TAG1, 2009/09/14, Bas Edixhoven, <u>From sums of squares to modular forms, continued</u>. (<u>Motto:</u> no formulas, but geometry!) 1.

0. <u>Recall from last week:</u>

$$\begin{tikzcd}
\mathbb{H} \arrow[d] \arrow[r, bend left, "\mathrm{sec}"] & B \arrow[l, "\mathrm{quot}"'] \arrow[d] \arrow[loop right, "\mathbb{C}^\times"] \\
\left\{ \begin{array}{l} \text{lattices in } \mathbb{C} \\ \text{up to } \mathbb{C}^\times \end{array} \right\} & L \arrow[l]
\end{tikzcd}$$

$B = \{(z_1, z_2) \in \mathbb{C}^2 \mid \mathbb{R}z_1 + \mathbb{R}z_2 = \mathbb{C}, \ \mathrm{Im}(z_1/z_2) > 0\}$

$= \{\text{lattices in } \mathbb{C} \text{ with oriented } \mathbb{Z}\text{-basis}\}$

$L = \{\text{lattices in } \mathbb{C}\}$

$\longleftarrow$: quotient for $\mathbb{C}^\times$, $\downarrow$: quotient for $\mathrm{SL}_2(\mathbb{Z})$.

For $k \in \mathbb{Z}$, $M_k(B) = \{f : B \to \mathbb{C} \mid \forall \lambda, \forall b: \ f(\lambda \cdot b) = \lambda^{-k} \cdot f(b)\}$

$\quad \wr \downarrow \mathrm{sec}^*$

$M_k(\mathbb{H}) = \{f : \mathbb{H} \to \mathbb{C}\}$.

- $f : \mathbb{H} \to \mathbb{C}$ gives $(z_1, z_2) \mapsto z_2^{-k} \cdot f(z_1/z_2)$ in $M_k(B)$.
- The right action by $\mathrm{GL}_2(\mathbb{R})^+$ on $M_k(B)$, becomes on $M_k(\mathbb{H})$:

$$\underbrace{(f \cdot g)}_{g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z) = \Big((z_1, z_2) \mapsto z_2^{-k} \cdot f(z_1/z_2)\Big)\left(g \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}\right) = (cz+d)^{-k} \cdot f\left(\frac{az+b}{cz+d}\right).$$

For $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ a subgroup, we have $M_k(\mathbb{H})^\Gamma = \{f \in M_k(\mathbb{H}) \mid \forall \gamma \in \Gamma : f \cdot \gamma = f\}$

Then $M_k(\mathbb{H})^\Gamma = M_k(B)^\Gamma = \{f \in M_k(B) \mid f \text{ depends only on the basis up to } \Gamma\text{-action}\}$

1. <u>Examples.</u>

(1) Let $k \in \mathbb{Z}_{\geq 3}$, and for each $\Lambda \subset \mathbb{C}$ a lattice, put: $G_k(\Lambda) = \sum_{0 \neq x \in \Lambda} x^{-k}$, this converges absolutely, and it is zero iff $k \equiv 1 \, (2)$.

Hence: $G_k \in M_k(B)^{\mathrm{SL}_2(\mathbb{Z})} = M_k(\mathbb{H})^{\mathrm{SL}_2(\mathbb{Z})}$.

G_k: Eisenstein series of weight k on $\mathrm{SL}_2(\mathbb{Z})$ (up to some normalising constant factor).

(1.2) Let $n \in \mathbb{Z}_{\geq 0}$, $n \equiv 0\ (8)$

Then, $\forall z \in \mathbb{H}$: $\Theta_{\mathbb{Z}^n}(-1/z) = (-iz)^{n/2}\,\Theta_{\mathbb{Z}^n}(z) = z^{n/2}\,\Theta_{\mathbb{Z}^n}(z)$,

hence $\Theta_{\mathbb{Z}^n} \in M_{n/2}(\mathbb{H})^{\Gamma}$, where $\Gamma = \left\langle \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}0&-1\\1&0\end{pmatrix} \right\rangle \subset SL_2(\mathbb{Z})$.

(1.3) Let $n \in \mathbb{Z}_{\geq 0}$, $n \equiv 0\ (2)$. Note that $\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}1&-2\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix} = \begin{pmatrix}1&0\\2&1\end{pmatrix}$.

