartan curves, etc. We leave that presentation as a reminder that a more precise determination could possibly be computed some day.

For a picture of the curve we refer to Figure 3: it actually represents the coarse quotient $X_{\mathrm{ns}}(p)$, but that does not affect the general shape.

Proof One starts from the description of the semistable model of $\overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))$ of Section 2 and takes the quotient by some fixed non-split torus $\Gamma_{\mathrm{ns}}(p)$ in $\mathrm{GL}_2(\mathbb{F}_p)$. Recall the vertical Igusa parts $\mathrm{Ig}_{i,P}$ are indexed by $\mathbb{F}_p^* \times \mathbb{P}^1(\mathbb{F}_p)$, and the action of $\mathrm{GL}_2(\mathbb{F}_p)$ on the latter set is given by

$$(i, (a\colon b)) \overset{g}{\mapsto} (i \det(g), g^{-1} \cdot (a\colon b)).$$

If $D \simeq \mathbb{F}_p^*$ denotes the subgroup of scalar matrices, the action of $\Gamma_{\mathrm{ns}}(p)$ on $\mathbb{P}^1(\mathbb{F}_p)$ factorizes via the quotient $\Gamma_{\mathrm{ns}}(p)/D \simeq \mathbb{Z}/(p+1)\mathbb{Z}$ and that action is free and transitive. (Indeed the orbits on $\mathbb{P}^1(\mathbb{F}_{p^2})$, say, have size $p + 1$ or 1, and $\mathbb{P}^1(\mathbb{F}_p)$ is preserved.) One can therefore choose as representatives for the cosets $(\mathbb{F}_p^* \times \mathbb{P}^1(\mathbb{F}_p))/\Gamma_{\mathrm{ns}}(p)$ the two elements $(1, (1\colon 0))$ and $(d, (1\colon 0))$ for d some non-square in $\mathbb{F}_p^*$. Each Igusa component $\mathrm{Ig}_{i,P}$ has stabilizer ± 1 in $\Gamma_{\mathrm{ns}}(p)$, so the two vertical components are isomorphic to $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$. That is for the first part of the Theorem.

Let us deal with the Drinfeld components. Recall that an equation for them in the bad fiber of $\overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))$ is given by

$$-a = \alpha\beta^p - \alpha^p\beta \tag{15}$$

for some a in $\overline{\mathbb{F}}_p^*$ (cf. Section 2, (4)). To be completely explicit we choose some multiplicative generator κ of $\mathbb{F}_{p^2}^*$ and pick $\lambda := \kappa^{p-1}$ as a generator of the cyclic subgroup of elements with norm 1 within $\mathbb{F}_{p^2}^*$. That subgroup is precisely the stabilizer in $\Gamma_{\mathrm{ns}}(p)$ of any Drinfeld component $D_{i,s}$ by Section 2.2.3. Set now $P_1 := \lambda e_1 + \lambda^p e_2$, $P_2 := \lambda^p e_1 + \lambda e_2$, for (e_1, e_2) the canonical basis, say, of $\mathbb{F}_p^2$. We can choose $\Gamma_{\mathrm{ns}}(p)$ so that it acts diagonally on this basis (P_1, P_2), that is, $\Gamma_{\mathrm{ns}}(p)$ can be written as $\left\{ \begin{pmatrix} a^p & 0 \\ 0 & a \end{pmatrix}, a \in \mathbb{F}_{p^2}^* \right\}$ with respect to (P_1, P_2). We perform the change of coordinates

$$\tilde{\alpha} := \lambda\alpha + \lambda^p\beta \quad \text{and} \quad \tilde{\beta} := \lambda^p\alpha + \lambda\beta. \tag{16}$$

Then if $N := (\lambda^{-2} - \lambda^2)$ one has

$$\alpha = \frac{1}{N}(-\lambda\tilde{\alpha} + \lambda^p\tilde{\beta}) \quad \text{and} \quad \beta = \frac{1}{N}(\lambda^p\tilde{\alpha} - \lambda\tilde{\beta}) \tag{17}$$

from which equation (15) becomes

$$aN = \tilde{\alpha}^{p+1} - \tilde{\beta}^{p+1}. \tag{18}$$

Now coordinates for the quotient curve $D_s := D_{i,s}/(\Gamma_{\mathrm{ns}}(p) \cap \mathrm{SL}_2(\mathbb{F}_p))$ are

$$\begin{cases} u_1 := \tilde{\alpha}^{p+1} \\ v_1 := \tilde{\alpha}\tilde{\beta} \end{cases} \tag{19}$$

(they are indeed stable under the action of $\Gamma_{\mathrm{ns}}(p) \cap \mathrm{SL}_2(\mathbb{F}_p)$, and the corresponding morphism of curves has due degree $p+1$) so an equation for D_s is

$$u_1^2 - v_1^{p+1} - aNu_1 = 0 \tag{20}$$

or, setting $U := u_1 - \frac{aN}{2}$ and $V := v_1$,

$$U^2 = V^{p+1} + \left(\frac{aN}{2}\right)^2 \tag{21}$$

which gives a hyperelliptic model for D_s.

Finally the assertion for the equations of singularities in the special fiber follows for instance from [14], Chapter 10.3, Proposition 3.48. $\qquad\square$

3.2 Stable model for $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{ns}}^+(p))$

Now for curves associated with the *normalizer* $\Gamma_{\mathrm{ns}}^+(p)$ of $\Gamma_{\mathrm{ns}}(p)$.

Theorem 3.3 *Let $p > 3$ be a prime, and let $[\Gamma_{\mathrm{ns}}^+(p)]$ be the moduli problem over $\mathbb{Z}[1/p]$ associated with $\Gamma_{\mathrm{ns}}^+(p)$. Let $\mathcal{P}$ be a moduli problem as in Theorem 3.1, and let $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{ns}}^+(p)) = \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))/\Gamma_{\mathrm{ns}}^+(p)$ be the corresponding compactified fine moduli space. We denote, as in Theorem 3.1, by $s_\mathcal{P}$ the number of supersingular points of $\overline{\mathcal{M}}(\mathcal{P})(\overline{\mathbb{F}}_p)$, and by W a totally ramified extension of $\mathbb{Z}_p$ of degree $p^2 - 1$.*

If $p \equiv 1 \mod 4$, then $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{ns}}^+(p))$ has a semistable model over W whose special fiber is made of two vertical parts, which are both isomorphic to the Igusa curve $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$, where C_4 denotes the cyclic subgroup of order 4 in $\mathbb{F}_p^*$. Those two parts are linked above each supersingular points of $\overline{\mathcal{M}}(\mathcal{P})$ by horizontal Drinfeld components.

If $p \equiv -1 \mod 4$, then $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{ns}}^+(p))$ has a semistable model over W whose special fiber is made of only one vertical part, which is isomorphic to the enhanced Igusa curve $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$. That vertical part is crossed at all $s_\mathcal{P}$ supersingular points by a horizontal component.

Wether p is 1 or $-1 \mod 4$, the $s_\mathcal{P}$ horizontal components of the special fiber are copies of some hyperelliptic curve D^+ for which an affine model is given by

$$Y^2 = X(X^{\frac{p+1}{2}} + A_{\mathrm{ns}}) \tag{22}$$

for A_{ns} in $\overline{\mathbb{F}}_p^*$.

The singular points have local equations either $W[[x,y]]/(xy - \pi^2)$, if $p \equiv -1 \mod 4$, or $W[[x,y]]/(xy - \pi^4)$ if $p \equiv 1 \mod 4$, for π a uniformizer of W.

The same caveat as in Remark 3.2 (regarding irreducibility of the vertical Igusa parts) is in order here.

As before, for a picture of the curve we refer to Figure 4, representing the coarse quotient $X_{\mathrm{ns}}(p)^+$.

Proof Use notations as in the above proof of Theorem 3.1: our basis (P_1, P_2) of $\mathbb{F}_{p^2}^2$ made of two $\mathbb{F}_{p^2}/\mathbb{F}_p$-conjugate vectors is such that $\Gamma_{\mathrm{ns}}(p) = \{ \begin{pmatrix} a^p & 0 \\ 0 & a \end{pmatrix}, a \in \mathbb{F}_{p^2}^* \}$ with respect to (P_1, P_2). Then the normalizer of $\Gamma_{\mathrm{ns}}(p)$ deprived from $\Gamma_{\mathrm{ns}}(p)$ is made of all elements

$$w_r := \begin{pmatrix} 0 & r \\ r^p & 0 \end{pmatrix}$$

for r running through $\mathbb{F}_{p^2}^*$. The element w_1 leaves stable the $\mathbb{F}_p$-line spanned by $(P_1 + P_2)$. Up to changing choices, one can assume that is the line $(1:0)$ chosen in our representative of $(\mathbb{F}_p^* \times \mathbb{P}^1(\mathbb{F}_p))/\Gamma_{\mathrm{ns}}(p)$. Therefore w_1 maps $(x, (1:0))$ to $(-x, (1:0))$, so that it exchanges the two orbits if and only if -1 is a non-square in $\mathbb{F}_p^*$.

Now for the Drinfeld components. One needs to compute the action of w_r for r satisfying $r^{p+1} = -1$. With notations as in (19), one checks that, independently of r and because of (20):

$$\begin{cases} u_1 \cdot w_r & = & \tilde{\alpha}^{p+1} \cdot w_r = r^{(p+1)p} \tilde{\beta}^{p+1} = -\tilde{\beta}^{p+1} = -\frac{v_1^{p+1}}{u_1} = -u_1 + aN \quad \text{and} \\ v_1 \cdot w_r & = & (r^p \tilde{\beta})(r\tilde{\alpha}) = -\tilde{\alpha}\tilde{\beta} = -v_1 \end{cases}$$

so that $U = (u_1 - \frac{aN}{2})$ and $V = v_1$ are mapped to their opposite. Therefore

$$\begin{cases} X := v_1^2 = V^2 \quad \text{and} \\ Y := (u_1 - \frac{aN}{2}) \times v_1 = U \times V \end{cases} \tag{23}$$

give coordinates for the image of any Drinfeld component in our $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{ns}}^+(p))_{\mathbb{F}_{p^2}}^{\mathrm{st}}$. From (20) we then check that a singular model for any Drinfeld component can now be given the equation

$$Y^2 = X(X^{\frac{p+1}{2}} + A_{\mathrm{ns}}).$$

The proof of the equations of singularities in the special fiber are straightforward and similar to that of Theorem 3.1. $\square$

3.3 Stable model for $X_{\mathrm{ns}}(p)_{\mathbb{Q}}$

Now we deal with the case of pure level p non-split Cartan. We therefore take quotients of our fine modular curves by the Galois group of the additional level structure $\mathcal{P}$ to produce the desired coarse moduli space.

