$\mathbf{GL_2}$. MODULAR FORMS, CLASSICALLY, AND $\mathbf{GL_2(A)}$

BAS EDIXHOVEN

Contents

Reference: Deligne's "Formes Modulaires et representations de $\mathrm{GL}_2(\mathbf{A})$" in Antwerp II, Springer Lecture Notes in Mathematics 239; and, pages 18 and 19 from Taylors long ICM text "Galois Representations" (available from his homepage). It is just 21 lines, but there is a lot to say...

Aim: To understand the 21 lines by Taylor.

1. The Harish-Chandra Isomorphism for $\mathfrak{gl}_2$

Recall that $\mathfrak{gl}_2 := \mathrm{Lie}(\mathrm{GL}_2(\mathbf{C}))$ is the Lie algebra $\mathrm{M}_2(\mathbf{C})$ with Lie-brackets

$$[X,Y] := XY - YX, \quad \forall X,Y \in \mathrm{M}_2(\mathbf{C}).$$

We pick the following $\mathbf{C}$-basis of $\mathfrak{gl}_2$:

$$c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad a_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The *universal enveloping algebra* $\mathrm{U}(\mathfrak{gl}_2)$ of $\mathfrak{gl}_2$ is

$$T(\mathfrak{gl}_2)/\langle a\otimes b - b\otimes a - [a,b] : a,b \in \mathfrak{gl}_2\rangle,$$

where $\mathrm{T}(\mathfrak{gl}_2) = \bigoplus_{\mathrm{n}\in\mathbf{N}} \mathfrak{gl}_2^{\otimes \mathrm{n}}$ is the tensor algebra. The algebra $\mathfrak{gl}_2$ maps into $\mathrm{T}(\mathfrak{gl}_2)$ by mapping $\mathfrak{gl}_2$ into $\mathfrak{gl}_2^{\otimes 1} \subset T(\mathfrak{gl}_2)$ via the identity map, and then to $\mathrm{U}(\mathfrak{gl}_2)$ via the quotient map $T(\mathfrak{gl}_2) \to \mathrm{U}(\mathfrak{gl}_2)$.

The algebra $\mathrm{U}(\mathfrak{gl}_2)$ satisfies the following properties

(1) the map $\mathfrak{gl}_2 \to \mathrm{U}(\mathfrak{gl}_2)$ transforms Lie-brackets into commutators;
(2) the algebra $\mathrm{U}(\mathfrak{gl}_2)$ is associative;
(3) the algebra $\mathrm{U}(\mathfrak{gl}_2)$ has a unit element,

and $\mathrm{U}(\mathfrak{gl}_2)$ is universal among all such algebras. In particular there is an isomorphism of categories

$$\{\mathfrak{gl}_2\text{-modules}\} \xrightarrow{\sim} \{\mathrm{U}(\mathfrak{gl}_2)\text{-modules}\},$$

because for every $\mathbf{C}$-vector space A

$$\mathrm{Hom}_{\text{Lie-algebras}/\mathbf{C}}(\mathfrak{gl}_2, \mathrm{End}(A)) = \mathrm{Hom}_{\mathbf{C}\text{-algebras}}(\mathrm{U}(\mathfrak{gl}_2), \mathrm{End}(A)).$$

Therefore the center $\mathfrak{z}_2$ of $\mathrm{U}(\mathfrak{gl}_2)$ is the endomorphism ring of the functor $\mathrm{id}_{\mathfrak{gl}_2\text{-modules}}$.

The Harish-Chandra isomorphism is an isomorphism of $\mathbf{C}$-algebras

$$\gamma_{HC} \colon \mathfrak{z}_2 \xrightarrow{\sim} \mathbf{C}[x_1, x_2]^{S_2} \quad (S_2\text{-acts by interchanging } x_1 \text{ and } x_2),$$

characterized as follows. We need the concept "Highest Weight Representations". Let

$$B = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \quad T = \{\left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)\} \cong \mathbf{G}_m^2, \quad W = N_T/T = S_2.$$

Under the action of T on $\mathfrak{gl}_2$ it holds $t \cdot a_+ = t_1 t_2^{-1} a_+$, so $(1, -1)$ is the positive root. It holds

$$X(T) = \mathrm{Hom}(T, \mathbf{G}_m) = \mathbf{Z}^2,$$

and the positive part is $\{a \in \mathbf{Z}^2 | a_1 \geq a_2\} \subset \mathbf{Z}^2$.

Theorem 1.1. *For all positive roots $a \in \mathbf{Z}^2$ there exists a unique (up to isomorphism) irreducible algebraic representation ρ_a of $\mathrm{GL}_{2,\mathbf{C}}$ such that a is the highest weight of $\rho_a|_T$.*

Explicitly one may take $\rho_a = \mathrm{Sym}^{a_1 - a_2}(\mathbf{C}^2) \otimes \det^{a_2}$.

The subgroup $T_{\mathbf{C}} \subset \mathrm{GL}_{n,\mathbf{C}}$ acts on the space of ρ_a via a character, and this character is given by

$$t = (t_1, t_2) \mapsto (t_1^{a_1 - a_2} + t_1^{a_1 - a_2 - 1} t_2 + \cdots + t_2^{a_1 - a_2}) \cdot t_1^{a_2} t_2^{a_2}.$$

Note $\mathrm{Sym}^d(\mathbf{C}^2) = \mathrm{Sym}^d(\mathbf{C}x \oplus \mathbf{C}y) = \mathbf{C}[x, y]_d$.

