Local ~~adele~~ adelic methods seminar. 2009/~~6~~01/05. (2 x 45 minutes) (I will need more).

Gl_2. Reference: Deligne's "Formes modulaires et repr. de $Gl(2)$" in Antwerp II, S.L.N.M. 349; and: p.18-19 of Taylor's long ICM text. (Modular forms, classically, and $Gl_2(\mathbb{A})$). just 21 lines, but there is a lot to say...

Aim: to understand the 19 lines by Taylor.

§1. The Harish-Chandra isomorphism for gl_2.

$$gl_2 = Lie(Gl_2(\mathbb{C})) = M_2(\mathbb{C}) = \mathbb{C}\cdot\underset{c}{\begin{pmatrix}1&0\\0&1\end{pmatrix}} \oplus \mathbb{C}\cdot\underset{\substack{h\\H}}{\begin{pmatrix}1&0\\0&-1\end{pmatrix}} \oplus \mathbb{C}\cdot\underset{\substack{a_+\\X}}{\begin{pmatrix}0&1\\0&0\end{pmatrix}} \oplus \mathbb{C}\cdot\underset{\substack{a_-\\Y}}{\begin{pmatrix}0&0\\1&0\end{pmatrix}}.$$

$$U(gl_2) = T(gl_2) \Big/ \langle a\otimes b - b\otimes a - [a,b]\rangle_{(a,b\in gl_2)}$$

Isom. of categories: "gl_2-mod" $\xrightarrow{\sim}$ $U(gl_2)$-mod.

So: $Z_2 := \text{center}(U(gl_2)) = End(id_{gl_2\text{-mod}})$.

H-C isom: $\gamma_{HC}: Z_2 \xrightarrow{\sim} \mathbb{C}[x_1,x_2]^{S_2} = \mathbb{C}[x_1+x_2, x_1x_2]$, charact. as follows.

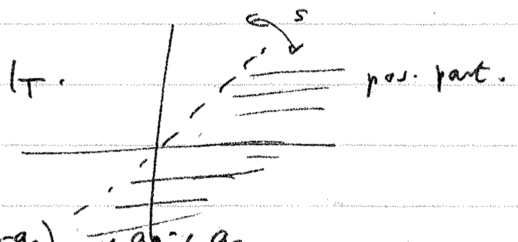
$$\begin{array}{ccc} Hom_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1,x_2]^{S_2}, \mathbb{C}) & = & S_2\backslash\mathbb{C}^2 \\ \uparrow & & \Uparrow \text{quot.} \\ Hom_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1,x_2], \mathbb{C}) & = & \mathbb{C}^2 \\ f & \longmapsto & (f(x_1), f(x_2)) \end{array}$$

"Highest weight representations": $Gl_2 \supset B \supset T = \mathbb{G}_m^2$, $W = N_T/T = S_2$, where $B = \left\{\begin{pmatrix}*&*\\0&*\end{pmatrix}\right\}$, $T = \left\{\begin{pmatrix}*&0\\0&*\end{pmatrix}\right\}$, $s := \begin{pmatrix}0&1\\1&0\end{pmatrix}$.

$T \circlearrowright gl_2$, ~~[illegible]~~ $t\cdot a_+ = t_1t_2^{-1}a_+$, so $(1,-1)$ is the positive root.

$X(T) = Hom(T, \mathbb{G}_m) = \mathbb{Z}^2$; positive part: $\{a\in\mathbb{Z}^2 \mid a_1 \geq a_2\}$

Theory: $\forall$ positive a $\exists!$ irred. alg. repr. ρ_a of $Gl_{2,\mathbb{C}}$ s.t. a is the highest weight of $\rho_a|_T$.



Explicitly: $\rho_a = Sym^{a_1-a_2}(\mathbb{C}^2)\otimes \det^{a_2}$

$$char(\rho_a|_T): t \mapsto \left(t_1^{a_1-a_2} + t_1^{a_1-a_2-1}t_2 + \dots + t_2^{a_1-a_2}\right)\cdot t_1^{a_2}t_2^{a_2} = t_1^{a_1}t_2^{a_2} + \dots$$

Note: $Sym^d(\mathbb{C}^2) = Sym^d(\mathbb{C}\cdot x\oplus\mathbb{C}\cdot y) = \mathbb{C}[x,y]_d$.

Now the characterisation: $\forall$ positive a:

$$\begin{tikzcd} Z_2 \arrow[r, "\chi_{HC}", "\sim"'] \arrow[dr, "\rho_a"'] & \mathbb{C}[x_1,x_2]^{S_2} \arrow[d, "\Theta_{H(\rho_a)}"] \arrow[dl, phantom, "\oslash"] \\ & \mathbb{C} \end{tikzcd} \qquad \begin{array}{c} f \\ \downarrow \\ f(a_1+\frac{1}{2}, a_2-\frac{1}{2}) \end{array}$$

$H(\rho_a) := \{a_1+\frac{1}{2}, a_2-\frac{1}{2}\}$

Note: $(a_1+\frac{1}{2}, a_2-\frac{1}{2}) = a + \frac{1}{2}\cdot\underbrace{(1,-1)}_{\text{sum of positive roots}}$.

This condition characterises χ_{HC} bec. $\{a+\frac{1}{2}(1,-1) \mid a \text{ positive in } \mathbb{Z}^2\}$ is Zariski dense in $\mathbb{C}^2$. (hence every autom. of $\mathbb{C}[x_1,x_2]^{S_2}$ ...).