We have, $\forall z \in \mathbb{H}$:

$$\Theta_{\mathbb{Z}^n}\left(\begin{pmatrix}1&0\\2&1\end{pmatrix}\cdot z\right) = \Theta_{\mathbb{Z}^n}\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix}\cdot \frac{2z+1}{-z}\right) = \left(\cancel{4}i\cdot\frac{2z+1}{\cancel{4}z}\right)^{n/2}\cdot\,\Theta_{\mathbb{Z}^n}\left(-2-\frac{1}{z}\right) =$$

$$= \left(i\cdot\frac{2z+1}{z}\right)^{n/2}\cdot\,\Theta_{\mathbb{Z}^n}\left(-\frac{1}{z}\right) = \left(i\cdot\frac{2z+1}{z}\right)^{n/2}\cdot(-iz)^{n/2}\cdot\,\Theta_{\mathbb{Z}^n}(z)$$

$$= (2z+1)^{n/2}\cdot\,\Theta_{\mathbb{Z}^n}(z).$$

So: $\Theta_{\mathbb{Z}^n} \in M_{n/2}(\mathbb{H})^{\Gamma}$, where $\Gamma = \left\langle \begin{pmatrix}1&0\\2&1\end{pmatrix}, \begin{pmatrix}1&2\\0&1\end{pmatrix} \right\rangle$.

2. <u>q-expansions of the G_k.</u> **Reference**: Diamond-Shurman, p. 4–6, and exercises 1.1.4–1.1.7.

Let $k \in \mathbb{Z}_{\geq 4}$, even.

We have $\forall z \in \mathbb{H}$: $G_k(z,1) = 2\cdot\sum_{n\in\mathbb{Z}_{>0}} n^{-k} + 2\cdot\sum_{\substack{m\in\mathbb{Z}_{\geq 1}\\ n\in\mathbb{Z}}} (n+mz)^{-k} =$

$$= 2\cdot\zeta(k) + 2\cdot\sum_{m\in\mathbb{Z}_{\geq 1}} f_k(mz),$$

where $f_k : \mathbb{H} \to \mathbb{C}$, $z \mapsto \sum_{n\in\mathbb{Z}} (n+z)^{-k}$, converges absolutely & uniformly on compact subsets of $\mathbb{C}-\mathbb{Z}$

$$\begin{tikzcd} \mathbb{H} \arrow[d, phantom, "\cap"] \arrow[r] & \mathbb{C} \\ \mathbb{C}-\mathbb{Z} \arrow[ur] \arrow[d, "q"] & \\ \mathbb{C}^\times-\{1\} \arrow[uur, bend right, "g_k"'] & \end{tikzcd} \qquad z \mapsto e^{2\pi i z}$$

holomorphic, and $\mathbb{Z}$-invariant.

g_k. Note: g_k is holomorphic, has a pole of order k at 1, and $\to 0$ as $|t|\to 0$ or $|t|\to\infty$ ($|\mathrm{Im}\, z|\to\infty$)

Hence: $g_2(t) = \dfrac{c\cdot t}{(t-1)^2}$ for some $c \in \mathbb{C}^\times$.

To find c, we use that for $z \to 0$: $f_k(z) = z^{-k} +$ holomorphic in z.

Now $f_2(z) = \dfrac{c\cdot e^{2\pi i z}}{(e^{2\pi i z}-1)^2} = \dfrac{c\cdot\left(1+2\pi i z+(2\pi i z)^2/2!+\cdots\right)}{(2\pi i z)^2\cdot\left(1+\frac{2\pi i z}{2!}+\cdots\right)^2}$,

hence $c = (2\pi i)^2 = -4\pi^2$.

So $f_2(z) = (2\pi i)^2\cdot\dfrac{e^{2\pi i z}}{(e^{2\pi i z}-1)^2}$, $\quad g_2(t) = \dfrac{(2\pi i)^2\cdot t}{(t-1)^2}$.

Note that, for $|t|<1$, $\dfrac{t}{(t-1)^2} = t\cdot\left(\dfrac{1}{1-t}\right)' = t\left(\sum_{n\geq 0} t^n\right)' = \sum_{n\geq 0} n t^n$

To conclude: $f_2 = -4\pi^2\cdot\dfrac{q}{(1-q)^2} = -4\pi^2\cdot\sum_{n\geq 0} n q^n$

For the f_k with $k\geq 2$, note:

$$\left(\frac{d}{dz}\right)^{k-2} f_2 = \left(\frac{d}{dz}\right)^{k-2}\left(\sum_{n\in\mathbb{Z}} (n+z)^{-2}\right) = (-1)^{k-2}\cdot(k-1)!\; f_k$$

