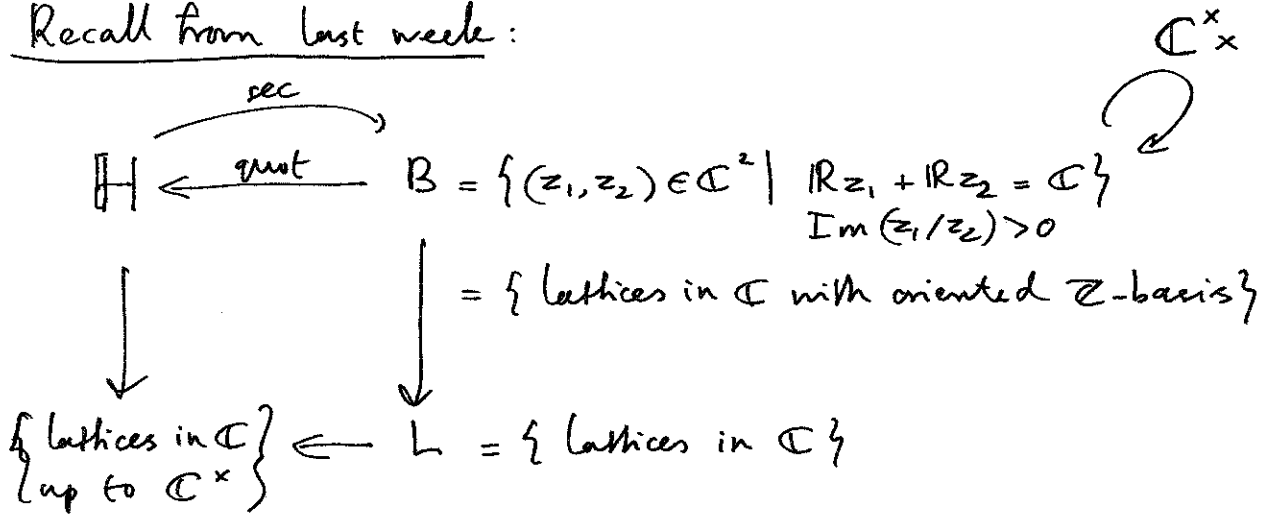


0. Recall from last week:



\leftarrow : quotient for \mathbb{C}^\times , \downarrow : quotient for $SL_2(\mathbb{Z})$.

For $k \in \mathbb{Z}$, $M_k(B) = \{ f: B \rightarrow \mathbb{C} \mid \forall \lambda, \forall b: f(\lambda \cdot b) = \lambda^{-k} \cdot f(b) \}$
 $\downarrow \text{sec}^\times$
 $M_k(\mathbb{H}) = \{ f: \mathbb{H} \rightarrow \mathbb{C} \}$.

• $f|_k: \mathbb{H} \rightarrow \mathbb{C}$ gives $(z_1, z_2) \mapsto z_2^{-k} \cdot f(z_1/z_2)$ in $M_k(B)$.

• The right action by $GL_2(\mathbb{R})^+$ on $M_k(B)$, becomes on $M_k(\mathbb{H})$:

$$(f \circ g)(z) = \left((z_1, z_2) \mapsto z_2^{-k} \cdot f(z_1/z_2) \right) \left(\begin{matrix} a & b \\ c & d \end{matrix} \cdot g(z) \right) = (cz+d)^{-k} \cdot f\left(\frac{az+b}{cz+d}\right).$$

For $\Gamma \subset SL_2(\mathbb{Z})$ a subgroup, we have $M_k(\mathbb{H})^\Gamma = \{ f \in M_k(\mathbb{H}) \mid \forall \gamma \in \Gamma: f \circ \gamma = f \}$

Then $M_k(\mathbb{H})^\Gamma = M_k(B)^\Gamma = \{ f \in M_k(B) \mid f \text{ depends only on the basis up to } \Gamma\text{-action} \}$

1. Examples. (1) Let $k \in \mathbb{Z}_{\geq 3}$, and for each $\Lambda \subset \mathbb{C}$ a lattice,

put: $G_k(\Lambda) = \sum_{0 \neq x \in \Lambda} x^{-k}$, this converges absolutely, and it is zero iff $k \equiv 1(2)$.

Hence: $G_k \in M_k(B)^{SL_2(\mathbb{Z})} = M_k(\mathbb{H})^{SL_2(\mathbb{Z})}$.