But this doesn't tell us what χ_{HC}, or Z_2, is.

Of course we have $c = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ in $gl_2 \subset U(gl_2)$. As $\mathrm{Lie}(\text{center } Gl_2(\mathbb{C})) = \mathbb{C}\cdot c$, $c \in Z_2$. We need 1 more element in Z_2.

Some generalities. Let G be a complex affine alg. group, $A := \mathcal{O}(G)$, $m :=$ max. id. of $e \in G(\mathbb{C})$. See Serre's book on Lie algebras & Lie groups.

Then $U(\mathrm{Lie}(G)) = \{$left-inv. diff. op's on $G\}$, ($\mathrm{Lie}(G)$ = left-inv. vect. fields)

$= $ "point-distr. at e": $\varinjlim_n (A/m^n)^\vee$,

with multipl. coming from $\mu: G\times G \to G$, $\mu^*: A \to A\otimes A$

(giving $(A/m^n)^\vee \otimes (A/m^n)^\vee \to (A/m^{2n})^\vee$.)

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A\otimes A \arrow[d] \\ A/m^{2n} \arrow[r] \arrow[dr] & A\otimes A/(m\otimes A + A\otimes m)^{2n} \arrow[d] \\ & A/m^n \otimes A/m^n \end{tikzcd}$$

So $U(\mathrm{Lie}\, G)$ is filtered: $\mathrm{Fil}_0 \subset \mathrm{Fil}_1 \subset \dots$,

with $gr_i = \mathrm{Sym}^i(\mathrm{Lie}\, G)$.

By construction: $\mathrm{center}(U(\mathrm{Lie}(G))) = U(\mathrm{Lie}(G))^{\mathrm{Lie}\, G} = U(\mathrm{Lie}(G))^{G^0}$.

For $G = Gl_n = GL(V)$: $\mathrm{Lie}(G) = \mathrm{End}(V) = V^\vee\otimes V = (V^\vee\otimes V)^\vee = \mathrm{End}(V)^\vee$,

~~and~~ $\mathrm{Sym}(\mathrm{End}(V)^\vee)^{Gl_n} = \{$conj. inv. polynomials on $\mathrm{End}(V)\}$

$= \mathbb{C}[$coeff. of char. pol.$]$, generators in degrees 1 (trace), 2, ..., n (determinant).

$\updownarrow$

id_V ($= c$ for $n=2$).

Casimir. $\qquad$ from sl_2.

So, for $n=2$, there is an element C in degree$_{\leq 2}$ s.t. c, C generate Z_2.

can take any semi-simple

Recipe for Casimir operator: take the Killing form $\langle \cdot,\cdot \rangle$ on sl_2 given by $\langle a,b\rangle = tr((ad\, a)(ad\, b))$ (here $ad\, a : sl_2 \to sl_2, \dots$); take any basis $(e_i)_{i\in I}$ of sl_2, then $C = \sum_{i\in I} e_i^\vee \otimes e_i$, where $(e_i^\vee)_{i\in I}$ is the dual basis w.r.t. the Killing form. We use $sl_2 = \mathbb{C}\cdot h \oplus \mathbb{C}\cdot a_+ \oplus \mathbb{C}\cdot a_-$.

We have: $[h, a_+] = 2a_+$, $[h, a_-] = -2a_-$, $[a_+, a_-] = h$,

Matrix of $\langle \cdot,\cdot \rangle$: $\begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}$, so $h^\vee = \frac{1}{8}h$, $a_+^\vee = \frac{1}{4}a_-$, $a_-^\vee = \frac{1}{4}a_+$.

So the Casimir operator is: $\frac{1}{4}\cdot\Big(\underbrace{\tfrac{1}{2}h^2 + a_+a_- + a_-a_+}_{C}\Big)$.

We have: $Z_2 = \mathbb{C}[c, C]$.

For $a \in \mathbb{Z}^2$ positive, $V_a = \mathbb{C}[x,y]_{a_1-a_2} \otimes (\mathbb{C}\cdot x\wedge y)^{\otimes a_2}$,

gl_2 acts on $\mathbb{C}[x,y]$ as : $c: x\partial_x + y\partial_y$, $h: x\partial_x - y\partial_y$, $a_+ : x\partial_y$, $a_-: y\partial_x$

$\mathbb{C}\cdot x\wedge y$ as : $c: 2$, $h: 0$, $a_+: 0$, $a_-: 0$

c acts on V_a as: $a_1 + a_2$

C acts on V_a as: $\left(\frac{1}{2}(x\partial_x + y\partial_y)(x\partial_x + y\partial_y + 2)\right) \otimes 1 = \frac{1}{2}\cdot(a_1-a_2)(a_1-a_2+2)$

$= \frac{1}{2}(a_1-a_2)^2 + a_1 - a_2$

Now we can make $\chi_{HC} : Z_2 \xrightarrow{\sim} \mathbb{C}[x_1,x_2]^{S_2}$ explicit.

$$\begin{tikzcd}
Z_2 \arrow[r, "\sim"] \arrow[d, equal] & \mathbb{C}[x_1,x_2]^{S_2} \arrow[d] & f \arrow[d, mapsto] \\
\mathbb{C}[c,C] \arrow[r, "\rho_a"] & \mathbb{C} & f(a_1+\frac{1}{2}, a_2-\frac{1}{2})
\end{tikzcd}$$

$c \mapsto a_1 + a_2$

$C \mapsto \frac{1}{2}(a_1-a_2)(a_1-a_2+2)$

So: $\chi_{HC} : \begin{cases} c \longmapsto x_1 + x_2 \\ C \longmapsto \frac{1}{2}(x_1-x_2-1)(x_1-x_2+1) = \frac{1}{2}\left((x_1-x_2)^2 - 1\right) \end{cases}$

§2. Very classical picture.