Now a characterisation of γ_{HC}. For all $a \in \mathbf{Z}^2$, positive, the diagram

$$\begin{array}{ccccc} \mathfrak{z}_2 & \xrightarrow{\gamma_{HC}} & \mathbf{C}[x_1, x_2]^{S_2} & \qquad & f \\ & \underset{\rho_a}{\searrow} & \downarrow{\scriptstyle \theta_{H(\rho_a)}} & & \downarrow \\ & & \mathbf{C} & & f(a_1 + \frac{1}{2}, a_2 - \frac{1}{2}), \end{array}$$

commutes. The subset

$$\{a + \frac{1}{2}(1, -1) | a \text{ positive in } \mathbf{Z}^2\} \subset \mathbf{C}^2$$

is Zariski dense, and so there can only be one isomorphism γ_{HC} fitting in the diagram above.

But this doesn't tell us what γ_{HC} or $\mathfrak{z}_2$ is. Of course we have $c = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathfrak{gl}_2 \subset \mathrm{U}(\mathfrak{gl}_2)$. As $\mathrm{Lie}(\mathrm{Center}(\mathrm{GL}_2(\mathbf{C}))) = \mathbf{C} \cdot c$ it holds $c \in \mathfrak{z}_2$. We need 1 more element in $\mathfrak{z}_2$.

Some generalities. Let G a complex affine algebraic group, $A := \mathcal{O}_G(G)$ the ring of global functions, $\mathfrak{m} \in \mathrm{Spec}(A)$ the maximal ideal corresponding to $e \in G(\mathbf{C})$. Then

$$\begin{aligned} \mathrm{U}(\mathrm{Lie}(G)) &= \{\text{left invariant differential operators on } G\} \\ &= \text{"point distributions at } e\text{"} = \varinjlim_n (A/\mathfrak{m}^n)^\vee, \end{aligned}$$

with multiplication on $\varinjlim_n (A/\mathfrak{m}^n)^\vee$ coming from the multiplication map $\mu\colon G \times G \to G$ on G in the following way. Let $\mu^*\colon A \to A \otimes A$ be the comultiplication (obtained from μ by applying the global sections functor to μ). The map $A \otimes A \to A/\mathfrak{m}^n \otimes A/\mathfrak{m}^n$ factors over $A \otimes A/(A \otimes \mathfrak{m} + \mathfrak{m} \otimes A)^{2n}$. [To see this, the ideal $(A \otimes \mathfrak{m} + \mathfrak{m} \otimes A)^{2n} \subset A \otimes A$ is generated by elements $t = \sum_{i+j=2n} x^i y^j$ with $x \in A \otimes \mathfrak{m}$ and $y \in \mathfrak{m} \otimes A$. All the terms in the sum t have either $i \geq n$ or $j \geq n$.] The maximal ideal of $A \otimes A$ corresponding to (e,e) is $\mathfrak{m} \otimes A + A \otimes \mathfrak{m}$, and μ maps (e,e) to e, so $\mu^*(\mathfrak{m}_e) \subset \mathfrak{m} \otimes A + A \otimes \mathfrak{m}$. We thus have the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A \otimes A \\ \downarrow & & \downarrow \\ A/\mathfrak{m}^{2n} & \longrightarrow & (A \otimes A)/(\mathfrak{m} \otimes A + A \otimes \mathfrak{m})^{2n} \\ & \searrow & \downarrow \\ & & A/\mathfrak{m}^n \otimes A/\mathfrak{m}^n. \end{array}$$

Take duals to find maps $(A/\mathfrak{m}^n)^\vee \otimes (A/\mathfrak{m}^n)^\vee \to (A/\mathfrak{m}^{2n})^\vee$; they define the multiplication on $\varinjlim_n (A/\mathfrak{m}^n)^\vee$.

Anyway $\mathrm{U}(\mathrm{Lie}G)$ is a filtered ring, and $\mathrm{Fil}_i \,\mathrm{U}(\mathrm{Lie}G)/\mathrm{Fil}_{i+1} \,\mathrm{U}(\mathrm{Lie}G) = \mathrm{Sym}^i(\mathrm{Lie}G)$ for all $i \in \mathbf{N}$.

By construction,

$$Z(\mathrm{U}(\mathrm{Lie}G)) = \mathrm{U}(\mathrm{Lie}(G))^{\mathrm{Lie}G} = \mathrm{U}(\mathrm{Lie}(G))^{G^\circ}.$$

For $G = \mathrm{GL}_n = \mathrm{GL}(V)$, with $V = \mathbf{C}^n$:

$$\mathrm{Lie}(G) = \mathrm{End}(V) = V^\vee \otimes V = (V^\vee \otimes V)^\vee = \mathrm{End}(V)^\vee,$$

and

$$\begin{aligned} \mathrm{Sym}(\mathrm{End}(V)^\vee)^{\mathrm{GL}_n} &= \{\text{conjugation invariant polynomials on } \mathrm{End}(V)\} \\ &= \mathbf{C}[\text{coefficients of characteristic polynomial}]. \end{aligned}$$

In particular, in case $n = 2$, there is an element C in degree ≤ 2 such that c and C generate $\mathfrak{z}_2$.