Eisenstein series of weight k on $SL_2(\mathbb{Z})$ (up to some normalizing constant factor).

(1.2) Let $n \in \mathbb{Z}_{>0}$, $n \equiv 0 \pmod{8}$

Then, $\forall z \in \mathbb{H}$: $\theta_{\mathbb{Z}^n}(-1/z) = (-iz)^{n/2} \theta_{\mathbb{Z}^n}(z) = z^{n/2} \theta_{\mathbb{Z}^n}(z)$,
 hence $\theta_{\mathbb{Z}^n} \in M_{n/2}(\mathbb{H})^\Gamma$, where $\Gamma = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle \subset SL_2(\mathbb{Z})$.

(1.3) Let $n \in \mathbb{Z}_{>0}$, $n \equiv 0 \pmod{2}$. Note that $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

We have, $\forall z \in \mathbb{H}$:

$$\begin{aligned} \theta_{\mathbb{Z}^n} \left(\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \cdot z \right) &= \theta_{\mathbb{Z}^n} \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \frac{2z+1}{z} \right) = \left(i \cdot \frac{2z+1}{z} \right)^{n/2} \cdot \theta_{\mathbb{Z}^n} \left(-2 - \frac{1}{z} \right) = \\ &= \left(i \cdot \frac{2z+1}{z} \right)^{n/2} \cdot \theta_{\mathbb{Z}^n} \left(-\frac{1}{z} \right) = \left(i \cdot \frac{2z+1}{z} \right)^{n/2} \cdot (-iz)^{n/2} \cdot \theta_{\mathbb{Z}^n}(z) \\ &= (2z+1)^{n/2} \cdot \theta_{\mathbb{Z}^n}(z). \end{aligned}$$

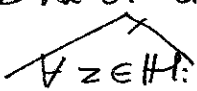
So: $\theta_{\mathbb{Z}^n} \in M_{n/2}(\mathbb{H})^\Gamma$, where $\Gamma = \langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \rangle$.

2. q-expansions of the G_k

Reference: Diamond-Shurman, p.4-6, and exercises 1.1.4-1.1.7.

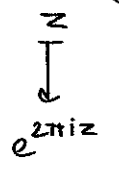
Let $k \in \mathbb{Z}_{>n}$, even.

We have: $G_k(z, 1) = 2 \cdot \sum_{n \in \mathbb{Z}_{>0}} n^{-k} + 2 \cdot \sum_{\substack{m \in \mathbb{Z}_{>1} \\ n \in \mathbb{Z}}} (n+mz)^{-k} =$



$$= 2 \cdot \zeta(k) + 2 \cdot \sum_{m \in \mathbb{Z}_{>n}} f_k(mz),$$

where $f_k: \mathbb{H} \rightarrow \mathbb{C}$, $z \mapsto \sum_{n \in \mathbb{Z}} (n+z)^{-k}$, converges absolutely & uniformly on compact subsets of $\mathbb{C} - \mathbb{Z}$.
 f_k is holomorphic and \mathbb{Z} -invariant.



Note: g_k is holomorphic, has a pole of order k at 1, and $\rightarrow 0$ as $|t| \rightarrow 0$ or $|t| \rightarrow \infty$ ($|\text{Im} z| \rightarrow \infty$)

Hence: $g_2(t) = \frac{c \cdot t}{(t-1)^2}$ for some $c \in \mathbb{C}^\times$.

To find c , we use that for $z \rightarrow 0$: $f_k(z) = z^{-k} + \text{holomorphic in } z$.

Now $f_2(z) = \frac{c \cdot e^{2\pi iz}}{(e^{2\pi iz} - 1)^2} = \frac{c \cdot (1 + 2\pi iz + (2\pi iz)^2/2! + \dots)}{(2\pi iz)^2 \cdot (1 + \frac{2\pi iz}{2!} + \dots)^2}$,

hence $c = (2\pi i)^2 = -4\pi^2$.

So $f_2(z) = (2\pi i)^2 \cdot \frac{e^{2\pi iz}}{(e^{2\pi iz} - 1)^2}$, $g_2(t) = (2\pi i)^2 \cdot \frac{t}{(t-1)^2}$.

Note that, for $|\epsilon| < 1$, $\frac{\epsilon}{(\epsilon-1)^2} = \epsilon \cdot \left(\frac{1}{\epsilon-1}\right)' = \epsilon \left(\sum_{n \geq 0} \epsilon^n\right)' = \sum_{n \geq 0} n \epsilon^n$. 3.