For $\Gamma \subset SL_2(\mathbb{Z})$ a congr. subgr. (i.e., containing some $\Gamma(n) = \ker(SL_2(\mathbb{Z}) \twoheadrightarrow SL_2(\mathbb{Z}/n\mathbb{Z}))$, $n \in \mathbb{Z}_{\geq 1}$, and $k \in \mathbb{Z}$, one defines:

$$S_k(\Gamma) := \left\{ f : \mathbb{H} \to \mathbb{C} \;\middle|\; \begin{array}{l} 1.\ f \text{ holomorphic} \\ 2.\ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \forall \tau \in \mathbb{H}:\ f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f(\tau) \\ 3.\ f \text{ vanishes at the cusps: } \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \\ \quad \left(\tau \mapsto (c\tau+d)^{-k} \cdot f\left(\frac{a\tau+b}{c\tau+d}\right)\right) \to 0 \text{ as } \operatorname{Im}\tau \to \infty. \end{array} \right\}.$$

What does the condition "$(c\tau+d)^{-k} \cdot f\left(\frac{a\tau+b}{c\tau+d}\right) = f(\tau)$" mean?

On $\mathbb{H}$ we have:

$$\left(\begin{tikzcd} \mathbb{Z}^2 \times \mathbb{H} \arrow[r, hook] \arrow[dr] & \mathbb{C} \times \mathbb{H} \arrow[r, two heads] \arrow[d] & \mathbb{E} \arrow[dl] \\ & \mathbb{H} \arrow[ur, bend right, "0"'] & \end{tikzcd}\right) \circlearrowleft SL_2(\mathbb{Z}) \text{ from left.}$$

To find the action, consider $\tau \in \mathbb{H}$, and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Then

$$\begin{tikzcd} \mathbb{E}_\tau = \mathbb{C}/(\mathbb{Z}\tau+\mathbb{Z}) = \mathbb{C}/(\mathbb{Z}(a\tau+b)+\mathbb{Z}\cdot(c\tau+d)) \arrow[d, "\wr"] & (z,\tau) \arrow[d, mapsto] \\ \mathbb{E}_{\gamma\tau} = \mathbb{C}/\left(\mathbb{Z}\frac{a\tau+b}{c\tau+d}+\mathbb{Z}\right) & \left(\frac{1}{c\tau+d}\cdot z, \frac{a\tau+b}{c\tau+d}\right) \end{tikzcd}$$

Note: $\left(\begin{pmatrix} n \\ m \end{pmatrix}, \tau\right) \longmapsto n\tau + m = \begin{pmatrix} n \\ m \end{pmatrix}^t \cdot \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \left(\gamma^{-1,t}\cdot\begin{pmatrix} n \\ m \end{pmatrix}\right)^t \cdot \underbrace{\gamma\cdot\begin{pmatrix} \tau \\ 1 \end{pmatrix}}_{=\begin{pmatrix} a\tau+b \\ c\tau+d \end{pmatrix}}$

One finds:

$$\begin{tikzcd} \left(\begin{pmatrix} n \\ m \end{pmatrix}, \tau\right) \arrow[d, mapsto, "\gamma\cdot"'] \\ \left(\gamma^{-1,t}\cdot\begin{pmatrix} n \\ m \end{pmatrix}, \gamma\tau\right) \end{tikzcd}$$

. This gives an inv. $\mathcal{O}_{\mathbb{H}}$-module $\underline{\omega} = 0^*\Omega^1_{\mathbb{E}/\mathbb{H}}$, with $SL_2(\mathbb{Z})$-action.

We have a global section: dz, and $(\gamma\cdot)^* dz = d((\gamma\cdot)^* z) = d\left(\frac{z}{c\tau+d}\right) = (c\tau+d)^{-1} \cdot dz$.

Then we have, for $f : \mathbb{H} \to \mathbb{C}$, $k \in \mathbb{Z}$, $\tau \in \mathbb{H}$, $\gamma \in GL_2(\mathbb{Z})$: $\left((\gamma\cdot)^*\left(f\cdot(dz)^{\otimes k}\right)\right)_\tau = \left((\gamma\cdot)^* f\right)(\tau)\cdot(c\tau+d)^{-k}\cdot(dz)^{\otimes k}$

Hence : $(\gamma\cdot)^* f\cdot(dz)^{\otimes k} = f\cdot(dz)^{\otimes k} \iff \forall \tau\in\mathbb{H}:\ f(\gamma\tau)\cdot(c\tau+d)^{-k} = f(\tau)$

So, condition (2) for $f \in S_k(\Gamma)$ means that $f\cdot(dz)^{\otimes k}$ is a Γ-invariant global section of $\underline{\omega}^{\otimes k}$: $f\cdot(dz)^{\otimes k} \in \left(\underline{\omega}^{\otimes k}(\mathbb{H})\right)^\Gamma$.