Recipe for the Casimir operator C: Take the *Killing form* $\langle \cdot, \cdot \rangle$ on $\mathfrak{sl}_2$ given by $\langle a, b \rangle = \mathrm{tr}((\mathrm{ad}\, a), (\mathrm{ad}\, b))$; take any basis $(e_i)_{i \in I}$ of $\mathfrak{sl}_2$, then $C = \sum_{i \in I} e_i^\vee \otimes e_i$, where $(e_i^\vee)_{i \in I}$ is the dual basis with respect to the Killing form. We use $\mathfrak{sl}_2 = \mathbf{C} \cdot h \oplus \mathbf{C} \cdot a_+ \oplus \mathbf{C} \cdot a_-$. We have:

$$[h, a_+] = 2a_+, \quad [h, a_-] = -2a_-, \quad [a_+, a_-] = h,$$

and the matrix of $\langle \cdot, \cdot \rangle$ is

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix},$$

so

$$h^\vee = \frac{1}{8} h, \quad a_+^\vee = \frac{1}{4} a_-, \quad a_-^\vee = \frac{1}{4} a_+.$$

So the Casismir operator is $\frac{1}{4} C$, with $C = \frac{1}{2} h^2 + a_+ a_- + a_- a_+$. We have $\mathfrak{z}_2 = \mathbf{C}[c, C]$.

For $a \in \mathbf{Z}^2$ positive, $V_a = \mathbf{C}[x,y]_{a_1-a_2} \otimes (\mathbf{C}\cdot x\wedge y)^{\otimes a_2}$, $\mathfrak{gl}_2$ acts

$$\begin{array}{lllll} \text{on } \mathbf{C}[x,y] \text{ as} & c : x\partial_x + y\partial_y, & h : x\partial_x - y\partial_y, & a_+ : x\partial_y, & a_- : y\partial_x \\ \text{on } \mathbf{C}\cdot x\wedge y \text{ as} & c : 2, & h : 0, & a_+ : 0, & a_- : 0, \end{array}$$

c acts on V_a as $a_1 + a_2$, and C acts on V_a as

$$(\frac{1}{2}(x\partial_x + y\partial_y)(x\partial_x + y\partial_y + 2)) \otimes 1 = \frac{1}{2}(a_1 - a_2)(a_1 - a_2 + 2) = \frac{1}{2}(a_1 - a_2)^2 + a_1 - a_2.$$

Now we can make the Harish Chandra isomorphism explicit. For all positive roots $a \in \mathbf{Z}^2$ the map ρ_a sends c to $a_1 + a_2$ and C to $\frac{1}{2}(a_1 - a_2)(a_1 - a_2 + 2)$. Hence γ_{HC} is given by

$$\begin{cases} c \mapsto x_1 + x_2 \\ C \mapsto \frac{1}{2}(x_1 - x_2 - 1)(x_1 - x_2 + 1) = \frac{1}{2}((x_1 - x_2)^2 - 1). \end{cases}$$

2. Very Classical Picture

Remember $\mathbf{H}$ is the space of $z \in \mathbf{C}$ with $\Im(z) > 0$, equipped with the induced topology.

Definition 2.1. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ a congruence subgroup[1], and $k \in \mathbf{Z}$. The *space of cuspidal modular forms of weight* k, notation $S_k(\Gamma)$, is the $\mathbf{C}$ vectorspace of complex valued functions $f\colon \mathbf{H} \to \mathbf{C}$ satisfying:

(1) f is holomorphic;
(2) $\forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \forall \tau \in \mathbf{H} : f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$;
(3) f vanishes at the cusps: $\forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$:

$$\left(\tau \mapsto (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right)\right) \longrightarrow 0 \quad \text{as } \Im(\tau) \longrightarrow \infty.$$

What does the condition "$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$" mean? (We want something more conceptual). On $\mathbf{H}$ we have

$$\begin{array}{ccccc} \mathbf{Z}^2 \times \mathbf{H} & \longrightarrow & \mathbf{C} \times \mathbf{H} & \longrightarrow & \mathbf{E} \\ & \searrow & \downarrow & \underset{0}{\rightleftarrows} & \\ & & \mathbf{H} & & \end{array}$$

The group $\mathrm{SL}_2(\mathbf{Z})$ acts on the left. To find the action, consider $\tau \in \mathbf{H}$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{Z})$. Then

$$\begin{array}{lcl} \mathbf{E}_\tau & = \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}) = \mathbf{C}/(\mathbf{Z}(a\tau+b) + \mathbf{Z}(c\tau+d)) & (z,\tau) \\ \downarrow \cong & & \downarrow \\ \mathbf{E}_{\gamma\tau} & = \mathbf{C}/\left(\mathbf{Z}\frac{a\tau+b}{c\tau+d} + \mathbf{Z}\right) & \left(\frac{1}{c\tau+d}z, \frac{a\tau+b}{c\tau+d}\right). \end{array}$$

[1]Congruence subgroup means that it contains the subgroup $\Gamma(N)$ for some $N \in \mathbf{Z}_{>1}$, where $\Gamma(N) := \ker(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}))$

Note:

$$\left(\begin{pmatrix} n \\ m \end{pmatrix}, \tau\right) \mapsto n\tau + m = \begin{pmatrix} n \\ m \end{pmatrix}^t \cdot \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \left(\gamma^{-1,t} \cdot \begin{pmatrix} n \\ m \end{pmatrix}\right)^t \cdot \gamma \cdot \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \left(\gamma^{-1,t} \cdot \begin{pmatrix} n \\ m \end{pmatrix}\right)^t \begin{pmatrix} a\tau + b \\ c\tau + d \end{pmatrix}.$$

One finds:

$$\left(\begin{pmatrix} n \\ m \end{pmatrix}, \tau\right) \overset{\gamma\cdot}{\mapsto} \left(\gamma^{-1,t} \begin{pmatrix} n \\ m \end{pmatrix}, \gamma\tau\right).$$