To conclude: $f_2 = -4\pi^2 \cdot \frac{q}{(1-q)^2} = -4\pi^2 \cdot \sum_{n \geq 0} n q^n$

For the f_k with $k \geq 2$, note:

$$\left(\frac{d}{dz}\right)^{k-2} f_2 = \left(\frac{d}{dz}\right)^{k-2} \left(\sum_{n \in \mathbb{Z}} (n+z)^{-2}\right) = (-1)^{k-2} \cdot (k-1)! f_k$$

And observe: $q = e^{2\pi i z}$, hence $dq = 2\pi i \cdot q \cdot dz$,

hence: $\frac{d}{dz} = 2\pi i \cdot q \cdot \frac{d}{dq}$

So: for $k \in \mathbb{Z}_{\geq 2}$: $f_k = (-1)^{k-2} \cdot \frac{1}{(k-1)!} \cdot (2\pi i)^{k-2} \cdot \left(q \frac{d}{dq}\right)^{k-2} \sum_{n \geq 0} n q^n$

$= \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 0} n^{k-1} \cdot q^n$

And for $k \in \mathbb{Z}_{\geq 3}$, k even:

$$G_k(z) = 2 \cdot \zeta(k) + \frac{2 \cdot (2\pi i)^k}{(k-1)!} \sum_{m \geq 1} \sum_{n \geq 0} n^{k-1} q^{mn}$$

$$= 2 \zeta(k) + \frac{2 \cdot (2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) \cdot q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$.

(recall the formulas for the $r_{\mathbb{Z}^n}(m)$, smaller n from last time!)

3. The E_k , and the $\zeta(k)$, k even.

We define $f_1: \mathbb{C} - \mathbb{Z} \rightarrow \mathbb{C}$, $z \mapsto \frac{1}{z} + \sum_{n \geq 1} \left(\frac{1}{z+n} + \frac{1}{z-n} \right)$
 (converges absolutely and uniformly on \dots).

Then $f_1' = \frac{d}{dz} f_1 = -f_2$, and, near 0, $f_1 = z^{-1} + \text{holomorphic } z$.

A small computation gives: $f_1 = \frac{2\pi i}{e^{2\pi i z} - 1} + \pi i$, (\mathbb{Z} -invariant!)
 (= $\pi \cdot \cot(\pi z)$), by the way.

Now we compute the coeff. of the power series of f_1 at 0.

$$f_1(z) = \frac{2\pi i}{e^{2\pi i z} - 1} + \pi i = z^{-1} + \sum_{n \geq 1} \frac{2z}{z^2 - n^2}$$

$$\begin{aligned} & \text{|| def} \\ & \pi i + \frac{1}{z} \cdot \sum_{k \geq 0} \frac{B_k}{k!} (2\pi i z)^k = - \sum_{n \geq 1} \frac{1}{n^2} \cdot \frac{2z}{1 - (z/n)^2} = \\ & \left(\frac{t}{e^t - 1} = \sum_{k \geq 0} \frac{B_k}{k!} t^k, \text{ in } \mathbb{Q}[[t]] \right) = - \sum_{n \geq 1} \frac{2z}{n^2} \cdot \sum_{m \geq 0} \left(\frac{z^2}{n^2} \right)^m \end{aligned}$$

(Bernoulli numbers)

$$\begin{aligned} & = -2z \cdot \sum_{n \geq 1} \sum_{m \geq 0} \frac{z^{2m}}{n^{2m+2}} \\ & = - \sum_{m \geq 0} 2 \cdot \zeta(2m+2) \cdot z^{2m+1} \end{aligned}$$

So, for $k \in \mathbb{Z}_{\geq 2}$ even: $-2 \zeta(k) = \text{coeff. of } z^{k-1} = \frac{B_k}{k!} \cdot (2\pi i)^k$

For $k \in \mathbb{Z}_{\geq 3}$, even, define:

$$E_k := \frac{1}{2 \zeta(k)} \cdot G_k = 1 - \frac{2k}{B_k} \cdot \sum_{n \geq 1} \sigma_{k-1}(n) \cdot q^n$$

Next: define the spaces $M_k(\Gamma)$ of holomorphic modular forms.

- determine the groups $\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \rangle$.
- dimensions of $M_k(\Gamma)$'s, bases. Deduce $r_{\mathbb{Z}^n}(m)$, $n \in \{2, 4, 6, 8, 10\}$.