If Γ acts freely on $\mathbb{H}$: $\Gamma \backslash \left(\begin{array}{c} \mathbb{E} \\ \downarrow \uparrow 0 \\ \mathbb{H} \end{array} \right) =: \begin{array}{c} \mathbb{E}(\Gamma) \\ \downarrow \uparrow 0 \\ Y(\Gamma) \end{array}$ univ. ell. curve + level Γ-structure

$$S_k(\Gamma) = \left\{ f \in \underline{\omega}^{\otimes k}_{\mathbb{E}(\Gamma)/Y(\Gamma)}(Y(\Gamma)) \;\middle|\; f \text{ extends over the cusps, and vanishes there} \right\}.$$

§3. Functions of lattices

The formulas in the preceding section are ugly! After Deligne, consider:

$$G := \{\varphi : \mathbb{Z}^2 \to \mathbb{C} \mid \varphi\ \mathbb{Z}\text{-lin.},\ \varphi(\mathbb{Z}^2) \text{ lattice}\}. \subset \mathbb{C}^2, \text{ open; 2 dim. } \mathbb{C}\text{-manif.}$$

$$\begin{array}{c} G \\ \wr\| \\ \mathrm{Isom}_{\mathbb{R}\text{-mod}}(\mathbb{R}^2, \mathbb{C}) \\ \wr\| \\ GL_2(\mathbb{R}). \end{array} \qquad GL_2(\mathbb{Z}) \times \mathbb{C}^\times \circlearrowright G.$$

$$\begin{array}{ccc} (\tau, 1) & \mathbb{C}^* \circlearrowright G & \varphi \\ \uparrow & q \downarrow & \downarrow \\ \tau & \mathbb{H}^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) \subset \mathbb{C} & (\varphi e_1 : \varphi e_2) = \varphi e_1/\varphi e_2 \end{array}$$

$\mathbb{C}^\times$ acts freely on G, quotient $G \to \mathbb{H}^\pm$.

<u>Note:</u> $G = \{(L, \varphi) \mid L \subset \mathbb{C} \text{ lattice}, \varphi : \mathbb{Z}^2 \xrightarrow{\sim} L\}$

$= \{(V, L, \alpha, \varphi) \mid V \text{ 1-dim. } \mathbb{C}\text{-v.sp.}, \alpha : \mathbb{C} \xrightarrow{\sim} V, L \subset V \text{ lattice}, \varphi : \mathbb{Z}^2 \xrightarrow{\sim} L\}/\cong$

$GL_2(\mathbb{Z})\backslash G/\mathbb{C}^\times = \{(V, L)\}/\cong \leftrightarrow \{E/\mathbb{C} \text{ ell. curve}\}/\cong$.

Then; for $k \in \mathbb{Z}$, $\Gamma \subset GL_2(\mathbb{Z})$: $\left\{ f : G \to \mathbb{C} \;\middle|\; \begin{array}{l} f(\varphi\gamma^t) = f(\varphi)\ \forall \varphi\in G, \forall\gamma\in\Gamma \\ f(\lambda\cdot\varphi) = \lambda^{-k}\cdot f(\varphi)\ \forall\varphi\in G, \forall\lambda\in\mathbb{C}^\times \end{array} \right\}$

$$\begin{array}{ccc} f & (f\circ q)\cdot(\varphi\mapsto\varphi e_2)^{-k} & \\ \downarrow & \uparrow & \wr \\ (\tau \mapsto f(\tau,1)) & f & \left\{ f : \mathbb{H}^\pm \to \mathbb{C} \;\middle|\; \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \forall \tau\in\mathbb{H} : f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\cdot f(\tau) \right\}. \end{array}$$

Correct this

§4. The "Shimura datum picture". (work with Gl_2, not Sl_2). (action of $Gl_2(\mathbb{A}^\infty)$).

For $K \subset Gl_2(\mathbb{A}^\infty)$ open, compact subgroup, we define:

$$Y_K := Gl_2(\mathbb{Q}) \backslash \left(\mathbb{H}^\pm \times Gl_2(\mathbb{A}^\infty)/K \right)$$

Example:
$$Y_{Gl_2(\hat{\mathbb{Z}})} = Gl_2(\mathbb{Q}) \backslash \left(\mathbb{H}^\pm \times Gl_2(\mathbb{A}^\infty)/Gl_2(\hat{\mathbb{Z}}) \right) = Gl_2(\mathbb{Z}) \backslash \mathbb{H}^\pm = Sl_2(\mathbb{Z}) \backslash \mathbb{H} = \{E/\mathbb{C}\}/\cong.$$

$\uparrow$ $Gl_2(\mathbb{Q})$ acts transitively on the set of $\hat{\mathbb{Z}}$-lattices in $(\mathbb{A}^\infty)^2$.

We want to interpret the Y_K as modulispaces. First analytically.

$Gl_2(\mathbb{A})$ via $\mathbb{R}$-basis of $\mathbb{C}$, either $(i,1)$, or $(1,i)$ ...