This gives an invariant $\mathcal{O}_{\mathbf{H}}$-module $\omega = 0^*\Omega^1_{\mathbf{E}/\mathbf{H}}$ with an $\mathrm{SL}_2(\mathbf{Z})$-action. We have a global section: $\mathrm{d}z$ (coming from the standard coordinate z on $\mathbf{C}$), and

$$(\gamma\cdot)^*\mathrm{d}z = \mathrm{d}((\gamma\cdot)^*z) = \mathrm{d}\left(\frac{z}{c\tau + d}\right) = (c\tau + d)^{-1}\mathrm{d}z.$$

Then we have, for $f\colon \mathbf{H} \to \mathbf{C}$:

$$\left((\gamma\cdot)^*\left(f\,(\mathrm{d}z)^{\otimes k}\right)\right)\tau = \left((\gamma\cdot)^*f\right)(\tau)\,(c\tau + d)^{-k} \cdot (\mathrm{d}z)^{\otimes k},$$

for $k \in \mathbf{Z}$, $\tau \in \mathbf{H}$, $\gamma \in \mathrm{SL}_2(\mathbf{Z})$. Hence

$$(\gamma\cdot)^*f(\mathrm{d}z)^{\otimes k} = f(\mathrm{d}z)^{\otimes k} \iff \forall \tau \in \mathbf{H} : f(\gamma\tau)(c\tau + d)^{-k} = f(\tau).$$

So condition (2) for $f \in S_k(\Gamma)$ means that $f(\mathrm{d}z)^{\otimes k}$ is a Γ-invariant global section of $\omega^{\otimes k}$:

$$f(\mathrm{d}z)^{\otimes k} \in (\omega^{\otimes k}(\mathbf{H}))^\Gamma.$$

If Γ acts freely on $\mathbf{H}$, then the quotient of $\mathbf{E}/\mathbf{H}$ by Γ together with the 0-section,

$$\begin{array}{c} \mathbf{E}(\Gamma) \\ \downarrow\uparrow 0 \\ Y(\Gamma), \end{array}$$

is the universal elliptic curve with Γ-level structure. And it holds:

$$S_k(\Gamma) = \{f \in \omega^{\otimes k}_{\mathbf{E}(\Gamma)/Y(\Gamma)}(Y(\Gamma)) | f \text{ extends over the cusps, and vanishes there}\}.$$

3. Functions of Lattices

The formulas in the preceding section are ugly! After Deligne, consider:

$$G := \{\varphi\colon \mathbf{Z}^2 \longrightarrow \mathbf{C} | \varphi \text{ is } \mathbf{Z}\text{-linear}, \varphi(\mathbf{Z}^2) = \text{lattice}\} \subset \mathbf{C}^2,$$

as an open 2-dimensional $\mathbf{C}$-submanifold of $\mathbf{C}^2$. It holds:

$$G \xrightarrow{\sim} \mathrm{Isom}_{\mathbf{R}\text{-mod}}(\mathbf{R}^2, \mathbf{C}) \xrightarrow{\sim} \mathrm{GL}_2(\mathbf{R}).$$

The group $\mathrm{GL}_2(\mathbf{Z}) \times \mathbf{C}^*$ acts on G. The action of $\mathbf{C}^*$ on G is free with quotient

$$q\colon G \longrightarrow \mathbf{H}^{\pm}, \quad \varphi \mapsto (\varphi e_1 : \varphi e_2) = \varphi e_1/\varphi e_2.$$

The quotient q has a section: $\tau \mapsto (\tau, 1)$.

Note:

$$G = \{(L,\varphi) | L \subset \mathbf{C} \text{ lattice }, \varphi\colon \mathbf{Z}^2 \xrightarrow{\sim} L\}$$
$$= (V, L, \varphi, \alpha) | V \text{ 1-dim. } \mathbf{C}\text{-v.sp.}, \alpha\colon \mathbf{C} \xrightarrow{\sim} V, L \subset V \text{ lattice}, \varphi\colon \mathbf{Z}^2 \xrightarrow{\sim} L\}/\cong$$
$$\mathrm{GL}_2(\mathbf{Z})\backslash G/\mathbf{C}^* = \{(V,L)\}/\cong \{E/\mathbf{C} \text{ elliptic curve}\}/\cong .$$

Then, for $k \in \mathbf{Z}$, $\Gamma \subset \mathrm{GL}_2(\mathbf{Z})$ the space

$$\left\{ f\colon G \longrightarrow \mathbf{C} \;\middle|\; \begin{array}{ll} f(\varphi\gamma^t) = f(\varphi) & \forall \varphi \in G, \forall \gamma \in \Gamma \\ f(\lambda\varphi) = \lambda^{-k} f(\varphi) & \forall \varphi \in G, \lambda \in \mathbf{C}^* \end{array} \right\},$$

is identified with the space

$$\left\{ f\colon \mathbf{H}^{\pm} \longrightarrow \mathbf{C} \;\middle|\; \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma, \forall \tau \in \mathbf{H} : f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \right\},$$

via

$$\begin{array}{ccc} f & & (f \circ q)\cdot(\phi \mapsto \phi(e_2))^{-k} \\ \downarrow & & \uparrow \\ (\tau \mapsto f(\tau, 1)) & & f, \end{array}$$