Theorem 3.4 *For $p > 3$ a prime, let $X_{\mathrm{ns}}(p)$ be the modular curve associated with a non-split Cartan subgroup in level p. Let $s = g(X_0(p)) + 1$ be the number of supersingular j-invariants in $\mathbb{F}_{p^2}$, where $g(X_0(p))$ is the genus of $X_0(p)$. Let W be a totally ramified extension of $\mathbb{Z}_p$ of degree $p^2 - 1$. Then $X_{\mathrm{ns}}(p)$ has a semistable model over W whose special fiber is made of two vertical irreducible components, which are linked in s points by s horizontal components, cf. Figure 3. The toric part of its jacobian therefore has dimension $s - 1 = g(X_0(p))$.*

Both vertical irreducible components, call them $\mathrm{Ig}(p)_1$ and $\mathrm{Ig}(p)_d$, are isomorphic to the coarse Igusa curve $\overline{M}(\mathrm{Ig}(p)/\{\pm1\})_{\overline{\mathbb{F}}_p}$.

The s horizontal (Drinfeld) components are all hyperelliptic smooth curve D_s for which an affine model is given by

$$U^2 = V^{\frac{p+1}{e(s)}} + A_{\mathrm{ns}} \tag{24}$$

for some A_{ns} in $\overline{\mathbb{F}}_p^*$, and $e(s)$ is the order of the geometric automorphism group $\mathrm{Aut}_{\overline{\mathbb{F}}_p}(s)/\{\pm1\}$ (which we recall to be 1 except when $j \equiv 1728$ or $0 \mod p$ where $e = 2$ or 3 respectively).

The singular points have local equations $W[[x, y]]/(xy - \pi^{2e(s)})$, for π a uniformizer of W.
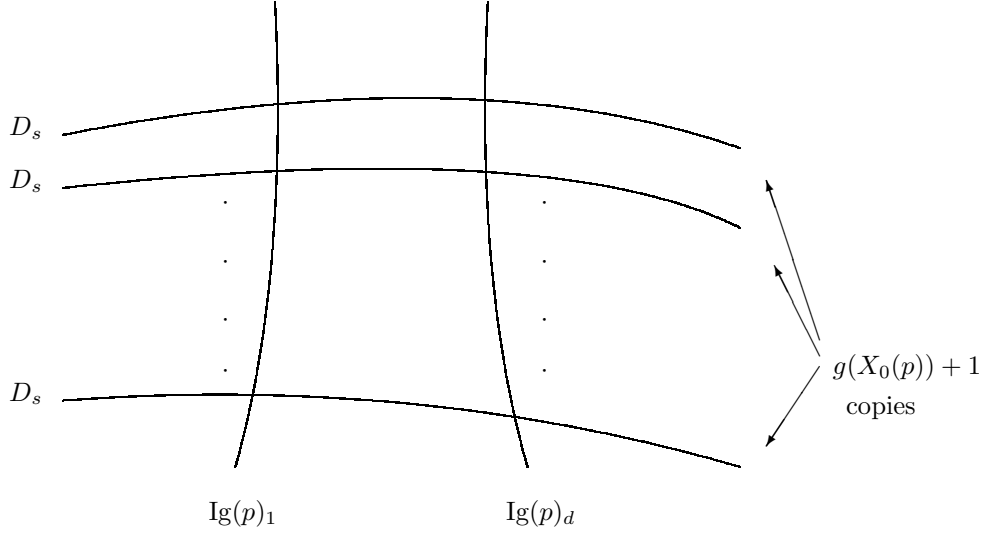
Figure 3: Special fiber of $X_{\mathrm{ns}}(p)^{\mathrm{st}}_{\overline{\mathbb{F}}_{p^2}}$

Proof After Theorem 3.1, what remains to do is to take the quotient of $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{ns})(p)^{\mathrm{st}}$ by the Galois group G of $\mathcal{P}$. The stabilizers in G have order $1, 2, 3, 4$ or 6, hence are prime to p, so the only thing to watch out is what happens on the locus of extra-automorphisms, that is, on Drinfeld components associated with supersingular j invariants equal to 1728 or 0. It then follows from Section 2.2.2 that the exceptional automorphism $[i]$ (respectively, $[\zeta]$) maps the parameters α and β to $i\alpha$ and $i\beta$ (respectively, $\zeta\alpha$ and $\zeta\beta$). Keeping track of those transformations through the computations of equations (15) to (21) shows that equations for the relevant quotients Drinfeld components have shape as given in (24). $\square$

3.4 Semistable model for $X_{\mathrm{ns}}^+(p)_{\mathbb{Q}}$

Theorem 3.5 *Let $p > 3$ be a prime, and keep same notations as in Theorem 3.4. Let w be the involution of the curve $X_{\mathrm{ns}}(p)$ associated with the quotient of the normalizer of the non-split Cartan subgroup by the Cartan itself, and $X_{\mathrm{ns}}^+(p) := X_{\mathrm{ns}}(p)/w$ the quotient curve.*

Then in the special fiber of the stable model $X_{\mathrm{ns}}(p)^{\mathrm{st}}$ given in Theorem 3.4, w fixes horizontal components, and switches the two vertical ones if and only if $p \equiv -1 \mod 4$. The dual graph of its special fiber is therefore topologically the same as that of $X_{\mathrm{ns}}(p)$ if $p \equiv 1 \mod 4$, or has trivial homology if $p \equiv -1 \mod 4$, cf. Figure 4. The vertical components are either both isomorphic to the Igusa curve $\overline{M}(\mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$ (where C_4 denotes the cyclic subgroup of order 4 in $\mathbb{F}_p^*$), in case $p \equiv 1 \mod 4$, or, if $p \equiv -1 \mod 4$, is isomorphic to $\overline{M}(\mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$. The toric rank is precisely

- *$t = \frac{p-13}{12}$ if $p \equiv 1 \mod 12$;*

- *$t = \frac{p-5}{12}$ if $p \equiv 5 \mod 12$;*

- *else $t = 0$.*

The $s = g(X_0(p)) + 1$ horizontal components are hyperelliptic curve D_s^+ for which an affine model is given, if the supersingular invariant s attached to D_s^+ is neither 0 or 1728, by

$$Y^2 = X(X^{\frac{p+1}{2}} + A_{\mathrm{ns}}) \tag{25}$$

for some A_{ns} in $\overline{\mathbb{F}}_p^*$. In the case $s = 0$, D_0^+ has a model

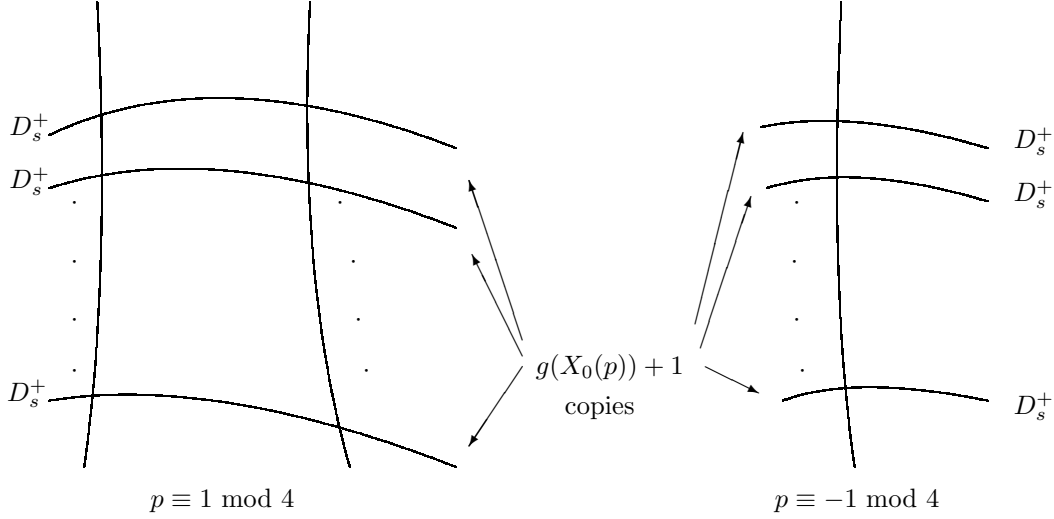$$Y^2 = X(X^{\frac{p+1}{6}} + A_{\mathrm{ns}}). \tag{26}$$

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Figure 4: Special fiber at $\mathbb{F}_{p^2}$ of the *semistable* model $X_{\mathrm{ns}}^+(p)$, depending on $p \equiv \pm 1 \bmod 4$

If $s = 1728$, D_{1728}^+ *is just a projective line* $\mathbb{P}^1{}_{\mathbb{F}_{p^2}}$.

The singular points have local equations $W[[x,y]]/(xy - \pi^{2e(s)})$ *if* $p \equiv -1 \bmod 4$, *else it is* $W[[x,y]]/(xy - \pi^{4e(s)})$, *where $e(s)$ denotes as usual the order of the geometric automorphism group* $\mathrm{Aut}_{\overline{\mathbb{F}}_p}(s)/\{\pm 1\}$, *and π is a uniformizer of W.*

Remark 3.6 Note that in the cases where projective lines are showing-up as Drinfeld components, the model we obtain is only semistable, and needs contracting the only rational curve in order to become stable.