$\wr\!\downarrow$

$\mathrm{Isom}_{\mathbb{R}\text{-v.sp}}(\mathbb{C}, \mathbb{R}^2) \times Gl_2(\mathbb{A}^\infty)$

$\downarrow \quad \uparrow$ inverses of eachother.

$$\left\{ (V, L, \alpha, \beta, \varphi) \;\middle|\; \begin{array}{l} V \text{ 1-dim. } \mathbb{C}\text{-v.sp.} \\ L \subset V \text{ 2 dim. } \mathbb{Q}\text{-v.sp. s.t. } \mathbb{R}\cdot L = V \\ \alpha : \mathbb{C} \xrightarrow{\sim} V \\ \beta : \mathbb{Q}^2 \xrightarrow{\sim} L \\ \varphi : (\mathbb{A}^\infty)^2 \xrightarrow{\sim} \mathbb{A}^\infty \otimes_{\mathbb{Q}} L \end{array} \right\} \Big/ \cong$$

notion of isom:

$$\begin{tikzcd} & \mathbb{C} \arrow[dl, "\alpha"', "\wr"] \arrow[dr, "\wr", "\alpha'"'] \arrow[d, phantom, "\oslash"] & \\ V \arrow[rr, "f_V", "\sim"'] & & V' \\ L \arrow[u, hook] \arrow[rr, "f_L", "\sim"'] & \oslash & L' \arrow[u, hook] \\ & \mathbb{Q}^2 \arrow[ul, "\beta", "\wr"'] \arrow[ur, "\wr"', "\beta'"'] \arrow[u, phantom, "\oslash"] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A}^\infty \otimes L \arrow[rr, "\mathrm{id} \otimes f_L"] & \oslash & \mathbb{A}^\infty \otimes L' \\ & (\mathbb{A}^\infty)^2 \arrow[ul, "\varphi", "\wr"'] \arrow[ur, "\wr", "\varphi'"'] & \end{tikzcd}$$

$$\left((\mathrm{id}_{\mathbb{R}} \otimes \beta)^{-1} \circ \alpha, (\mathrm{id}_{\mathbb{A}^\infty} \otimes \beta)^{-1} \circ \varphi \right)$$

$\uparrow$

$(V, L, \alpha, \beta, \varphi)$

$$\mathbb{R}^2 \xrightarrow[\sim]{\mathrm{id}_{\mathbb{R}} \otimes \beta} \mathbb{R} \otimes L = V \xleftarrow[\sim]{\alpha} \mathbb{C}$$

$$(\mathbb{A}^\infty)^2 \xrightarrow[\sim]{\mathrm{id}_{\mathbb{A}^\infty} \otimes \beta} \mathbb{A}^\infty \otimes L \xleftarrow[\sim]{\varphi} (\mathbb{A}^\infty)^2$$

(g_∞, g^∞)

$\downarrow$

$$\left(\mathbb{C},\ g_\infty^{-1}\mathbb{Q}^2,\ \mathrm{id}_{\mathbb{C}},\ g_\infty^{-1}|_{\mathbb{Q}^2},\ (\mathrm{id}_{\mathbb{A}^\infty} \otimes g_\infty^{-1}|_{\mathbb{Q}^2}) \circ g^\infty \right)$$

<u>Consequence:</u> $Y_K = \{ (V, L, \varphi) \mid \begin{array}{l} V \text{ 1-dim. } \mathbb{C}\text{-v.sp.} \\ L \subset V \text{ 2 dim } \mathbb{Q} \text{ v.sp. s.t. } \mathbb{R}L = V \\ \varphi \in \mathrm{Isom}((\mathbb{A}^\infty)^2, \mathbb{A}^\infty \otimes L) / K \end{array} \} / \cong .$

"(V, L) + level str. φ."

What kind of category do the (V, L) give us ??

<u>Answer</u>: complex elliptic curves "up to isogeny". $\mathbb{Q} \otimes \mathrm{Ell}(\mathbb{C})$

same objects as $\mathrm{Ell}(\mathbb{C})$, tensor Hom's with $\mathbb{Q}$.

$\mathrm{Ell}(\mathbb{C}) \overset{\text{equiv.}}{\rightleftarrows} (V, L_0)$, V 1 dim. $\mathbb{C}$-v.sp., $L_0 \subset V$ lattice

$\mathrm{Hom}((V, \mathbb{Q}\otimes L_0), (V', \mathbb{Q}\otimes L_0')) = \mathbb{Q} \otimes \mathrm{Hom}((V, L_0), (V', L_0'))$.

Now algebraically: $Y_K(\overline{\mathbb{Q}}) = \{ (E, \varphi) \mid \begin{array}{l} E \in \mathrm{Ob}\ \mathbb{Q}\otimes \mathrm{Ell}(\overline{\mathbb{Q}}) \\ \varphi \in \mathrm{Isom}((\mathbb{A}^\infty)^2, V(E))/K \end{array} \} / \cong$

$$\begin{tikzcd} \mathrm{Ell}(\overline{\mathbb{Q}}) \arrow[r] \arrow[d, "T"'] \arrow[dr, "V"] & \mathbb{Q}\otimes \mathrm{Ell}(\overline{\mathbb{Q}}) \arrow[d, dashed, "V"] \\ \hat{\mathbb{Z}}\text{-mod} \arrow[r] & \mathbb{A}^\infty\text{-mod}. \end{tikzcd}$$

For $g \in GL_2(\mathbb{A}^\infty)$: $\varphi \mapsto \varphi \cdot g$ gives

$$Y_K(\overline{\mathbb{Q}}) \xrightarrow[\cdot g]{\sim} Y_{g^{-1}Kg}(\overline{\mathbb{Q}}).$$

For $K \subset GL_2(\hat{\mathbb{Z}})$; $\varphi \in \mathrm{Isom}((\mathbb{A}^\infty)^2, V(E))/K$: $\varphi \hat{\mathbb{Z}}^2 \subset V(E)$ is a lattice, gives a genuine elliptic curve E' : $T(E') = \varphi \hat{\mathbb{Z}}^2$ in $V(E)$.