4. The Shimura Datum Picture

For $K \subset \mathrm{GL}_2(\mathbf{A}^\infty)$ a compact open subgroup, we define:

$$Y_K := \mathrm{GL}_2(\mathbf{Q})\backslash(\mathbf{H}^{\pm} \times \mathrm{GL}_2(\mathbf{A}^\infty)/K).$$

Example 4.1.

$$Y_{\mathrm{GL}_2(\hat{\mathbf{Z}})} = \mathrm{GL}_2(\mathbf{Q})\backslash(\mathbf{H}^{\pm} \times \mathrm{GL}_2(\mathbf{A}^\infty)/\mathrm{GL}_2(\hat{\mathbf{Z}}))$$
$$\stackrel{*}{=} \mathrm{GL}_2(\mathbf{Z})\backslash\mathbf{H}^{\pm} = \mathrm{SL}_2(\mathbf{Z})\backslash\mathbf{H} = \{\mathrm{E}/\mathbf{C}\}/\cong$$

To see the equality marked $*$ note that $\mathrm{GL}_2(\mathbf{Q})$ acts transitively on the set of $\hat{\mathbf{Z}}$-lattices in $(\mathbf{A}^\infty)^2$.

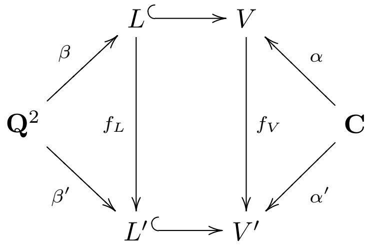
We want to interpret the Y_K as moduli spaces. First analytically. Fix an $\mathbf{R}$-basis of $\mathbf{C}$, either $(i,1)$, or $(1,i)$ to get an isomorphism $\mathrm{GL}_2(\mathbf{R}) \cong \mathrm{Isom}_{\mathbf{R}\text{-vsp}}(\mathbf{C},\mathbf{R}^2)$. Hence:

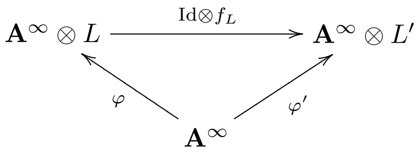
$$\mathrm{GL}_2(\mathbf{A}) \xrightarrow{\cong} \mathrm{Isom}_{\mathbf{R}\text{-vsp}}(\mathbf{C},\mathbf{R}^2) \times \mathrm{GL}_2(\mathbf{A}^\infty) \underset{\Phi_1}{\overset{\Phi_2}{\rightleftarrows}} \{(V,L,\alpha,\beta,\varphi)\}/\cong,$$

where the space on the right is the set of isomorphism classes of tuples $(V, L, \alpha, \beta, \varphi)$, where

- V is a 1-dimensional $\mathbf{C}$-vector space;
- $L \subset V$ is a 2-dimensional $\mathbf{Q}$-vector space, such that $\mathbf{R}L = V$;
- $\alpha\colon \mathbf{C} \to V$ an isomorphism;
- $\beta\colon \mathbf{Q}^2 \to L$ an isomorphism;
- $\varphi\colon (\mathbf{A}^\infty)^2 \to \mathbf{A}^\infty \otimes_{\mathbf{Q}} L$ an isomorphism.

An isomorphism of two tuples $(V, L, \alpha, \beta, \varphi)$ and $(V', L', \alpha', \beta', \varphi')$ is two commutative diagrams:





The maps Φ_i are given by

$$\Phi_1\colon (g_\infty, g^\infty) \mapsto (\mathbf{C}, g_\infty^{-1}\mathbf{Q}^2, \mathrm{id}_\mathbf{C}, g_\infty^{-1}|_{\mathbf{Q}^2}, (\mathrm{id}_{\mathbf{A}^\infty} \otimes g_\infty^{-1}|_{\mathbf{Q}^2}) \circ g^\infty),$$
$$\Phi_2\colon (V, L, \alpha, \beta, \varphi) \mapsto \left((\mathrm{id}_\mathbf{R} \otimes \beta)^{-1} \circ \alpha, (\mathrm{id}_{\mathbf{A}^\infty} \otimes \beta)^{-1} \circ \varphi\right).$$

Consequence: Y_K is the set of isomorphism classes of triples (V, L, φ), where V is a 1-dimensional $\mathbf{C}$-vector space, $L \subset V$ is a 2-dimensional $\mathbf{Q}$-vector space such that $\mathbf{R}L = V$, and where $\varphi \in \mathrm{Isom}((\mathbf{A}^\infty)^2, \mathbf{A}^\infty \otimes L)/K$.

What kind of category do the (V, L) give us?? Answer complex elliptic curves "up to isogeny": $\mathbf{Q} \otimes \mathrm{Ell}(\mathbf{C})$, i.e. the category whose objects are elliptic curves and the Hom-sets are tensorred with $\mathbf{Q}$.

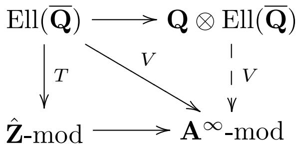
The category $\mathrm{Ell}(\mathbf{C})$ is equivalent with the category of pairs (V, L_0), where V is a 1-dimensional $\mathbf{C}$-vector space and $L_0 \subset V$ is a lattice. We have

$$\mathrm{Hom}((V, \mathbf{Q} \otimes L_0), (V', \mathbf{Q} \otimes L_0') = \mathbf{Q} \otimes \mathrm{Hom}((V, L_0), (V', L_0')).$$

Now algebraically:

$$Y_K(\overline{\mathbf{Q}}) = \{(E, \varphi) | E \in \mathrm{Ob}(\mathbf{Q} \otimes \mathrm{Ell}(\overline{\mathbf{Q}})), \varphi \in \mathrm{Isom}((\mathbf{A}^\infty)^2, V(E))/K\}/\cong$$

We have



For $K \subset \mathrm{GL}_2(\hat{\mathbf{Z}})$, $\varphi \in \mathrm{Isom}((\mathbf{A}^\infty)^2, V(E))/K$ the space $\varphi\hat{\mathbf{Z}}^2 \subset V(E)$ is a $\hat{\mathbf{Z}}$-lattice and gives a genuine elliptic curve E' such that

$$T(E') = \varphi\hat{\mathbf{Z}}^2,$$

in $V(E)$.