Proof We use Theorem 3.3, applying similar arguments as in the proof of Theorem 3.4. Again the only delicate point is to follow the effect of exceptional automorphisms on relevant Drinfeld components, associated with some supersingular elliptic curve E_0. So write ζ_n ($n = 4$ or 6) for our generator of $\mu_n(\mathbb{F}_{p^2}) = \mathrm{Aut}_{\mathbb{F}_{p^2}}(E_0)$. Keeping track of the action of ζ_n on parameters α, β as given in Section 2.2.2, and the subsequent parameters given in (23), one sees that $[\zeta_n]$ maps X to $\zeta_n^4 X$ and Y to $\zeta_n^2 Y$. (One readily checks that equations (25) are preserved.) In the case $n = 6$, parameters for the quotient Drinfeld component by the action of $[\zeta_6]$ are clearly $\mathcal{X} := X^3$ and $\mathcal{Y} := XY$, so one deduces from equation (23): $Y^2 = X(X^{(p+1)/2}) + A_{\mathrm{ns}})$ that a model for that quotient Drinfeld component is $\mathcal{Y}^2 = \mathcal{X}(\mathcal{X}^{(p+1)/6} + A_{\mathrm{ns}})$. When $n = 4$ however, parameters for the quotient Drinfeld component by the action of $[i]$ are $\mathcal{X} := X$ and $\mathcal{Y} := Y^2$. From (23) we therefore see that taking quotient by the action of $[i]$ gives the projective line, $[i]$ being the hyperelliptic involution of D_{1728}. $\square$

Corollary 3.7 *Let $J = \mathrm{Jac}(X_{\mathrm{ns}}^+(p))$ be the jacobian of $X_{\mathrm{ns}}^+(p)$ over the fraction field of a totally ramified extension W of $\mathbb{Z}_p$ of degree $p^2 - 1$. Set $n := \mathrm{num}((p-1)/12)$ and let $S := g(X_0(p)) + 1$ be the number of supersingular points in characteristic p. Then the Néron model of J over W has a component group at the special fiber which is isomorphic to*

$$(\mathbb{Z}/8n\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^{S-2}$$

if $p \equiv 1 \mod 4$ (and is trivial if $p \equiv -1 \mod 4$).

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Proof This follows from Theorem 3.5 together with classical results of Raynaud. (Notice that for $p \equiv 1 \mod 4$ our theorem shows that the metrized dual graph of $X_{\mathrm{ns}}^+(p)$ above p is exactly the same as that of $X_0(p)$ at the special fiber of some totally ramified extension of $\mathbb{Q}_p$ with degree 8; then one can use for instance [13], Proposition 2.11.) Of course one could similarly write the component groups over any extension of $\mathrm{Frac}(W)$. $\square$

Remark 3.8 As a safety check one can compute that, assuming $p \equiv 5 \mod 12$ to fix ideas, the genus of the Igusa components $\overline{M}(\mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$ is $(p-5)(p-17)/96$, that of the $(p+7)/12$ Drinfeld components is $(p-5)/4$ for all but the $j=0$ one, for which it is $(p-5)/12$, and the toric rank is $(p-5)/12$ too. The total genus therefore sums up to

$$2\frac{(p-5)(p-17)}{96} + \left(\frac{p+7}{12} - 1\right) \cdot \frac{p-1}{4} + \frac{p-5}{12} + \frac{p-5}{12} = \frac{(p-5)^2}{24}$$

which indeed is the genus of $X_{\mathrm{ns}}^+(p)$ computed as a Riemann surface (check for instance [16] p. 117).

Over $\mathbb{Z}[1/p]$, a nice modular interpretation of $X_{\mathrm{ns}}^+(p)$ has been given in [18]. It is however hard to see what survives of it here above p.

We remark that when $p \equiv -1 \mod 4$, the Néron model of the jacobian of $X_{\mathrm{ns}}^+(p)$ over W gives an example of an abelian scheme which yet has "bad reduction" above p as a polarized abelian variety, in the sense that it decomposes as the product of abelian varieties with the induced (reducible) product polarization.

Note also that we could have derived the mere toric (and abelian) dimensions of stable models for $X_{\mathrm{ns}}^+(p)$ from the corresponding description for $X_{\mathrm{s}}^+(p)$ as recalled in next section, using Chen isogeny between $\mathrm{Jac}(X_{\mathrm{s}}^+(p))^{p-\mathrm{new}}$ and $\mathrm{Jac}(X_{\mathrm{ns}}^+(p))$ (cf. [6], [3]). We come-back to that point in Remark 4.5 below.

4 Split Cartan structures

In this section we describe the bad fibers of $X_{\mathrm{s}}(p)$ and $X_{\mathrm{s}}^+(p)$, following the same paths as for the non-split Cartan cases. Recall that those models (for the split Cartan curves) had already been described in the first author's thesis ([7], [8]).

4.1 Stable model for $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_s(p))$

Theorem 4.1 *Let $p > 3$ be a prime, and let $[\Gamma_{\mathrm{s}}(p)]$ be the moduli problem over $\mathbb{Z}[1/p]$ associated with a split Cartan subgroup $\Gamma_{\mathrm{s}}(p)$ in level p (not its normalizer). Let $\mathcal{P}$ be a representable Galois moduli problem, which is finite étale over $(\mathrm{Ell})_{/\mathbb{Z}_p}$. Let $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{s}}(p)) = \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))/\Gamma_{\mathrm{s}}(p)$ be the associated compactified fine moduli space. We denote by W a totally ramified extension of $\mathbb{Z}_p$ of degree $p^2 - 1$.*

Then $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{s}}(p))$ has a semistable model over W whose special fiber is made of four vertical Igusa parts, which are linked by horizontal Drinfeld components above each supersingular points via the projection $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{s}}(p)) \to \overline{\mathcal{M}}(\mathcal{P})$.

The two central parts are isomorphic to enhanced quotients of Igusa curves $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$. We call them $\mathrm{Ig}(p)_1$ and $\mathrm{Ig}(p)_d$ (for d some non-square in $\mathbb{F}_p^*$). The two outer vertical parts are simply copies of $\overline{\mathcal{M}}(\mathcal{P})$.

If $s_{\mathcal{P}}$ is the number of supersingular points in $\overline{\mathcal{M}}(\mathcal{P})$, the $s_{\mathcal{P}}$ horizontal (Drinfeld) components are all copies of some hyperelliptic curves for which an affine model is

$$U^2 = V^{p+1} + A_{\mathrm{s}} \tag{27}$$

for some non-zero A_{s} in $\overline{\mathbb{F}}_p$.

The singular points on the central Igusa components have local equations $W[[x,y]]/(xy - \pi^2)$, and those on the two rational outer vertical components, have equations $W[[x,y]]/(xy - \pi^{p-1})$, for π a uniformizer of W.

Proof This is very akin to the proof of Theorem 3.1. We compute the quotient of the vertical Igusa parts $\mathrm{Ig}_{i,P}$ indexed by $\mathbb{F}_p^* \times \mathbb{P}^1(\mathbb{F}_p)$. Fixing a split torus

$$\Gamma_{\mathrm{s}}(p) = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a, b \in \mathbb{F}_p^* \}$$

and writing again $D \simeq \mathbb{F}_p^*$ the subgroup of diagonal matrices, $\Gamma_{\mathrm{s}}(p)$ acts on $\mathbb{P}^1(\mathbb{F}_p)$ via its quotient $\Gamma_{\mathrm{s}}(p)/D \simeq \mathbb{Z}/(p-1)\mathbb{Z}$. That action has two fixed points say $(1\colon 0)$ and $(0\colon 1)$, and one orbit of size $p-1$. One chooses as representatives for the coset $(\mathbb{F}_p^* \times \mathbb{P}^1(\mathbb{F}_p))/\Gamma_{\mathrm{s}}(p)$ the four elements $(1,(1\colon 0))$, $(1,(0\colon 1))$, $(1,(1\colon 1))$ and $(d,(1\colon 1))$ for d some non-square in $\mathbb{F}_p^*$. The Igusa parts attached with the first two representatives, have stabilizer $\Gamma_{\mathrm{s}}(p) \cap \mathrm{SL}_2(\mathbb{F}_p) = \{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, t \in \mathbb{F}_p^* \}$. The stabilizer of the other two components are ± 1. So two vertical parts are isomorphic to $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\mathbb{F}_p^*\})_{\mathbb{F}_{p^2}} \simeq \overline{\mathcal{M}}(\mathcal{P})_{\mathbb{F}_{p^2}}$, and two are isomorphic to $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\mathbb{F}_{p^2}}$. This is for the first part of the Theorem.

Let us deal with the Drinfeld components. Recall (cf. (4)) that an equation for them in the bad fiber of the semistable model $\overline{\mathcal{M}}(\Gamma(p))^{\mathrm{st}}$ is given by

$$-a = \alpha\beta^p - \alpha^p\beta \tag{28}$$

for some a in $\overline{\mathbb{F}}_p^*$. The stabilizer in $\Gamma_{\mathrm{s}}(p)$ of any component $D_{i,s}$ is $\Gamma_{\mathrm{s}}(p) \cap \mathrm{SL}_2(\mathbb{F}_p)$, its action on coordinates of $D_{i,s}$ is given by $(\alpha,\beta) \mapsto (t\alpha, t^{-1}\beta)$, so coordinates on $D_{i,s}/\Gamma_{\mathrm{s}}(p) \cap \mathrm{SL}_2(\mathbb{F}_p)$ can be chosen as $(u,v) = (\alpha^{p-1}, \alpha\beta)$. From that, equation (28) becomes

$$v^p - u^2 v + a \cdot u = 0 \tag{29}$$

and the change of variables $U := uv - a/2$ (and $v = V$) yields:

$$U^2 = V^{p+1} + \frac{a^2}{4}$$

as an affine hyperelliptic model for D_s. The assertion about the width of singularities follows from similar arguments as those in the proof of Theorem 3.1. $\square$

4.2 Stable model for $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_s^+(p))$

Theorem 4.2 *Let $p > 3$ be a prime, and let $[\Gamma_{\mathrm{s}}^+(p)]$ be the moduli problem over $\mathbb{Z}[1/p]$ associated with the normalizer $\Gamma_{\mathrm{s}}^+(p)$ of a split Cartan subgroup in level p. Let $\mathcal{P}$ a moduli problem as in Theorem 3.1, and let $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{s}}^+(p)) = \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))/\Gamma_s^+(p)$ be the corresponding compactified fine moduli space. Let W be a totally ramified extension of $\mathbb{Z}_p$ of degree $p^2 - 1$, and $s_{\mathcal{P}}$ be the number of supersingular points of $\overline{\mathcal{M}}(\mathcal{P})(\overline{\mathbb{F}}_p)$.*

If $p \equiv 1 \bmod 4$, then $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{s}}^+(p))$ has a semistable model over W whose special fiber is made of three vertical parts. Two neighbor vertical parts are isomorphic to the enhanced Igusa curve $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$, where C_4 denotes the cyclic subgroup of order 4 in $\mathbb{F}_p^*$. One outer part is a copy of $\overline{\mathcal{M}}(\mathcal{P})$. Those three parts are linked above supersingular points of $\overline{\mathcal{M}}(\mathcal{P})$ by $s_{\mathcal{P}}$ horizontal components, cf. first case of Figure 6.

