1. The quotients $\Gamma\backslash\mathbb{H}$ as $\mathbb{C}$-manifolds, $\Gamma \subset SL_2(\mathbb{Z})$ subgroup.

Def. For $\Gamma \subset SL_2(\mathbb{Z})$, subgroup, $Y(\Gamma) := \Gamma\backslash\mathbb{H}$, quot: $\mathbb{H} \to Y(\Gamma)$ the quotient map, in (Sets).

Topology: give $Y(\Gamma)$ the quotient topology, i.e., $U \subset Y(\Gamma)$ open $\Longleftrightarrow$ $\mathrm{quot}^{-1}U \subset \mathbb{H}$ is open. Then quot : $\mathbb{H} \to Y(\Gamma)$ is a quotient in (Top).

$\mathbb{C}$-manifold: for $U \subset Y(\Gamma)$ open:

$$\mathcal{O}_{Y(\Gamma)}(U) := \{ f : U \to \mathbb{C} \mid f \circ \mathrm{quot} : \mathrm{quot}^{-1}U \to \mathbb{C} \text{ is holomorphic} \}. \qquad (\mathrm{quot}^{-1}U \subset \mathbb{H})$$

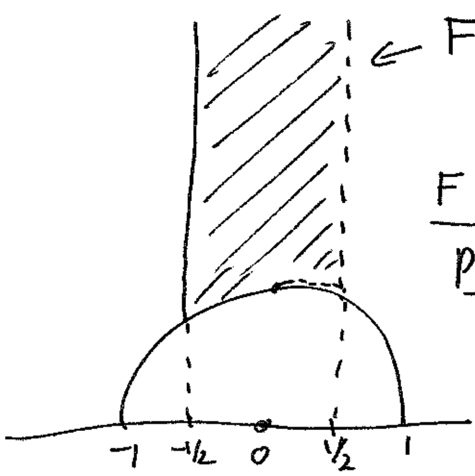
Claim $(Y(\Gamma), \mathcal{O}_{Y(\Gamma)})$ is locally isom. to $(D, \mathcal{O}_D)$, $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$.
To show this, we need "local coordinates", at all points of $Y(\Gamma)$.

Recipe for local coordinates. Let $\tau \in \mathbb{H}$. Let $z : \mathbb{H} \to \mathbb{C}$, $\tau \mapsto \tau$.
(standard global coordinate on $\mathbb{H}$).

Put: $z_\tau := \prod_{\gamma \in \Gamma_\tau/(\{\pm 1\} \cap \Gamma_\tau)} \gamma^*(z - \tau)$, $\quad e_\tau := \#\left(\Gamma_\tau / (\{\pm 1\} \cap \Gamma_\tau)\right)$. (finite)

Then $v_\tau(z_\tau) = e_\tau$, $\exists\, t_\tau \in \mathcal{O}_{\mathbb{H},\tau}$ with $t_\tau^{e_\tau} = z_\tau$, $v_\tau(t_\tau) = 1$,
$\Gamma_\tau / (\Gamma_\tau \cap \{\pm 1\}) = \mu_{e_\tau}$, multiplies t_τ by root of 1.

To get the required open subsets on which z_τ is a local coordinate we study the standard fundamental domain for $SL_2(\mathbb{Z})$.



$$F = \left\{ \tau \in \mathbb{H} \;\middle|\; \begin{array}{l} |\tau| \geq 1,\ -1/2 \leq \mathrm{Re}(\tau) < 1/2 \\ |\tau| = 1 \Rightarrow \mathrm{Re}(\tau) \leq 0 \end{array} \right\}$$

F is a fund. domain for $SL_2(\mathbb{Z}) \circlearrowright \mathbb{H}$.

Proof. Let $\tau \in \mathbb{H}$. Then $SL_2(\mathbb{Z}) \cdot \tau =$
$= \{ z_1/z_2 \mid (z_1, z_2)$ is an oriented $\mathbb{Z}$-basis of $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau \}$

Let $z_2 \in \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau$ be a shortest non-zero element, and (z_1, z_2) a $\mathbb{Z}$-basis. Then $|z_1/z_2| \geq 1$, and, putting $\tau' := z_1/z_2$, $z_2^{-1}\cdot(\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau) = \mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau'$.

Let $\tau'' := \tau' - n$, s.t. ~~Re~~ $-1/2 \le \mathrm{Re}(\tau'') < 1/2$; (unique n).
Then $|\tau''| \geq 1$. If $|\tau''| = 1$ and $\mathrm{Re}(\tau'')>0$, then $-\tau''$ is also a shortest element of $\mathbb{Z}\cdot 1 + \mathbb{Z}\cdot\tau''$, and applying our construction above gives $-1/\tau'' \in F$.
So, we have shown that F intersects all orbits.
Let us now show that it intersects each orbit exactly <u>once</u>.
See my Trieste notes (homepage, William Stein...), p. 3.
That also gives, for $\tau \in F$: $e_\tau = 1$ if $\tau \notin \{i, e^{2\pi i/3}\}$

$e_i = 2$, $e_{e^{2\pi i/3}} = 3$.