Now modular forms in this perspective. For $k \in \mathbf{Z}$ and $K \subset \mathrm{GL}_2(\mathbf{A}^\infty)$ the space $\mathcal{S}_k(\Gamma)$ is the set of functions $f\colon \mathrm{GL}_2(\mathbf{A}) \to \mathbf{C}$ such that

(1) f is holomorphic;
(2) For all $\lambda \in \mathbf{C}^\times$ we have $f \circ (\cdot\lambda) = \lambda^k f$;
(3) f is $\mathrm{GL}_2(\mathbf{Q})$-invariant on the left;
(4) f is right K-invariant;
(5) f is of moderate growth and zero in the cusps.

To see this last property, f extends over the cusps of $Y_{\mathbf{C}}$, and vanishes there.

For $n \in \mathbf{Z}_{\geq 1}$, let $K_{1,n}$ be defined by the Cartesian diagram

$$\begin{array}{ccc} \mathrm{GL}_2(\hat{\mathbf{Z}}) & \twoheadrightarrow & \mathrm{GL}_2(\mathbf{Z}/n\mathbf{Z}) \\ \uparrow & & \uparrow \\ K_{1,n} & \twoheadrightarrow & \{\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)\} \end{array}$$

Then $Y_{K_{1,n}} = Y(\Gamma_1(n))$. Hence for $k \in \mathbf{Z}$ we have $\mathcal{S}_k(\Gamma_1(n)) = \mathcal{S}_k(K_{1,n})$ and we have

$$\mathcal{S}_k(\Gamma_1(n)) \hookrightarrow \varinjlim_K \mathcal{S}_k(K) =: \mathcal{S}_k = \{f \colon \mathrm{GL}_2(\mathbf{A}) \longrightarrow \mathbf{C} : \text{above list}\}.$$

Important point: $\mathrm{GL}_2(\mathbf{A}^\infty)$ acts on $\mathcal{S}_k$.

For $K \subset \mathrm{GL}_2(\mathbf{A}^\infty)$ open compact subgroup: $\mathcal{S}_k(K)$ equals $(\mathcal{S}_k)^K$; the $\mathrm{GL}_2(\mathbf{A}^\infty)$-action gives Hecke operators on the $\mathcal{S}_k(K)$ (via $\mathbf{C}[K_1 \backslash \mathrm{GL}_2(\mathbf{A}^\infty)/K_2]$).

Suppose now that f is a normalised weight k newform of level n: $a_1(f) = 1$, and its system of eigenvalues does not occur in a level $< n$ form. Then f generates an irreducible $\mathrm{GL}_2(\mathbf{A}^\infty)$-submodule of $\mathcal{S}_k$,

$$V_f^\infty := \langle \{gf | g \in \mathrm{GL}_2(\mathbf{A}^\infty)\} \rangle,$$

$V_f^\infty = \bigotimes_p{}' V_f, p$, where $V_{f,p} \subset \varinjlim_{K \subset \mathrm{GL}_2(\mathbf{Q}_p)} \mathcal{S}_k(K \times K_{1,n}^p)$ is the subrepresentation of $\mathrm{GL}_2(\mathbf{Q}_p)$ generated by f.

5. GL_2 and modular forms

Extra reference: S. Kudla, From modular forms to automorphic representations in a book "An introduction to the Langlands program". But it only gives formulas...

Let G be the set $\mathrm{Isom}_{\mathbf{R}\text{-vsp}}(\mathbf{C}, \mathbf{R}^2)$. The group $\mathrm{GL}_2(\mathbf{R})$ acts on the left on this set, and the group $\mathrm{GL}_{\mathbf{R}}(\mathbf{C})$ acts on the right on this set. Note that $\mathrm{GL}_{\mathbf{R}}(\mathbf{C})$ contains the group $\mathbf{C}^\times$, so the group $\mathbf{C}^\times$ has a right action on G.

The quotient for the $\mathbf{C}^\times$-action on G is given by

$$q \colon G \longrightarrow \mathbf{H}^\pm, \quad \varphi \mapsto \varphi^{-1}e_1/\varphi^{-1}e_2, \tag{(.1)}$$

where $\mathbf{H}^\pm := \{z \in \mathbf{C} | \Im(z) \neq 0\}$.

We can give a moduli interpretation of this map in the following manner. The set G is also the set of isomorphism classes of tuples (V, L, ψ, α), where V is a one-dimensional $\mathbf{C}$-vectorspace, $L \subset V$ a $\mathbf{Z}$-lattice, ψ an isomorphism from $\mathbf{Z}^2$ to L, and $\alpha \colon \mathbf{C} \xrightarrow{\sim} V$ an isomorphism of $\mathbf{C}$-vectorspaces.