If $p \equiv -1 \bmod 4$, then $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathrm{s}}^+(p))$ has a semistable model over W whose special fiber is made of only two vertical parts. One is isomorphic to the Igusa curve $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$. The second vertical part is again a copy of $\overline{\mathcal{M}}(\mathcal{P})$. Those components are linked above supersingular points of $\overline{\mathcal{M}}(\mathcal{P})$ by $s_{\mathcal{P}}$ horizontal components, cf. second case of Figure 6.

Wether p is 1 or $-1 \bmod 4$, the $s_{\mathcal{P}}$ horizontal components D_s^+ of the special fiber are copies of some hyperelliptic curve for which an affine model is given by

$$Y^2 = X(X^{\frac{p+1}{2}} + A_{\mathrm{s}}) \tag{30}$$

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for some A_s in $\overline{\mathbb{F}}_p^*$.

Crossing points on the trivial vertical part (which is a copy of $\overline{\mathcal{M}}(\mathcal{P})$) have local equations $W[[x,y]]/(xy - \pi^{p-1})$, where π is some uniformizer of W. As for the genuine Igusa components, singularities have equations $W[[x,y]]/(xy - \pi^4)$, if $p \equiv 1 \mod 4$, or $W[[x,y]]/(xy - \pi^2)$ if $p \equiv -1 \mod 4$.

Proof This is again also very similar to the proof of Theorem 3.3. We take the further quotient of the curve $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_s(p))$ by the normalizer $\Gamma_s(p)^+$. Fricke's involution w is here given by the set $\{w_{a,b} := \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}, a, b \in \mathbb{F}_p^*\}$. Therefore w switches the two outer vertical parts. As for the central ones, their representatives $(1, (1:1))$ and $(d, (1:1))$ are mapped to $(-1, (1:1))$ and $(-d, (1:1))$ respectively by w. It follows that the components $\mathrm{Ig}(p)_1$ and $\mathrm{Ig}(p)_d$ of Theorem 4.1 are switched if and only if -1 is a non-square in $\mathbb{F}_p^*$.

With notations as in (29) one checks that, for $w_{t,-t^{-1}}$ in $\mathrm{SL}_2(\mathbb{F}_p)$:

$$(u,v) \cdot w_{t,-t^{-1}} = (\alpha^{p-1}, \alpha\beta) \cdot w_{t,-t^{-1}} = ((t\beta)^{p-1}, -\alpha\beta) = (\frac{v^{p-1}}{u}, -v) = (u - \frac{a}{v}, -v)$$

so the coordinates $U := uv - a/2$ and $V := v$ and mapped to their opposite by $w_{t,-t^{-1}}$. Therefore

$$\begin{cases} X := V^2 \\ Y := UV \end{cases} \tag{31}$$

are coordinates for the image of any Drinfeld component in our $X_s^+(p)_{\overline{\mathbb{F}}_p}$, and we conclude as in the proof of Theorem 3.3. $\square$

4.3 Stable model for $X_s(p)_{\mathbb{Q}}$

Now for the coarse case.

Theorem 4.3 *For $p > 3$ a prime, let $X_s(p)$ be the modular curve associated with a split Cartan subgroup $\Gamma_s(p)$ in level p. Let $s = g(X_0(p)) + 1$ be the number of supersingular j-invariants in characteristic p, where $g(X_0(p))$ is the genus of $X_0(p)$. Then $X_s(p)$ has a semistable model whose special fiber is made of four vertical irreducible components, which are linked in s points by s horizontal components, cf. Figure 5. The toric part of its jacobian has therefore dimension $3(s-1) = 3g(X_0(p))$.*

The two central vertical components, call them $\mathrm{Ig}(p)_1$ and $\mathrm{Ig}(p)_d$, are isomorphic to the quotient coarse Igusa curve $\overline{M}(\mathrm{Ig}(p)/\{\pm 1\})_{\mathbb{F}_{p^2}})$. The two outer vertical components are projective lines.

The s horizontal Drinfeld components are all hyperelliptic smooth curves for which an affine model is given by

$$U^2 = V^{\frac{p+1}{e(s)}} + A_s \tag{32}$$

for some A_s in $\overline{\mathbb{F}}_p$, and $e(s) = \mathrm{Card}(\mathrm{Aut}_{\overline{\mathbb{F}}_p}(s)/\{\pm 1\})$.

Singular points on the rational vertical components have equations $W[[x,y]]/(xy - \pi^{(p-1)e(s)})$ and those on Igusa components have equations $W[[x,y]]/(xy - \pi^{2e(s)})$.

(Note that, abusing a bit notations, we have used the same labels $\mathrm{Ig}(p)_1$ and $\mathrm{Ig}(p)_d$ as in Theorem 4.1. Note also that equations (32) define genuinely hyperelliptic curves only when p is not to small.)
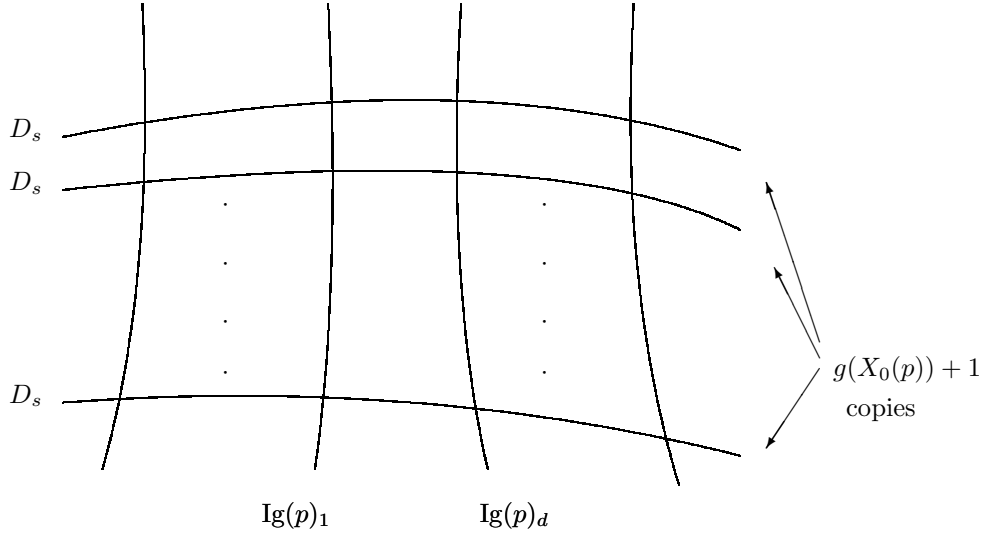
Figure 5: Special fiber of $X_{\mathrm{s}}(p)$

Proof   Here we parallel the proof of Theorem 3.4. Indeed Theorem 4.1 shows that we only need taking the quotient of our semistable model $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_s(p))^{\mathrm{st}}$ by the Galois group G for $\mathcal{P}$, and check what happens on the locus of extra-automorphisms, that is, on Drinfeld components associated with supersingular j invariants equal to 1728 or 0. Section 2.2.2 shows that the exceptional automorphism $[\zeta_n]$ (for $n = 4$ or 6) maps the parameters α and β to $\zeta_n \alpha$ and $\zeta_n \beta$ respectively. Keeping track of those transformations through the computations of equations (29) and around, and doing the math, shows that the relevant quotients Drinfeld components are indeed given by (32). $\square$

4.4   Stable model for $X_s^+(p)_{\mathbb{Q}}$

Theorem 4.4 *Let $p > 3$ be a prime, and use the same notations as in Theorem 4.1 above. Let w be the involution of the curve $X_{\mathrm{s}}(p)$ defined by the action of the normalizer $\Gamma_{\mathrm{s}}^+(p)$, and let $X_{\mathrm{s}}^+(p) := X_{\mathrm{s}}(p)/w$ be the quotient curve.*

Then w leaves the horizontal components of $X_{\mathrm{s}}(p)$ stable and exchanges the two outer vertical (rational) components. It switches the two central vertical ones if $p \equiv -1 \bmod 4$, else it leaves them stable. The special fiber of $X_{\mathrm{s}}^+(p)^{\mathrm{st}}$ therefore has a dual graph as in Figure 6. Its toric rank is explicitely