And observe: $q = e^{2\pi i z}$, hence $dq = 2\pi i\cdot q\cdot dz$,

hence: $\dfrac{d}{dz} = 2\pi i\cdot q\cdot\dfrac{d}{dq}$

So: for $k\in\mathbb{Z}_{\geq 2}$:
$$f_k = (-1)^{k-2}\cdot\frac{1}{(k-1)!}\cdot(2\pi i)^{k-2}\cdot\left(q\frac{d}{dq}\right)^{k-2}(2\pi i)^2\sum_{n\geq 0} n q^n$$
$$= \frac{(-2\pi i)^k}{(k-1)!}\sum_{n\geq 0} n^{k-1}\cdot q^n$$

And for $k\in\mathbb{Z}_{\geq 3}$, k even:

$$G_k(z) = 2\cdot\zeta(k) + 2\cdot\frac{(2\pi i)^k}{(k-1)!}\sum_{m\geq 1}\sum_{n\geq 0} n^{k-1} q^{mn}$$
$$= 2\,\zeta(k) + 2\cdot\frac{(2\pi i)^k}{(k-1)!}\sum_{n\geq 1}\sigma_{k-1}(n)\cdot q^n,$$

where $\sigma_{k-1}(n) = \sum_{d\mid n} d^{k-1}$. (recall the formulas for the $r_{\mathbb{Z}^n}(m)$, small n from last time!)

3. The E_k, and the $\zeta(k)$, k even.

We define $f_1 : \mathbb{C}-\mathbb{Z} \to \mathbb{C}$, $z \mapsto \frac{1}{z} + \sum_{n\geq 1} \underbrace{\left(\frac{1}{z+n} + \frac{1}{z-n}\right)}_{= \frac{2z}{z^2-n^2}}$

(converges absolutely and uniformly on .....).

Then $f_1' = \frac{d}{dz} f_1 = -f_2$, and, near 0, $f_1 = z^{-1} + \text{holomorphic} \cdot z$,

A small computation gives: $f_1 = \frac{2\pi i}{q-1} + \pi i$, ($\mathbb{Z}$-invariant!)

$(= \pi \cdot \cot(\pi z)$, by the way.)

Now we compute the coeff. of the power series of f_1 at 0.

$$f_1(z) = \frac{2\pi i}{e^{2\pi i z}-1} + \pi i = z^{-1} + \boxed{\sum_{n\geq 1} \frac{2z}{z^2-n^2}}$$

$$\frac{2\pi i}{e^{2\pi i z}-1} + \pi i \overset{\text{def}}{=} \pi i + \frac{1}{z} \cdot \sum_{k \geq 0} \frac{B_k}{k!} (2\pi i z)^k$$

$$\left(\frac{\epsilon}{e^{\epsilon}-1} = \sum_{k\geq 0} \frac{B_k}{k!} \epsilon^k, \text{ in } \mathbb{Q}[[\epsilon]] \right).$$

(Bernoulli numbers)

$$\sum_{n\geq 1} \frac{2z}{z^2-n^2} = -\sum_{n\geq 1} \frac{1}{n^2} \cdot \frac{2z}{1-(z/n)^2} =$$

$$= -\sum_{n\geq 1} \frac{2z}{n^2} \cdot \sum_{m\geq 0} \left(z^2/n^2\right)^m$$

$$= -2z \cdot \sum_{\substack{n\geq 1\\ m\geq 0}} \frac{z^{2m}}{n^{2m+2}}$$

$$= -\sum_{m\geq 0} 2 \cdot \zeta(2m+2) \cdot z^{2m+1}$$

So, for $k \in \mathbb{Z}_{\geq 2}$ even: $\boxed{-2\zeta(k) = \text{coeff. of } z^{k-1} = \frac{B_k}{k!} (2\pi i)^k}$

For $k \in \mathbb{Z}_{\geq 3}$, even, define:

$$\boxed{E_k := \frac{1}{2\zeta(k)} \cdot G_k = 1 - \frac{2k}{B_k} \cdot \sum_{n\geq 1} \sigma_{k-1}(n) \cdot q^n}$$

Next: • define the spaces $M_k(\Gamma)$ of holomorphic modular forms.
• determine the groups $\left\langle \begin{pmatrix}1&0\\2&1\end{pmatrix}, \begin{pmatrix}1&2\\0&1\end{pmatrix} \right\rangle$. (
• dimensions of $M_k(\Gamma)$'s, bases. Deduce $r_{\mathbb{Z}^n}(m)$, $n \in \{2,4,6,8,10\}$.