The set $\mathbf{H}^{\pm}$ is the set of isomorphism classes of tuples (V, L, ψ) (same definitions as above). And under this interpretation the quotient map q in (.1) is the map sending a tuple (V, L, ψ, α) to (V, L, ψ).

Remark 5.1. On $\mathbf{H}^{\pm}$ we have the line bundle $\omega := 0^*\Omega^1_{\mathbf{E}/\mathbf{H}^{\pm}}$, which is $\mathrm{GL}_2(\mathbf{R})$-equivariant, but it has no invariant trivialisation (stabilisers act non-trivially). But, on G, α gives us a $\mathrm{GL}_2(\mathbf{R})$-invairant trivilisation of ω. It is this that makes it possible to describe modular forms as functions, and not as sections of a line bundle.

Concretely, let $f \in \mathcal{S}_k(\Gamma)$, where $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ is a congruence subgroup. Important: Here we view f as a modular form on the double half plane $\mathbf{H}^{\pm} \subset \mathbf{C}$. Let $\varphi \in G$, let $w_1 := \varphi^{-1}e_1$, $w_2 := \varphi^{-1}e_2$, $\tau := w_1/w_2$ and $L := \mathbf{Z}w_1 \oplus \mathbf{Z}w_2 \subset \mathbf{C}$.

We have the commutative diagram

$$\begin{array}{ccc} \mathbf{C}/L & \xrightarrow{\quad\sim\quad} & \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z}) \\ \uparrow & & \uparrow \\ \mathbf{C} & \xrightarrow[\sim]{\omega_2^{-1}\cdot} & \mathbf{C} \end{array}$$

$$\tilde{f}(\varphi)(\mathrm{d}z)^{\otimes k} := f_{\mathbf{H}}(w_1/w_2)w_2^{-k}(\mathrm{d}z))^{\otimes k} \longleftarrow\!\!\shortmid f_{\mathbf{H}}(\mathrm{d}z)^{\otimes k}$$

The set

$$((.2)) \qquad \left\{\tilde{f}\colon G \longrightarrow \mathbf{C} | \forall\gamma \in \Gamma \forall\varphi \in G | \tilde{f}(\gamma\varphi) = \tilde{f}(\varphi), \forall\lambda \in \mathbf{C}^{\times}, \forall\varphi \in G : \tilde{f}(\varphi\lambda) = \lambda^k \tilde{f}(\varphi)\right\},$$

is in bijection with the set

$$((.3)) \qquad \left\{f_{\mathbf{H}}\colon \mathbf{H}^{\pm} \longrightarrow \mathbf{C} | \forall\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma, \forall\tau \in \mathbf{H}^{\pm} : f_{\mathbf{H}}\left(\frac{a\tau + b}{c\tau + b}\right) = (c\tau + d)^k f_{\mathbf{H}}(\tau)\right\}.$$

This bijection given as follows. Let $\tilde{f}$ be an element of the set (.2), then

$$f_{\mathbf{H}}\colon \mathbf{H}^{\pm} \longrightarrow \mathbf{C}, \quad \tau \mapsto \tilde{f}(\tau \mapsto e_2, 1 \mapsto e_1)),$$

is the corresponding element of the set (.3). Inversely, if $f_{\mathbf{H}}$ is an element of the set (.3), then the corresponding element of the set (.2) is given by

$$\tilde{f} := (q^* f_{\mathbf{H}})w_2^{-k}).$$

We use $i \in \mathbf{H}^{\pm}$ as base point and (hence) $\varphi_0\colon \mathbf{C} \xrightarrow{\sim} \mathbf{R}^2$ with $\varphi_0^{-1}e_1 = i$ and $\varphi_0^{-1}e_2 = 1$ as base point in G. In particular,

$$\mathrm{GL}_2(\mathbf{R}) \xrightarrow{\sim} G, \quad g \mapsto g\varphi_0.$$

Then, for $g \in \mathrm{GL}_2(\mathbf{R})$ we have

$$w_1(g\varphi_0) = ai + b, \quad w_2(g\varphi_0) = ci + d,$$

when $g^{-1,t} = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$. Hence:

$$\tilde{f}(g\varphi_0) = f\left(\frac{ai + b}{ci + d}\right)(ci + d)^{-k}.$$

And let me say it in yet another way how $f \in \mathcal{S}_{k,K} \subset H^0(Y_K, \omega^{\otimes k})$ gives $\tilde{f}$. Let $f(V, L, \alpha, \beta, \varphi) \in (V^\vee)^{\otimes k}$. Then α is an isomorphism $\mathbf{C} \xrightarrow{\sim} V$, and it induces an isomorphism

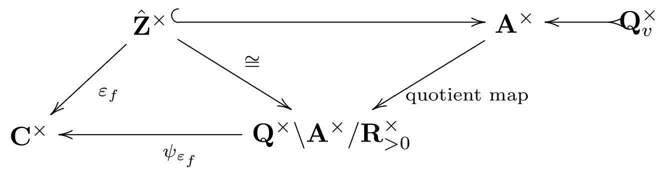
$$(V^\vee)^{\otimes k} \underset{(\alpha^\vee)^{\otimes k}}{\xrightarrow{\sim}} (\mathbf{C}^\vee)^{\otimes k} = \mathbf{C}.$$

Via this isomorphism, we get $\tilde{f}(V, L, \alpha, \beta, \varphi) \in \mathbf{C}$. Clearly,

$$\forall \lambda \in \mathbf{C}^\times : \tilde{f}(V, L, \alpha \circ (\lambda \cdot), \beta, \varphi) = \lambda^k f(V, L, \alpha, \beta, \varphi).$$

Let us compute the action of the center $\mathbf{A}^\times$ on $\mathbf{C} \cdot \tilde{f}$. Let $\varepsilon_f : \hat{\mathbf{Z}}^\times \to \mathbf{C}^\times$ be the character of f.