- $t = \frac{p-13}{6}$ *if $p \equiv 1 \bmod 12$;*

- $t = \frac{p-5}{6}$ *if $p \equiv 5 \bmod 12$;*

- $t = \frac{p-7}{12}$ *if $p \equiv 7 \bmod 12$;*

- $t = \frac{p+1}{12}$, *if $p \equiv 11 \bmod 12$.*

One vertical component of $X_{\mathrm{s}}^+(p)^{\mathrm{st}}$ is therefore a projective line. Each of the two other vertical components, in the case $p \equiv 1 \bmod 4$, is isomorphic to the quotient coarse Igusa curve $\overline{M}(\mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$, for C_4 the scalar subgroup of order 4. When $p \equiv -1 \bmod 4$, the remaining non-rational vertical component is $\overline{M}(\mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$.
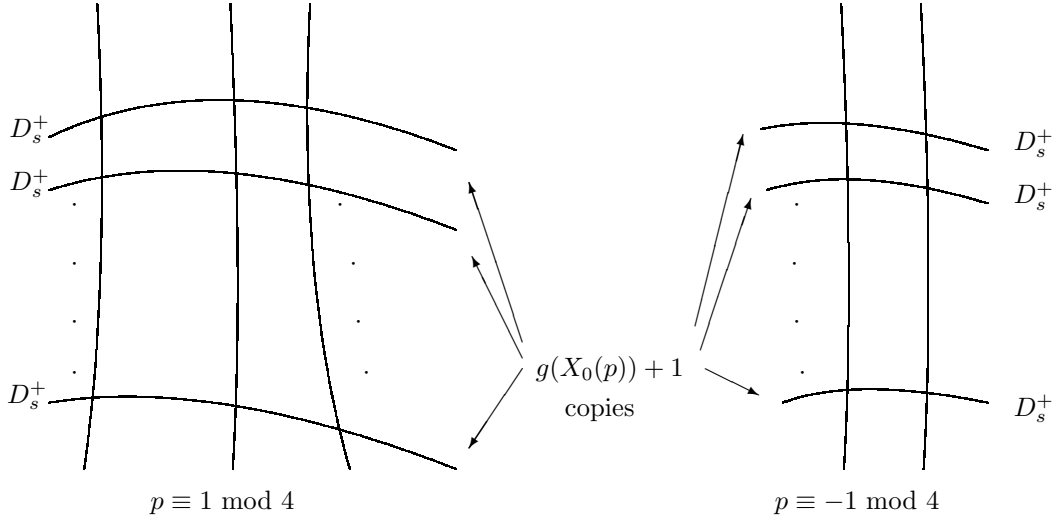
Figure 6: Special fiber of $X_{\mathrm{s}}^+(p)$, depending on $p \equiv \pm 1 \bmod 4$

The s Drinfeld horizontal components, above supersingular invariants different from 0 and 1728, are hyperelliptic curves for which an affine model is given by

$$Y^2 = X(X^{\frac{p+1}{2}} + A_{\mathrm{s}}) \tag{33}$$

for some A_{s} in $\overline{\mathbb{F}}_p^*$. If the supersingular invariant is 0, D_s^+ has a model

$$Y^2 = X(X^{\frac{p+1}{6}} + A_{\mathrm{ns}}). \tag{34}$$

If the supersingular invariant is 1728, then D_{1728}^+ is just a projective line $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$.

Crossing points on the rational vertical component have equations $W[[x,y]]/(xy - \pi^{(p-1)e(s)})$, for $e(s) = \mathrm{Card}(\mathrm{Aut}_{\overline{\mathbb{F}}_p}(s)/\{\pm 1\})$. As for Igusa components, singularities have local equations given by $W[[x,y]]/(xy - \pi^{4e(s)})$ if $p \equiv 1 \bmod 4$, and $W[[x,y]]/(xy - \pi^{2e(s)})$ if $p \equiv -1 \bmod 4$.

Proof   This time what we mimic is Theorem 3.5: use Theorem 4.2, applying similar arguments as in the proof of Theorem 4.3. $\square$

Remark 4.5 It follows from Chen-Edixhoven's theorem ([3], [6]) that

$$\mathrm{Jac}(X_{\mathrm{s}}^+(p)) \sim \mathrm{Jac}(X_{\mathrm{ns}}^+(p)) \times \mathrm{Jac}(X_0(p))$$

so for $p = 13$ the split and non-split Cartan curves curves $X_{\mathrm{s}}^+(13)$ and $X_{\mathrm{ns}}^+(13)$ have isogenous jacobians. But in [2], Burcu Baran computed models showing that they even are *isomorphic* (for some isomorphism which does not seem to have any natural modular interpretation - for instance, the packet of six $\mathbb{Q}$-valued CM points and the rational cusp on the former curve are mapped to seven rational CM points on the latter (and those sets are proven in [1] to be the full $X_{\mathrm{s}}^+(13)(\mathbb{Q})$ and $X_{\mathrm{ns}}^+(13)(\mathbb{Q})$ respectively)). Our two models however look like having different bad fibers: both have one horizontal component, but $X_{\mathrm{s}}^+(13)_{\mathbb{F}_{13}}$ has three vertical ones, whereas $X_{\mathrm{ns}}^+(13)_{\mathbb{F}_{13}}$ has only two. A closer look however shows that the all vertical components are rational. After contracting the $\mathbb{P}^1$s only the horizontal component of each model therefore survives, and both happen to be geometrically isomorphic to the genus-3 curves with affine model $Y^2 = X^8 + X$. This finally shows that our isomorphic modular curves have potentially good reduction everywhere.

5 Exceptional subgroups

We finally do the computations for modular curves in prime level p, associated with linear groups $\Gamma_{\mathfrak{A}_4}(p)$, $\Gamma_{\mathfrak{S}_4}(p)$ and $\Gamma_{\mathfrak{A}_5}(p)$ having projective image the permutation groups $\mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$ respectively.

Things go essentially the same way as for the Cartan cases, to the only exception that equations for the Drinfeld components are more delicate to write down explicitly. It seems in particular that writing them as *quotients*, as we did for the Cartan subgroups, is hardly doable with bare hands. So instead of giving closed expressions we describe in next paragraph an algorithmic method to obtain them. Then we review the other features of special fibers (topology of the dual graph, vertical components...) for the three exceptional cases, and each time display some numerical examples of those Drinfeld equations.

5.1 Computation of Drinfeld components

Starting from the affine equation (4) for the generic Drinfeld component D on $X(p)$, or better the smooth projective model

$$x^p y - x y^p = z^{p+1} \tag{35}$$

we see that the projection $(x, y, z) \mapsto (x, y)$ presents it as a μ_{p+1}-covering of the projective line, for μ_{p+1} the group of $p+1$st roots of unity, which is ramified precisely above $\mathbb{P}^1(\mathbb{F}_p)$. We also see that D is endowed with an action of $G := \mathrm{SL}_2(\mathbb{F}_p) \times \mu_{p+1}$ defined as

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \alpha \right) \cdot (x, y, z) = (ax + cy, bx + dy, \alpha z) \tag{36}$$

(recall the "transposed" action of $\mathrm{SL}_2(\mathbb{F}_p)$ as described in (10)). Clearly the two actions of $\mathrm{SL}_2(\mathbb{F}_p)$ and μ_{p+1} commute. The group G does not act faithfully, but its quotient by $\{\pm 1\} = \mu_2(\mathbb{F}_p)$ (embedded diagonally), does. Therefore if H is any subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ (containing -1) we have the commutative diagram

$$
\begin{array}{ccc}
D/\mu_2 & \xrightarrow{\ \pi\ } & \mathbb{P}^1_{\mathbb{F}_p} \\
\downarrow{\scriptstyle q} & & \downarrow \\
D/H & \longrightarrow & (\mathbb{P}^1_{\mathbb{F}_p})/H
\end{array}
$$

where the (smooth) curves D/μ_2 and $(D/\mu_2)/H = D/H$ on the left-hand side are endowed with an action of $\mu_{p+1}/\mu_2 = \mu_{\frac{p+1}{2}}$, the quotients by which are precisely the projective lines on the right-hand side. This diagram is co-cartesian by the universal properties of the quotient morphisms, and cartesian exactly away from the locus in $(\mathbb{P}^1_{\mathbb{F}_p})/H$ where both maps are ramified (above such points the fibered product has a 2-dimensional tangent space).

Let us first make the quotient $(\mathbb{P}^1_{\mathbb{F}_p})/H$ explicit by giving a rational function $\phi \colon \mathbb{P}^1_{\mathbb{F}_p} \to \mathbb{P}^1_{\mathbb{F}_p}$ that realises it. We can take

$$\phi(t) = \frac{\prod_{P \in O_1} (t - t(P))^{\#H_1}}{\prod_{P \in O_2} (t - t(P))^{\#H_2}}, \tag{37}$$

where the O_i are two distinct H-orbits of elements of $\mathbb{P}^1(\overline{\mathbb{F}}_p)$, not containing ∞, and the H_i are their respective isotropy groups. The diagram above now has become

$$
\begin{array}{ccc}
D/\mu_2 & \xrightarrow{\ \pi\ } & \mathbb{P}^1_{\mathbb{F}_p} \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle \phi} \\
D/H & \xrightarrow{\ \overline{\pi}\ } & \mathbb{P}^1_{\mathbb{F}_p}
\end{array}
$$

Via $\overline{\pi}$, D/H is a $\mu_{\frac{p+1}{2}}$-covering of $\mathbb{P}^1_{\overline{\mathbb{F}}_p}$, hence it can be given a (singular) equation of shape $u^{(p+1)/2} = f(t)$, with $\zeta \in \mu_{\frac{p+1}{2}}$ sending u to ζu, say, and we need to spot such an f. We can multiply f by arbitrary non-zero $(p+1)/2$th powers, so we just need to determine $\text{div}(f)$ with coefficients modulo $(p+1)/2$. For that, we observe that, at each fixed point, $\zeta \in \mu_{p+1}$ acts on the cotangent space of D by ζ (use equation (36)). Therefore, at each fixed point of D/μ_2, $\zeta \in \mu_{(p+1)/2}$ acts on the cotangent space by ζ. Now let $P \in D/\mu_2$ be a fixed point for $\mu_{(p+1)/2}$, let $Q := q(P)$ and let $e = \#H_P$ be the ramification index of q at P. Then $\zeta \in \mu_{(p+1)/2}$ acts on the cotangent space at Q by ζ^e. We note that $\#H_P = \#H_{\pi(P)}$, the ramification index of ϕ at $\pi(P)$, and that $v_{\overline{\pi}(Q)}(f) \cdot (p+1)/2 = v_Q(\overline{\pi}^* f) = v_Q(u) \cdot (p+1)/2$, hence $v_{\overline{\pi}(Q)}(f) = v_Q(u)$. It follows that $\zeta \in \mu_{(p+1)/2}$ sends u to $\zeta^{v_Q(u)e} \cdot u$, which we know to be ζu itself. Hence:

$$v_{\overline{\pi}(Q)}(f) = v_Q(u) = e^{-1} = (\#H_{\pi(P)})^{-1} \quad \text{in } \mathbb{Z}/((p+1)/2)\mathbb{Z}.$$

We finally obtain for our Drinfeld component D/H over $\overline{\mathbb{F}}_p$ the equation

$$u^{\frac{p+1}{2}} = \prod_{R \in H \backslash \mathbb{P}^1(\mathbb{F}_p), \, \phi(R) \neq \infty} (t - \phi(R))^{1/\#H_R}, \tag{38}$$

where the product is over a set of representatives R with $\phi(R) \neq \infty$ for the H-orbits of $\mathbb{P}^1(\mathbb{F}_p)$, and where $1/\#H_R$ is lift in $\mathbb{Z}$ of the inverse of $\#H_R$ in $(\mathbb{Z}/((p+1)/2)\mathbb{Z})^\times$.