<u>Now modular forms in this perspective</u>. " $\underbrace{\mathrm{Isom}(\mathbb{C}, \mathbb{R}^2)}_{\cap \mathbb{C}^2} \times GL_2(\mathbb{A}^\infty)$

For $k \in \mathbb{Z}$, $K \subset GL_2(\mathbb{A}^\infty)$: $S_k(K) = \{ f : GL_2(\mathbb{A}) \to \mathbb{C} \mid$
- f is holomorphic,
- $\forall \lambda \in \mathbb{C}^\times$: $f \circ (\cdot \lambda) = \lambda^{+k} \cdot f$
- f is $GL_2(\mathbb{Q})$-inv. on the left
- f is right K-invariant
- f is of moderate growth, and zero in the cusps $\}$

say like this: f extends over the cusps of Y_K, and vanishes there.

For $n \in \mathbb{Z}_{\geq 1}$, let

$$\begin{tikzcd} GL_2(\hat{\mathbb{Z}}) \arrow[r, two heads] & GL_2(\mathbb{Z}/n\mathbb{Z}). \\ K_{1,n} \arrow[u, hook] \arrow[r, two heads] \arrow[ur, phantom, "\square"] & \left\{\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}\right\} \arrow[u, hook] \end{tikzcd}$$

Then $Y_{K_{1,n}} = Y(\Gamma_1(n))$.

Hence, for $k \in \mathbb{Z}$: $S_k(\Gamma_1(n)) = S_k(K_{1,n})$, and we have:

$$S_k(\Gamma_1(n)) \hookrightarrow \varinjlim_K S_k(K) =: S_k = \{f : Gl_2(\mathbb{A}) \to \mathbb{C} : \text{"list"}\}$$
$$\circlearrowleft_{Gl_2(\mathbb{A}^\infty)}$$

For $K \subset Gl_2(\mathbb{A}^\infty)$ open compact subgrp: $S_k(K) = (S_k)^K$; the $Gl_2(\mathbb{A}^\infty)$-action gives Hecke operators on the $S_k(K)$ ($\mathbb{C}[K_1 \backslash Gl_2(\mathbb{A}^\infty)/K_2]_*$).

Suppose now that f is a normalised weight k newform of level n: $a_1(f) = 1$, and its system of eigenvalues does not occur in a level $< n$.
Then f generates an irreducible $Gl_2(\mathbb{A}^\infty)$-submodule of S_k,
$V_f^\infty := \langle \{g \cdot f \mid g \in Gl_2(\mathbb{A}^\infty)\} \rangle$,

$$V_f^\infty = \otimes_p' V_{f,p}, \quad V_{f,p} \subset \varinjlim_{K \subset Gl_2(\mathbb{Q}_p)} S_k(K \times K_{1,n}^p), \text{ the subrepr. of } Gl_2(\mathbb{Q}_p) \text{ gen. by } f.$$

~~§5. Including the ∞ place~~

~~$V_f :=$ the $(gl_2, O(2))$-module generated by $V_f^\infty \subset \{f : Gl_2(\mathbb{A}) \to \mathbb{C}\}$, i.e. $Gl_2(\mathbb{A}^\infty) \times \mathcal{U}(gl_2) \cdot \{f, s \cdot f\}$ where $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.~~
~~Then $V_f = V_f^\infty \otimes V_{f,\infty}$, and $V_{f,\infty} =$ the $(gl_2, O(2))$-sub-module in $\{f : Gl_2(\mathbb{R}) \to \mathbb{C} \mid f \text{ smooth}\}$ generated by f.~~
~~$Gl_2(\mathbb{R}) = \mathrm{Isom}(\mathbb{C}, \mathbb{R}^2) \circlearrowleft Gl_{\mathbb{R}}(\mathbb{C})$ (from right), $SO_2 = S^1 = U_1 = \{u \in \mathbb{C} \mid |u| = 1\}$, $s = (z \mapsto \bar{z})$.~~

~~A suitable inner automorphism of gl_2 sends c, h, a_+, a_- to $c' = c$, h', a_+', a_-', s.t. $\mathrm{Lie}(U_1)_{\mathbb{C}} = \mathbb{C} \cdot h'$.~~

Gl_2 and modular forms (3). 2009/02/09.

Extra reference: S. Kudla, From modular forms to autom. represns, in a book "An introd. to the ~~Eis~~ Langlands program". (But it only gives formulas)

I have a correction ~~to~~ make, in the part "§3, functions of lattices". (And I change the direction of the isom., so my notation is no longer that of Deligne.)

$G = \mathrm{Isom}_{\mathbb{R}\text{-v.sp.}}(\mathbb{C}, \mathbb{R}^2)$, $\quad G \hookrightarrow \mathbb{C}^2 : \varphi \mapsto (\varphi^{-1}e_1, \varphi^{-1}e_2)$

$Gl_2(\mathbb{R}) \circlearrowright G \circlearrowleft Gl_{\mathbb{R}}(\mathbb{C}) \supset \mathbb{C}^\times$; $\quad$ quotient for $\mathbb{C}^\times$: $\quad G \xrightarrow{q} \mathbb{H}^\pm : \varphi \mapsto \varphi^{-1}e_1/\varphi^{-1}e_2$.

Moduli-interpretation: $G = \{(V, L, \psi, \alpha)\}/\cong$, $\quad$ V: 1dim. $\mathbb{C}$-v.sp., $L \subset V$ $\mathbb{Z}$-lattice, $\psi: \mathbb{Z}^2 \xrightarrow{\sim} L$, $\alpha: \mathbb{C} \xrightarrow{\sim} V$.

$$\downarrow$$

$\mathbb{H}^\pm = \{(V, L, \psi)\}/\cong$.