Consider the commutative diagram



where v is a place of $\mathbf{Q}$. We denote with $\psi_{\varepsilon_f, v}$ the composition

$$\mathbf{Q}_v^\times \longrightarrow \mathbf{A}^\times \longrightarrow \mathbf{Q}^\times\backslash\mathbf{A}^\times/\mathbf{R}_{>0}^\times \longrightarrow \mathbf{C}^\times.$$

Note $\varepsilon_f(-1) = (-1)^k$, hence $\psi_{\varepsilon_f,\infty}(\lambda) = \left(\frac{\lambda}{|\lambda|}\right)^k$. We claim that $\mathbf{R}^\times \subset \mathbf{A}^\times$ acts via $\lambda \mapsto \lambda^k = |\lambda|^k \psi_{\varepsilon_f,\infty}$ and $\mathbf{Q}_p^\times$ via $|\cdot|_p^k \psi_{\varepsilon_f,p}$. Hence $\mathbf{A}^\times$ acts as $||\cdot||^k \psi_{\varepsilon_f}$ and indeed $\mathbf{Q}^\times$ acts trivially.

Here is the proof. Let $f(V, L, \varphi) \in (V^\vee)^{\otimes k}$, $f(V, L, p\varphi) \in (V^\vee)^{\otimes k}$, so

$$(p\cdot)^* : (V^\vee)^{\otimes k} \longrightarrow (V^\vee)^{\otimes k}, \quad f(V, L, p\varphi) \mapsto f(V, L, \varphi),$$

so

$$\prod_{\ell \neq p} \varepsilon_{f,l}(p) \cdot f(V, L, p_p \cdot \varphi) = f(V, L, p \cdot \varphi) = p^{-k} f(V, L, \varphi).$$

So

$$f(V, L, p_p \varphi) = |p|_p^k \cdot \prod_{\ell \neq p} \varepsilon_{f,\ell}(p)^{-1} f(V, L, \varphi).$$

Now comes a strange thing. I thought that $\tilde{f}$ would be in $\mathcal{A}_{\{s,t\}}(\mathrm{GL}_2(\mathbf{Q})\backslash\mathrm{GL}_2(\mathbf{A}))$ for suitable s and t. But computing s and t gives $(s,t) = (k - \frac{1}{2}, \frac{1}{2})$: not integers.

Here's the computation. Recall $\mathrm{Lie}(Gl_2(\mathbf{R}))$; c, h, a_+, a_- and $C = \frac{1}{2}h^2 + a_+a_- + a_-a_+$,

$$\gamma_{\mathrm{HC}} : \mathfrak{z} \xrightarrow{\sim} \mathbf{C}[x_1, x_2]^{\mathfrak{S}_2}$$
$$c \longmapsto x_1 + x_2$$
$$C \mapsto \frac{1}{2}((x_1 - x_2)^2 - 1)$$

Let $\theta_{(s,t)} : \mathbf{C}[x_1, x_2]^{\mathfrak{S}_2}$ be the morphism which evaluates a polynomial $P \in \mathbf{C}[x_1, x_2]^{\mathfrak{S}_2}$ at $x_1 = s$ and $x_2 = t$.

Now $h\tilde{f} = k\tilde{f}$ and $c\tilde{f} = k\tilde{f}$ as $\tilde{f} \circ (\lambda\cdot) = \lambda^k \tilde{f}$ for all $\lambda \in \mathbf{C}^\times$. Moreover $a_-\tilde{f} = 0$ as $\tilde{f}$ is holomorphic, and a_0 consists of anti-holomorphic derivation (see Deligne, or compute it).

Then $c\tilde{f} = k\tilde{f}$, and

$$C\tilde{f} = (\frac{1}{2}h^2 - (a_+a_- - a_-a_+))\tilde{f} = (\frac{1}{2}h^2 - h)\tilde{f} = \frac{1}{2}(k^2 - 2k)\tilde{f}.$$

So, for $\tilde{f}$ we have $s + t = k$ and $\frac{1}{2}((s-t)^2 - (s-t)) = \frac{1}{2}(k^2 - 2k)$. Therefore, $s + t = k$ and $s - t = k - 1$. We conlcude that $s = k - \frac{1}{2}$ and $t = \frac{1}{2}$.

So this means that we should consider $\tilde{\tilde{f}} := \tilde{f}|| \det(\cdot)||^{-\frac{1}{2}}$. Then, for $\tilde{\tilde{f}}$ we have $s = k - 1$ and $t = 0$, so $\tilde{\tilde{f}} \in \mathcal{A}^\circ_{(k-1,v)}$.

I'm glad to see this $(k-1, 0)$ as the HS $\sim f$ is of type $\{(k-1,0),(0,k-1)\}$, or also, the Hodge-Tate weights of the Galois representation corresponding to f are $k - 1$ and 0.

The automorphic representation corresponding to f: The function $\tilde{\tilde{f}}$ generates a $(\mathfrak{gl}_2, O(2))$-module $V_{f,\infty}$, with basis

$$\{a_+^n\tilde{\tilde{f}}|n \geq 0\} \coprod \{a_-^n\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\tilde{\tilde{f}}|n \geq 0\},$$

and $V_f = V_{f,\infty} \otimes V_f^\infty$.