(Notice that in all cases below, H_R is the isotropy group of the *intersection* of our exceptional groups *with* $\text{SL}_2(\mathbb{F}_p)$. In particular, the cases $p \equiv 11$ or $19 \bmod 12$ in Section 5.3 below (group $\mathfrak{S}_4$) should cause no worries with respect to the condition that $\#H_R$ is invertible mod $(p+1)/2$.)

In next sections we illustrate this method by providing a few numerical examples, constructing explicitly some ϕ and equations (38) for each case $H = \mathfrak{A}_4$, $\mathfrak{S}_4$ or $\mathfrak{A}_5$.

5.2 $\mathfrak{A}_4$

We first notice that the fact $\mathfrak{A}_4$ has no subgroup of index 2 implies $\mathfrak{A}_4$ in fact belongs to $\text{SL}_2(\mathbb{F}_p)/\{\pm 1\}$. The smallest number field over which the corresponding modular curve has a geometrically connected model is therefore the quadratic subfield of $\mathbb{Q}(\mu_p)$.

It follows from [12], proof of Theorem 2.3 on p. 186 of Chapter XI, that there are three orbits of elements in $\mathbb{P}^1(\mathbb{F}_{p^2})$ with non-trivial isotropy subgroups for the action of $\mathfrak{A}_4$ in $\text{PGL}_2(\mathbb{F}_{p^2})$. Those isotropy subgroups have order 2, 3 and 3 (cf. case (iii) of the Lemma after Theorem 2.3 quoted above); we call them G_2, $G_{3,1}$ and $G_{3,2}$. In $\mathbb{P}^1(\mathbb{F}_{p^2})$, there is therefore one orbit of size 6 (call it O_2), two of size 4 (call them $O_{3,1}$ and $O_{3,2}$), and $(p^2 - 13)/12$ orbits of size 12 (homogeneous spaces under action of $\mathfrak{A}_4$). Restricting that combinatorics to $\mathbb{P}^1(\mathbb{F}_p)$ sums-up as:

$p \bmod 12$	exceptional orbits in $\mathbb{P}^1(\mathbb{F}_p)$	total number N_p of orbits under $\mathfrak{A}_4$
1	O_2, $O_{3,1}$, $O_{3,2}$	$(p+23)/12$
5	O_2	$(p+7)/12$
7	$O_{3,1}$, $O_{3,2}$	$(p+17)/12$
11	none	$(p+1)/12$

Theorem 5.1 *Let $p > 3$ be a prime, and let $[\Gamma_{\mathfrak{A}_4}(p)]$ be the moduli problem over $\mathbb{Z}[1/p]$ associated with $\Gamma_{\mathfrak{A}_4}(p)$. Let $\mathcal{P}$ be a representable Galois moduli problem, which is finite étale over $(\text{Ell})_{/\mathbb{Z}_p}$. Let $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathfrak{A}_4}(p)) = \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))/\Gamma_{\mathfrak{A}_4}(p)$ be the associated compactified fine moduli space. Let W be a totally ramified extension of $\mathbb{Z}_p$ of degree $p^2 - 1$ (e.g. $W = \mathbb{Z}_p[(1 - \zeta_p)^{1/(p+1)}]$).*

Then $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathfrak{A}_4}(p))$ has a semistable model over W whose special fiber is made of $2N_p$ vertical Igusa parts (for $N_p \sim p/12$ as in the above array) which are linked by horizontal Drinfeld components above each supersingular points of $\overline{\mathcal{M}}(\mathcal{P})$ via the projection $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathfrak{A}_4}(p)) \to \overline{\mathcal{M}}(\mathcal{P})$. The geometric vertical parts are almost all copies of quotient enhanced Igusa curves $\overline{\mathcal{M}}(\mathcal{P}, \text{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$, except that:

- if $p \equiv 1 \bmod 12$, two of them are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$, and four are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$, for C_* a cyclic automorphism group of order $*$;

- if $p \equiv 5 \bmod 12$, there are two exceptional Igusa parts, which are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$;

- if $p \equiv 7 \bmod 12$, there are four exceptional Igusa parts, copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$.

Singular points located on components of shape $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_r)_{\overline{\mathbb{F}}_p}$ have local equations

$$W[[x, y]]/(xy - \pi^r), r = 2, 4 \text{ or } 6$$

for π as usual a uniformizer of W (e.g. $\pi = (1 - \zeta_p)^{1/(p+1)}$).

Proof As already remarked, $\mathfrak{A}_4$ has no subgroup of index 2 so that $\mathfrak{A}_4$ in fact belongs to $\mathrm{SL}_2(\mathbb{F}_p)/\{\pm 1\}$, and the image under the determinant of the full group $\Gamma_{\mathfrak{A}_4}(p)$ consists of all the squares of $\mathbb{F}_p^*$. Therefore vertical parts of our quotient curve can be indexed by the set of pairs $\{(\mathcal{O}, d)\}$, where $\mathcal{O}$ runs through the set of orbits of $\mathbb{P}^1(\mathbb{F}_p)$ under the action of $\mathfrak{A}_4$, and d runs through $\mathbb{F}_p^*$ modulo squares.

Igusa parts associated with orbits having the generic trivial isotropy group have stabilizer ± 1 in $\Gamma_{\mathfrak{A}_4}(p)$, so they are isomorphic to $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$. As for the Igusa components associated with G_2 and $G_{3,i}$, they are isomorphic to some $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$ and $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$ respectively, for C_r a cyclic automorphism group of oder r. The rest goes as in the proof of the previous theorems. $\square$

Now for equations of Drinfeld components. It follows from Theorem 6.1 of [4] that a system of generators for the image of $\mathfrak{A}_4$ in $\mathrm{SL}_2(\mathbb{F}_p)/\{\pm 1\}$ can be taken as any (S, T) with S of (projective) order 3, T of order 4, and ST has order 3, that is, S, T and ST are matrices with determinant 1 and trace ± 1, 0 and ± 1 respectively. One readily checks with that numerical criterion that $\mathfrak{A}_4$ has a model in $\mathrm{SL}_2(\mathbb{F}_p)/\{\pm 1\}$ (although not in $\mathrm{SL}_2(\mathbb{Z})/\{\pm 1\}$). When $p \equiv 1 \bmod 3$, one can for instance take

$$S = \begin{pmatrix} \zeta_3 & 0 \\ -1 & \zeta_3^2 \end{pmatrix}; \ T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

for ζ_3 some primitive third root of unity. In order to write-down a function ϕ as in equation (37) it is enough to find representatives a_1 and a_2 of two different $\mathfrak{A}_4$-orbits in $\mathbb{P}^1(\mathbb{F}_p)$ and take

$$\phi(t) = \prod_{g \in \mathfrak{A}_4} (t - g \cdot a_1)(t - g \cdot a_2)^{-1}$$

from which can write explicit forms of equation (38). As for numerical examples with $p = 13$ or $p = 103$ we compute that:

if $p = 13$, the set $\mathbb{P}^1(\mathbb{F}_{13})$ decomposes in three orbits: $O_{3,1} = \{0, \infty, 9, 10\}$, $O_{3,2} = \{1, 2, 6, 12\}$ and $O_2 = \{3, 4, 5, 7, 8, 11\}$ under our action of $\mathfrak{A}_4$, whence, as in (37), for instance a function ϕ:

$$\phi(t) = \frac{[(t-1)(t-2)(t-6)(t-12)]^3}{[(t-3)(t-4)(t-5)(t-7)(t-8)(t-11)]^2}$$

for which $\phi(\mathbb{P}^1(\mathbb{F}_p)) = \{0, 1, \infty\}$ (the images of the orbits of 1, ∞ and 3), with ramification indices 3, 3 and 2, respectively. The inverses in $\mathbb{Z}/7\mathbb{Z}$ of these are 5, 5 and 4, respectively. We therefore obtain from (38) the affine singular model:

$$u^7 = t^5(t-1)^5$$

for $\mathfrak{A}_4$-exceptional Drinfeld components in level 13 (with suitable rigidification).

If $p = 103$, we consider for instance the orbits of 0 and 1 to obtain

$$\phi(t) = \frac{[t(t-56)(t-57)]^3}{[(t-1)(t-10)(t+1)(t-72)]^3}$$

whence

$$\phi(\mathbb{P}^1(\mathbb{F}_p)) = \{0, \infty, 3, -3, -1, 22, 39, -14, -39, 10\}$$

with respective ramification indices $3, 3, 1, 1, 1, 1, 1, 1, 1, 1$ having inverses $35, 35, 1, 1, 1, 1, 1, 1, 1, 1$ mod 52, and an equation for $\mathfrak{A}_4$-Drinfeld components in characteristic 103 which is

$$u^{52} = t^{35}(t-3)(t+3)(t+1)(t-22)(t-39)(t+14)(t+39)(t-10).$$

5.3 $\mathfrak{S}_4$

Notice that $\mathfrak{S}_4$ belongs to $\mathrm{SL}_2(\mathbb{F}_p)/\{\pm 1\}$ if (and only) if $p \equiv \pm 1$ mod 8, cf. Theorems 1 & 2 of Feit's appendix in [12], pp. 201 & 202). If $p \equiv 3$ or 5 mod 8 then $\mathfrak{S}_4 \cap \mathrm{SL}_2(\mathbb{F}_p)/\{\pm 1\} = \mathfrak{A}_4$, so the relevant curve $X_{\mathfrak{S}_4}(p)$ is a form of the curve $X_{\mathfrak{A}_4}(p)$ studied in paragraph 5.2 above, and the former curve does have a geometrically integral model over $\mathbb{Q}$.