Note: on $\mathbb{H}^\pm$ we have the line bundle $\underline{\omega}$, $Gl_2(\mathbb{R})$-equivariant, but it has no invariant trivialisation (stabilisers act non-trivially).
But, on G, α gives us a $Gl_2(\mathbb{R})$-invariant trivialisation of $\underline{\omega}$.
It is this that makes it possible to describe modular forms as functions, and not as sections of a line bundle.

Concretely: let $f \in S_k(\Gamma)$, $\Gamma \subset Sl_2(\mathbb{Z})$ congr. subgroup; gives $\tilde{f}: G \to \mathbb{C}$.
Let $\varphi \in G$, let $\omega_1 := \varphi^{-1}e_1$, $\omega_2 := \varphi^{-1}e_2$, $\tau := \omega_1/\omega_2$,
$L := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \subset \mathbb{C}$,

$$\begin{array}{ccc} \mathbb{C}/L & \xrightarrow{\sim} & \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}) \\ \uparrow & & \uparrow \\ \mathbb{C} & \xrightarrow[z]{\omega_2^{-1}\cdot} & \mathbb{C} \end{array}$$

$$\begin{array}{ccc} G & \subset & \mathbb{C}^2 \\ & \searrow^{\omega_{1,2}} & \downarrow pr_{1,2} \\ & & \mathbb{C} \end{array}$$

$$\tilde{f}(\varphi) = f_{\mathbb{H}}(\omega_1/\omega_2)\cdot\omega_2^{-k}\cdot(dz)^{\otimes k} \longleftarrow f_{\mathbb{H}}(\tau)\,(dz)^{\otimes k}$$

$$\left\{\tilde{f}: G \to \mathbb{C} \;\middle|\; \begin{array}{l} \forall \gamma \in \Gamma, \forall \varphi \in G: \tilde{f}(\gamma\varphi) = \tilde{f}(\varphi) \\ \forall \lambda \in \mathbb{C}^\times, \forall \varphi \in G: \tilde{f}(\varphi\lambda) = \lambda^{+k}\cdot\tilde{f}(\varphi) \end{array}\right\} \qquad \tilde{f} = (q^* f_{\mathbb{H}})\cdot\omega_2^{-k} \qquad \tilde{f}$$

$$\uparrow\downarrow \qquad\qquad \uparrow \qquad\qquad \downarrow$$

$$\left\{f_{\mathbb{H}}: \mathbb{H}^\pm \to \mathbb{C} \;\middle|\; \begin{array}{l} \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \\ \forall \tau \in \mathbb{H}^\pm \end{array} : f_{\mathbb{H}}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot f_{\mathbb{H}}(\tau)\right\} \qquad f_{\mathbb{H}} \qquad \left(\tau \mapsto \tilde{f}\begin{pmatrix}\tau \mapsto e_2 \\ 1 \mapsto e_1\end{pmatrix}\right.$$

We use $i \in \mathbb{H}^{\pm}$ as base point and (hence) $\varphi_0 : \mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$ with $\varphi_0^{-1} e_1 = i$, $\varphi_0^{-1} e_2 = 1$ as base point in G. In particular, $GL_2(\mathbb{R}) \xrightarrow{\sim} G$, $g \mapsto g \cdot \varphi_0$.

Then, for $g \in GL_2(\mathbb{R})$, we have

$$\omega_1(g\cdot\varphi_0) = ai+b, \qquad \text{when: }$$
$$\omega_2(g\cdot\varphi_0) = ci+d \qquad g^{-1,t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Hence: $\tilde{f}(g\cdot\varphi_0) = f\left(\frac{ai+b}{ci+d}\right)\cdot(ci+d)^{-k}$.

And let me say it in yet another way how $f \in S_{k,K} \in H^0(Y_K, \underline{\omega}^{\otimes k})$ gives $\tilde{f}$:

$$\begin{tikzcd} f(V,L,\alpha,\beta,\varphi) \in (V^\vee)^{\otimes k} \arrow[r, "\sim"', "(\alpha^\vee)^{\otimes k}"] \arrow[rd, bend right, mapsto, "\alpha : \mathbb{C} \xrightarrow{\sim} V"'] & \mathbb{C}^{\vee\otimes k} = \mathbb{C} \\ & \tilde{f}(V,L,\alpha,\beta,\varphi) \end{tikzcd}$$

Clearly: $\tilde{f}(V,L,\alpha\circ(\lambda\cdot),\beta,\varphi) = \lambda^k \cdot f(V,L,\alpha,\beta,\varphi)$.
$\forall \lambda \in \mathbb{C}^\times$:

Let us compute the action of the center $\mathbb{A}^\times$ on $\mathbb{C}\cdot\tilde{f}$; $\underline{\underline{\varepsilon_f : \hat{\mathbb{Z}}^\times \to \mathbb{C}^\times}}$.

$\mathbb{R}^\times$: $\lambda \mapsto \lambda^k = |\lambda|^k \cdot \psi_{\varepsilon_f,\infty}$

$\mathbb{Q}_p^\times$: $|\cdot|_p^k \cdot \psi_{\varepsilon_f,p}$

$$\begin{tikzcd} \hat{\mathbb{Z}}^\times \arrow[r, hook] \arrow[d, "\varepsilon_f"'] \arrow[dr, "\sim"] & \mathbb{A}^\times \arrow[d, "\mathrm{quot}"] & \mathbb{Q}_p^\times \arrow[l, hook'] \arrow[dll, bend left=60, "\psi_{\varepsilon_f,p}"] \\ \mathbb{C}^\times & \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}^\times_{>0} \arrow[l, "\psi_{\varepsilon_f}"] & \end{tikzcd}$$

Hence $\mathbb{A}^\times$ acts as $\|\cdot\|^k\cdot\psi_{\varepsilon_f}$, and indeed $\mathbb{Q}^\times$ acts trivially.