Local model for $\mathbb{H}$ at $\tau \in F$:

$$\begin{tikzcd}
\mathbb{H} \arrow[d, "\mathrm{quot}"] & \{a\in\mathbb{H} \mid |t_\tau(a)|<\varepsilon\} \arrow[r, "t_\tau", "\sim"'] \arrow[d, "\mathrm{quot}"] & D(0,\varepsilon) \ni w \arrow[d] \\
Y(SL_2(\mathbb{Z})) & \{\qquad\} \arrow[r, "z_\tau", "\sim"'] & D(0, \varepsilon^{e_\tau}) \ni w^{e_\tau}
\end{tikzcd}$$

$\varepsilon \in \mathbb{R}_{>0}$ suff. small.

For more details: see Diamond–Shurman, Ch. 2.

2. The compactifications $Y(\Gamma) \hookrightarrow X(\Gamma)$.

Here, we assume that $\Gamma \subset SL_2(\mathbb{Z})$ is of <u>finite index</u>.
<u>Reference</u>: Trieste, §1.3.

<u>Step 1.</u> $\Gamma = SL_2(\mathbb{Z})$. Let $\mathcal{U} = \{\tau\in\mathbb{H} \mid \mathrm{Im}(\tau) > 1\}$.
Then, for $\tau, \tau' \in \mathcal{U}$: $\tau' \in SL_2(\mathbb{Z}).\tau \iff \tau' \in \tau + \mathbb{Z}$.

So:

$$\begin{tikzcd}
\mathbb{H} \arrow[d, "\mathrm{quot}"'] & \mathcal{U} \arrow[l, hook'] \arrow[d, "q"] & \tau \arrow[d, mapsto] \\
Y(SL_2(\mathbb{Z})) & D(0, e^{-2\pi})^\times \arrow[l, hook'] & e^{2\pi i \tau}
\end{tikzcd}$$

punctured disk: $\{z\in\mathbb{C} \mid 0<|z|<e^{-2\pi}\}$.

open immersion.

Let $\overline{F}$:= closure of F in $\mathbb{H}$. Then $\overline{F} \cap \{\tau\in\mathbb{H} \mid \mathrm{Im}(\tau) \leq 2\}$ is compact. Hence: $Y(SL_2(\mathbb{Z})) - D(0, e^{-2\pi})$ is compact.

$$\begin{tikzcd}
Y(SL_2(\mathbb{Z})) \arrow[d, hook] & D(0, e^{-2\pi})^\times \arrow[l, hook'] \arrow[d, hook] \\
X(SL_2(\mathbb{Z})) & D(0, e^{-2\pi}) \arrow[l, hook']
\end{tikzcd}$$

<u>push-out</u>. (co-Cartesian).

<u>compact</u>, connected, simply connected.

So: $\begin{cases} X(SL_2(\mathbb{Z})) \cong \mathbb{P}^1(\mathbb{C}) \\ Y(SL_2(\mathbb{Z})) \cong \mathbb{A}^1(\mathbb{C}). \end{cases}$

<u>Step 2</u> $\Gamma \subset SL_2(\mathbb{Z})$ of finite index.

Then take $\Gamma' \subset \Gamma$ s.t. $\Gamma' \triangleleft SL_2(\mathbb{Z})$ with finite index.

Then

$$\begin{tikzcd} Y(\Gamma') \arrow[r] \arrow[dr, "SL_2(\mathbb{Z})/\Gamma'"'] & Y(\Gamma) \arrow[d] \\ & Y(SL_2(\mathbb{Z})) \end{tikzcd}$$

is quotient for Γ/Γ'

Hence: all these maps (in (Top)) are <u>proper</u>: inverse image of compact is compact (equivalently: separated & universally closed).

$$\begin{tikzcd} Y(\Gamma) \arrow[d, "f"'] \arrow[r, hookleftarrow] \arrow[dr, phantom, "\square"] & \cancel{\ } f^{-1}D^\times \arrow[d] \\ Y(SL_2(\mathbb{Z})) \arrow[r, hookleftarrow] & D^\times \end{tikzcd}$$

$\leftarrow$ unramified cover. (étale)

as above

Hence $f^{-1}D^\times$ is a disjoint union of connected unram. covers of $D^\times$.

Note: $\pi_1(D^\times) = \mathbb{Z}$.

Now read Trieste §1.3 completely.