Now [12], proof of Theorem 2.3 on p. 187 of Chapter XI, gives that there are three orbits of elements in $\mathbb{P}^1(\mathbb{F}_{p^2})$ with non-trivial isotropy subgroups for the action of $\mathfrak{S}_4$ in $\mathrm{PGL}_2(\mathbb{F}_{p^2})$, and those isotropy subgroups have order 2, 3 and 4 (cf. case (iv) of the Lemma after Theorem 2.3 quoted above): we shall denote them by G_2, G_3 and G_4 respectively. In $\mathbb{P}^1(\mathbb{F}_{p^2})$, there is therefore one orbit of size 12, one of size 8, one of size 6, and $(p^2-25)/24$ of size 24 (which are homogeneous spaces under action of $\mathfrak{S}_4$). We denote by O_2, O_3 and O_4 the exceptional orbits of order 12, 8 and 6 respectively. Restricting that combinatorics to $\mathbb{P}^1(\mathbb{F}_p)$ gives

p mod 24	exceptional orbits in $\mathbb{P}^1(\mathbb{F}_p)$	total number N_p of orbits under $\mathfrak{S}_4$
1	O_2 , O_3, O_4	$(p+47)/24$
5	O_4	$(p+19)/24$
7	O_3	$(p+17)/24$
11	O_2	$(p+13)/24$
13	O_3, O_4	$(p+35)/24$
17	O_2, O_4	$(p+31)/24$
19	O_2, O_3	$(p+5)/24$
23	none	$(p+1)/24$

.

Theorem 5.2 *Let $p > 3$ be a prime which is congruent to $\pm 1 \mod 8$, and let $[\Gamma_{\mathfrak{S}_4}(p)]$ be the moduli problem over $\mathbb{Z}[1/p]$ associated with $\Gamma_{\mathfrak{S}_4}(p)$. Let $\mathcal{P}$ be a representable Galois moduli problem, which is finite étale over $(\mathrm{Ell})_{/\mathbb{Z}_p}$. Let $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathfrak{S}_4}(p)) = \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))/\Gamma_{\mathfrak{S}_4}(p)$ be the associated compactified fine moduli space. Let W be a totally ramified extension of $\mathbb{Z}_p$ of degree $p^2 - 1$.*

Then $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathfrak{S}_4}(p))$ has a semistable model over W whose special fiber is made of vertical Igusa parts, which are linked by horizontal Drinfeld components above each supersingular points of $\overline{\mathcal{M}}(\mathcal{P})$ via the projection $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathfrak{S}_4}(p)) \to \overline{\mathcal{M}}(\mathcal{P})$.

Almost all vertical parts are isomorphic to quotient enhanced Igusa curves $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$, except that:

- *if $p \equiv 1$ mod 24, there are six exceptional Igusa part; two are copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$, two are isomorphic to $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$, and two are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_8)_{\overline{\mathbb{F}}_p}$, where C_* denotes a cyclic automorphism group of order $*$. The total number of Igusa parts is $2N_p$ (for $N_p \simeq p/24$ the number of orbits as indicated in the array before our theorem);*

- *if $p \equiv 5 \mod 24$, there is only one exceptional Igusa part, which is $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$. The total number of Igusa parts is $2N_p - 1$;*

- *if $p \equiv 7 \mod 24$, there are two exceptional Igusa parts, which are a copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$. The total number of Igusa parts is $2N_p$;*

- *if $p \equiv 11 \mod 24$, there is no exceptional Igusa part. The total number of Igusa parts is $2N_p - 1$;*

- *if $p \equiv 13 \mod 24$, there are three exceptional Igusa parts, of which two are copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$, and one is isomorphic to $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$. The total number of Igusa parts is $2N_p - 1$;*

- *if $p \equiv 17 \mod 24$, there are four exceptional Igusa parts, of which two are copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$ and two are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_8)_{\overline{\mathbb{F}}_p}$. The total number of Igusa parts is $2N_p$;*

- *if $p \equiv 19 \mod 24$, there are two exceptional Igusa parts, which are copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$. The total number of Igusa parts is $2N_p - 1$;*

- *if $p \equiv 23 \mod 24$, the total number of Igusa parts is $2N_p$, and none is exceptional.*

Singular points located on components of shape $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_r)_{\overline{\mathbb{F}}_p}$ have local equations

$$W[[x,y]]/(xy - \pi^r), r = 2, 4, 6 \text{ or } 8$$

for π as usual a uniformizer of W (e.g. $\pi = (1 - \zeta_p)^{1/(p+1)}$).

Proof One first readily checks that the isotropy groups G_n, $n = 2, 3$ or 4, are all cyclic, with order n. So a generator for G_n can be taken as $\begin{pmatrix} \zeta_n & 0 \\ 0 & 1 \end{pmatrix}$, for ζ_n some primitive n^{th}-root of unity in $\mathbb{F}_p$.

For $p \equiv 1, 7, 17 \mod 24$, one computes at hand that the determinant of those generators are squares in $\mathbb{F}_p$. (This again could also have been derived from the fact that $\mathfrak{S}_4$ belongs to $\mathrm{SL}_2(\mathbb{F}_p)/\{\pm 1\}$ if (and only) if $p \equiv \pm 1 \mod 8$, cf. Theorems 1 & 2 in [12], pp. 201 & 202). Whence the vertical components, for primes in those congruences classes (and $p \equiv 23 \mod 24$) in our theorem.

For the remaining classes we proceed with case-by-case examinations.

If $p \equiv 5 \mod 24$, a generator for the non-trivial isotropy group G_4 in $\mathfrak{S}_4$ can be taken as $\begin{pmatrix} \zeta_4 & 0 \\ 0 & 1 \end{pmatrix}$, whose determinant is not a square in $\mathbb{F}_p$. So there is one exceptional Igusa part, which is a copy of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$.

If $p \equiv 11 \mod 24$, a generator for the non-trivial isotropy group G_2 in $\mathfrak{S}_4$ can be taken as $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, whose determinant is not a square in $\mathbb{F}_p$. So the corresponding Igusa part is just a plain copy of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$.

If $p \equiv 13 \mod 24$, the elements of G_3 in $\mathfrak{S}_4$ have square determinant, so the corresponding orbits give rise to two exceptional Igusa parts which are copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$. On the other hand, the determinant of $\begin{pmatrix} \zeta_4 & 0 \\ 0 & 1 \end{pmatrix}$ is a non-square in $\mathbb{F}_p$. So G_4 gives rise to a unique exceptional Igusa part, isomorphic to $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$.

If $p \equiv 19 \mod 24$, the group G_3, in a similar fashion to the previous case, gives rise to two exceptional Igusa parts which are copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$. Similarly to the case $p \equiv 13 \mod 24$, on the other hand, G_2 gives rise to one plain Igusa part $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$.

The shape of singularities easily follows from our description of local isotropy groups. $\qquad \square$

We compute equations of Drinfeld components. Using Theorem 6.1 of [4], we can take as a system of generators for the image of $\mathfrak{S}_4$ in $\mathrm{SL}_2(\mathbb{C})/\{\pm 1\}$ any set (S, T) whose traces satisfy

$$t_S^2 + t_T^2 + t_{ST}^2 - t_S t_T t_{ST} = 3 \quad \text{and} \quad t_S, t_T, t_{ST} \in \{0, \pm 1, \pm \sqrt{2}\}.$$

One readily checks with that numerical criterion that $\mathfrak{S}_4$ has for instance a model in $\mathrm{SL}_2(\overline{\mathbb{Z}})/\{\pm 1\}$ with generators:

$$S = \begin{pmatrix} \sqrt{2} & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & i \\ i & 0 \end{pmatrix}$$

which when $p \equiv 1 \bmod 8$ gives by reduction an easy model in $\mathrm{SL}_2(\mathbb{F}_p)/\{\pm 1\}$.

If $p = 73$, we can compute the orbit of 0 and that of 1, which respectively are

$$A := \mathfrak{S}_4 \cdot 0 = \{0, 5, 16, 17, 26, 32, 39, 46, 52, 61, 62, \infty\}$$

and

$$B := \mathfrak{S}_4 \cdot 1 = \{1, 4, 6, 13, 18, 19, 23, 27, 28, 31, 33, 34, 36, 42, 44, 45, 47, 50, 51, 55, 59, 60, 65, 72\}$$

so that, setting $\phi(t) = \prod_{a \in A \setminus \{\infty\}} (t - a)^2 \prod_{a \in B} (t - b)^{-1}$, we have $\phi(\mathbb{P}^1(\mathbb{F}_p)) = \{48, 14, 0, 58, \infty\}$, whose elements have respective ramification indices $4, 3, 2, 1$ and 1. The inverse of the latter mod 37 are respectively $28, 25, 19, 1$ and 1, whence the explicit forms

$$u^{37} = (t + 25)^{28}(t - 14)^{25} t^{19}(t + 15)$$

of equation (38) for the $\mathfrak{S}_4$-Drinfeld components in level 73.

5.4 $\quad \mathfrak{A}_5$

One knows that, for any prime-power q, whenever $\mathfrak{A}_5$ can be realized as a subgroup of some $\mathrm{GL}_2(\mathbb{F}_q)/\mathbb{F}_q^*$ then it belongs to $\mathrm{SL}_2(\mathbb{F}_q)/\{\pm 1\}$, and that is the case if and only if q is ± 1 mod 5 (cf. [10], Theorem 1 on p. 201). Let us henceforth assume for that subsection 5.4 that $q = p$ is a prime satisfying that congruence condition. (We therefore remark that the smallest number field over which the corresponding modular curve has a geometrically integral model is $\mathbb{Q}(\sqrt{5})$.) Again [12] (proof of Theorem 2.3 on p. 186 of Chapter XI) gives that there are three orbits of elements in $\mathbb{P}^1(\mathbb{F}_{p^2})$ with non-trivial isotropy subgroups for the action of $\mathfrak{A}_5$ in $\mathrm{GL}_2(\mathbb{F}_{p^2})/\mathbb{F}_{p^2}^*$, and those isotropy subgroups have order 2, 3 and 5 (cf. case (v) of the Lemma after Theorem 2.3 quoted above): call them G_2, G_3 and G_5. In $\mathbb{P}^1(\mathbb{F}_{p^2})$, there is therefore one orbit of size 30, one of size 20, one of size 12, and $(p^2 - 61)/60$ orbits of size 60. We denote by O_2, O_3 and O_5 the exceptional orbits of order 30, 20 and 12 respectively. The combinatorics implies that, restricting to $\mathbb{P}^1(\mathbb{F}_p)$:

$p \bmod 60$	exceptional orbits in $\mathbb{P}^1(\mathbb{F}_p)$	total number N_p of orbits under $\mathfrak{A}_5$
1	O_2 , O_3, O_5	$(p + 119)/60$
11	O_5	$(p + 49)/60$
19	O_3	$(p + 41)/60$
29	O_2	$(p + 31)/60$
31	O_3 , O_5	$(p + 89)/60$
41	O_2, O_5	$(p + 79)/60$
49	O_2, O_3	$(p + 71)/60$
59	none	$(p + 1)/60$

.