<u>Note</u>: $\varepsilon_f(-1) = (-1)^k$, hence $\psi_{\varepsilon_{f,\infty}}(\lambda) = \left(\frac{\lambda}{|\lambda|}\right)^k$.

<u>Here's the proof</u>. $f(V,L,\varphi) \in (V^\vee)^{\otimes k}$, $f(V,L,p\cdot\varphi) \in (V^\vee)^{\otimes k}$,

so $(p\cdot)^* : (V^\vee)^{\otimes k} \to (V^\vee)^{\otimes k}$, $f(V,L,p\varphi) \mapsto f(V,L,\varphi)$,

so: $f(V,L,p\cdot\varphi) = p^{-k}\cdot f(V,L,\varphi)$

$$= \prod_{l\neq p} \varepsilon_{f,l}(p)\cdot f(V,L,p_p\cdot\varphi),$$

$$\left(\begin{tikzcd} V \arrow[r, "p\cdot", "\sim"'] & V \\ L \arrow[u, phantom, "\cup" rotate=90] \arrow[r, "p\cdot"] & L \arrow[u, phantom, "\cup" rotate=90] \\ \mathbb{A}^\infty\otimes L \arrow[r, "p\cdot"] & \mathbb{A}^\infty\otimes L \\ & (\mathbb{A}^\infty)^2 \arrow[ul, "\varphi"] \arrow[u, "p\cdot\varphi"'] \end{tikzcd}\right)$$

so: $f(V,L,p_p\cdot\varphi) = |p|_p^k \cdot \prod_{l\neq p}\varepsilon_{f,l}(p)^{-1}\cdot f(V,L,\varphi)$.

Now comes a strange thing. I thought that $\tilde{f}$ would be in $\mathcal{A}^{\circ}_{\{s,t\}}(Gl_2(\mathbb{Q})\backslash Gl_2(\mathbb{A}))$ for suitable s and t.
But computing s & t gives $(s,t) = (k-\frac{1}{2}, \frac{1}{2})$: <u>not</u> integers.

Here's the computation. Recall Lie $Gl_2(\mathbb{R})$: c, h, a_+, a_-;

$$C = \tfrac{1}{2}h^2 + a_+a_- + a_-a_+ , \quad z_2 \xrightarrow{\chi_{HC}\sim} \mathbb{C}[x_1,x_2]^{S_2} \quad f \mapsto f(s,t), \quad \theta_{(s,t)} : \mathbb{C}[x_1,x_2]^{S_2} \to \mathbb{C}$$

$$\begin{cases} c \longmapsto x_1 + x_2 \\ C \longmapsto \frac{1}{2}((x_1-x_2)^2 - 1) \end{cases}$$

Now $\begin{cases} h.\tilde{f} = k.\tilde{f} \text{ and } c.\tilde{f} = k.\tilde{f}, \text{ as } \tilde{f}\circ(t\cdot) = t^k\cdot\tilde{f} \ \forall t \in \mathbb{C}^\times. \\ a_-.\tilde{f} = 0 \text{ as } \tilde{f} \text{ is holom. and } a_- \text{ consists of anti-hol. derivatives} \end{cases}$

(see Deligne, or compute it).

Then: $c.\tilde{f} = k.\tilde{f}$, $C.\tilde{f} = (\frac{1}{2}h^2 - (a_+a_- - a_-a_+))\tilde{f} =$
$= (\frac{1}{2}h^2 - h)\tilde{f} = \frac{1}{2}(k^2-2k).\tilde{f}$.

So, for $\tilde{f}$: $\begin{cases} s+t = k \\ \frac{1}{2}((s-t)^2 - 1) = \frac{1}{2}(k^2-2k) \end{cases}$ i.e., $\begin{cases} s+t = k \\ s-t = k-1 \end{cases}$ $\begin{cases} s = k-\frac{1}{2} \\ t = \frac{1}{2} \end{cases}$.

So, this means that we should consider $\tilde{\tilde{f}} := \tilde{f}\cdot\|\det(\cdot)\|^{-1/2}$.
Then, for $\tilde{\tilde{f}}$: $\begin{cases} s = k-1 \\ t = 0 \end{cases}$, so $\tilde{\tilde{f}} \in \mathcal{A}^{\circ}_{(k-1,0)}$.

I'm glad to see this $(k-1,0)$, as the HS $\sim f$ is of type $\{(k-1,0),(0,k-1)\}$, or also: the HT ~~numbers~~ weights of the Gal. reps $\sim f$ are $k-1$ and 0.

The $\overset{\text{autom.}}{\text{repr.}}$ $\sim f$: $\tilde{\tilde{f}}$ generates a $(\mathfrak{gl}_2, O(2))$-module $V_{f,\infty}$, with basis $\{a_+^n.\tilde{\tilde{f}} \mid n \geq 0\} \amalg \{a_-^n.\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\tilde{\tilde{f}} \mid n \geq 0\}$

$$V_f = V_{f,\infty} \otimes V_f^{\infty}.$$