Theorem 5.3 *Let $p > 3$ be a prime, and let $[\Gamma_{\mathfrak{A}_5}(p)]$ be the moduli problem over $\mathbb{Z}[1/p]$ associated with $\Gamma_{\mathfrak{A}_5}(p)$. Let $\mathcal{P}$ be a representable Galois moduli problem, which is finite étale over $(\mathrm{Ell})_{/\mathbb{Z}_p}$. Let $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathfrak{A}_5}(p)) = \overline{\mathcal{M}}(\mathcal{P}, \Gamma(p))/\Gamma_{\mathfrak{A}_5}(p)$ be the associated compactified fine moduli space. Let W be a totally ramified extension of $\mathbb{Z}_p$ of degree $p^2 - 1$.*

Then $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathfrak{A}_5}(p))$ has a semistable model over W whose special fiber is made of $2N_p$ vertical Igusa parts (for $N_p \sim p/60$ as in the above array), which are linked by horizontal Drinfeld components above each supersingular points of $\overline{\mathcal{M}}(\mathcal{P})$ via the projection $\overline{\mathcal{M}}(\mathcal{P}, \Gamma_{\mathfrak{A}_5}(p)) \to \overline{\mathcal{M}}(\mathcal{P})$. Almost all vertical parts are geometrically isomorphic to quotient enhanced Igusa curves $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/\{\pm 1\})_{\overline{\mathbb{F}}_p}$, except that:

- *If $p \equiv 1 \bmod 60$, two exceptional Igusa parts are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$, two are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$, and two are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_{10})_{\overline{\mathbb{F}}_p}$, for C_* a cyclic automorphism group of order $*$.*

- *If $p \equiv 11 \bmod 60$, there are two exceptional Igusa parts, which are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_{10})_{\overline{\mathbb{F}}_p}$;*

- *If $p \equiv 19 \bmod 60$, two exceptional Igusa parts are copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$;*

- *If $p \equiv 29 \bmod 60$, two Igusa parts are copies of $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$;*

- *If $p \equiv 31 \bmod 60$, two Igusa parts are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$ and two are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_{10})_{\overline{\mathbb{F}}_p}$;*

- *If $p \equiv 41 \bmod 60$, two Igusa parts are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$ and two are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_{10})_{\overline{\mathbb{F}}_p}$;*

- *If $p \equiv 49 \bmod 60$, two Igusa parts are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_4)_{\overline{\mathbb{F}}_p}$ and two are $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_6)_{\overline{\mathbb{F}}_p}$.*

Singular points located on components of shape $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_r)_{\overline{\mathbb{F}}_p}$ have local equations

$$W[[x, y]]/(xy - \pi^r), r = 2, 4, 6 \text{ or } 10$$

for π as usual a uniformizer of W (e.g. $\pi = (1 - \zeta_p)^{1/(p+1)}$).

Proof   As $\mathfrak{A}_5$ in fact belongs to $\mathrm{SL}_2(\mathbb{F}_p)/\{\pm 1\}$, the image under the determinant of the full group $\Gamma_{\mathfrak{A}_5}(p)$ consists in all the squares of $\mathbb{F}_p^*$. The quotient of the set of vertical Igusa components, indexed by $\mathbb{F}_p^* \times \mathbb{P}^1(\mathbb{F}_p)$, therefore has twice the number of elements as indicated in the list above, depending on the class of p mod 60.

Igusa components associated with orbits of size $n = 60, 30, 20$ or 12, have stabilizer $C_{(120/n)}$ in $\Gamma_{\mathfrak{A}_5}(p)$ so they are isomorphic to $\overline{\mathcal{M}}(\mathcal{P}, \mathrm{Ig}(p)/C_{(120/n)})_{\overline{\mathbb{F}}_p}$. The rest goes as in the proof of the previous theorems. $\square$

As for Drinfeld components: using Theorem 6.1 of [4], we can take as a system of generators for the image of $\mathfrak{A}_5$ in $\mathrm{SL}_2(\overline{\mathbb{Z}})/\{\pm 1\}$ any set (S, T) whose traces satisfy

$$t_S^2 + t_T^2 + t_{ST}^2 - t_S t_T t_{ST} \in \{2 + \mu, 3, 2 - \mu^{-1}\} \text{ and } t_S, t_T, t_{ST} \in \{0, \pm\mu, \pm 1, \pm\mu^{-1}\}, \text{ for } \mu = \frac{1 + \sqrt{5}}{2}.$$

Taking $p = 421$, we can choose S and T as the reduction of the generators displayed in the introduction to [4], that is

$$S = \begin{pmatrix} 211 & 196 \\ 316 & 100 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 100 & 70 \\ 306 & 210 \end{pmatrix}$$

in $\mathrm{SL}_2(\mathbb{F}_{421})/\{\pm 1\}$. Drawing the graph of the homographic action of S and T on the elements 0 and 1 in $\mathbb{P}^1(\mathbb{F}_{421})$ yields the respective orbits

$A := \mathfrak{A}_5 \cdot 0 = \{0, 2, 3, 14, 17, 20, 29, 50, 51, 55, 72, 83, 94, 101, 146, 152, 153, 156, 163, 166, 177,$
$182, 190, 191, 192, 203, 206, 209, 210, 211, 212, 215, 218, 220, 222, 225, 230, 234, 236, 242, 250,$
$257, 264, 266, 279, 284, 293, 319, 326, 335, 343, 352, 355, 357, 359, 392, 396, 418, 419, \infty\}$

and

$B := \mathfrak{A}_5 \cdot 1 = \{1, 5, 23, 25, 26, 27, 35, 40, 60, 61, 81, 92, 93, 105, 107, 115, 127, 128, 137, 143,$
$154, 159, 160, 164, 172, 173, 189, 193, 195, 202, 223, 227, 233, 235, 243, 246, 252, 256, 259, 273,$
$274, 289, 294, 306, 323, 324, 325, 327, 348, 350, 361, 363, 370, 373, 374, 379, 382, 388, 389, 409\}.$

Setting $\phi(t) = \prod_{a \in A \setminus \{\infty\}} (t-a)^2 \prod_{a \in B} (t-b)^{-1}$ one computes

$$\phi(\mathbb{P}^1(\mathbb{F}_{421})) = \{0, 23, 47, 144, 161, 228, 292, 317, \infty\}$$

with ramification indices: $1, 2, 1, 3, 1, 1, 1, 5$ and 1 respectively. The list of their inverse mod 211 is $1, 106, 1, 141, 1, 1, 1, 169$ and 1, so that an equation as in (38) for the generic $\mathfrak{A}_5$-Drinfeld component in level 421 is finally

$$u^{211} = t(t-23)^{106}(t-47)(t-144)^{141}(t-161)(t-228)(t-292)(t-317)^{169}.$$

6 Decomposition of semiabelian jacobians

We can use the explicit description of our semistable models to write up explicitly the decomposition, up to isogeny, of the neutral component of the Néron model of their jacobian above the bad prime p, using Raynaud's theorem about Picard functors. The only non straightforward fact concerns the jacobian of the Drinfeld components, and it will be enough to deal with those of $X(p)^{\mathrm{st}}$.

Proposition 6.1 *The Drinfeld components of $X(p)^{\mathrm{st}}$ are supersingular.*

Proof From (4) we know that those horizontal components have geometric projective equation $X^p Y - XY^p = Z^{p+1}$. Hurwitz formula shows their genus is $g = \frac{1}{2}p(p-1)$. Considering the form H_s given by the equation $X^p Y - XY^p = aZ^{p+1}$, for $a \in \mathbb{F}_{p^2}$ some non-trivial square root of an element in $\mathbb{F}_p$, one checks that the number $\#H_s(\mathbb{F}_{p^2})$ of points of H_s with values in $\mathbb{F}_{p^2}$ is $(p^3 + 1)$. (This is most easily done by computing the fibers of the projection of H_s to the x-line.) This therefore means that H_s is *maximal over* $\mathbb{F}_{p^2}$:

$$\#H_s(\mathbb{F}_{p^2}) = 1 + p^2 + 2pg$$

is the maximum allowed by Weil's bound. Now if $(a_i)_{1 \leq i \leq 2g}$ is the set of eigenvalues of the Frobenius endomorphism of $\mathrm{Jac}(H_s)$ over $\mathbb{F}_p$, we have $\#H_s(\mathbb{F}_{p^2}) = 1 + p^2 - \sum_{i=1}^{2g} a_i^2$, and Riemann's hypothesis $|a_i| \leq \sqrt{p}$ implies that $a_i^2 = -p$, so that Frobenius has characteristic polynomial $(X^2 + p)^g$. The latter is the characteristic polynomial of Frobenius on E^g, for E a supersingular elliptic curve over $\mathbb{F}_p$. $\square$

This implies

Corollary 6.2 *Above p, the (non-rational) horizontal components of our semistable models for $X_{\mathrm{s}}(p)$, $X_{\mathrm{ns}}(p)$, $X_{\mathrm{s}}^+(p)$, and $X_{\mathrm{ns}}^+(p)$ are all supersingular.*

Remark 6.3 The horizontal components of Weinstein's stable model for $X(p^n N)$ in [20] are easily seen to be supersingular (if not projective lines) in the same fashion. (By the way, one can check that the curves labelled 2. $xy^p - x^p y = 1$ and 3. $y^p + y = x^{p+1}$ of Theorem 1.1 in loc. cit. are $\mathbb{F}_{p^3}$-forms of each other, and that 4. $y^p - y = x^2$ is a quotient of the former.)

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