TAG1, 2009/10/12, Bas Edixhoven. esp. Introd. of EGA I, Springer edition.



Introduction to schemes. References: Hartshorne AG, EGA's, ....

Advice: find a healthy balance between (commutative) algebra, geometry, and category theory; all 3 are essential, and, e.g., too little cat. th. makes it harder to understand. Cat. theory is a very efficient language, and has many basic principles (adjoints —)). Too much comm. alg. makes it dull. Always think of applications!

a. Sheaves. For $X$ in Top, a presheaf of sets on $X$ is a contravariant functor $\mathcal{F}\colon \mathrm{Open}(X) \longrightarrow \mathrm{Set}$. Such an $\mathcal{F}$ is called a sheaf if: $\forall\, U \subset X$ open, $\forall$ open cover $U = \bigcup_{i\in I} U_i$, the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I^2} \mathcal{F}(U_i \cap U_j) \quad \text{is exact.}$$

$$s \longmapsto ((i,j) \mapsto s_i|_{U_i\cap U_j})$$
$$\longmapsto ((i,j) \mapsto s_j|_{U_i\cap U_j})$$

Exercise. For $\mathcal{F}$ a sheaf on $X$, show that $\mathcal{F}(\emptyset) = \{\emptyset\}$, a 1-elmt set.

We have categories: $\mathrm{Presh}_{\mathrm{Set}}(X)$, $\mathrm{Sh}_{\mathrm{Set}}(X)$, $\mathrm{Sh}_{\mathbb{Z}\text{-mod}}(X)$, etc., morphisms of presheaves are just morphisms of functors, and $\mathrm{Sh}(X)$ is a full subcat. of $\mathrm{Presh}(X)$.

For $f\colon X \to Y$ in Top: $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, $(f_*\mathcal{F})\colon \underset{\substack{\cup\\ Y}}{U} \mapsto \mathcal{F}(f^{-1}U)$.

$f^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$, $f^{-1}\mathcal{F} = \left(U \mapsto \varinjlim_{V\supset fU} \mathcal{F}(V)\right)^{+}$

$f_*$ & $f^{-1}$ are adjoints: $\mathrm{Hom}_{\mathrm{Sh}(Y)}(\mathcal{G}, f_*\mathcal{F}) = \mathrm{Hom}_{\mathrm{Sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F})$

$$\Vert$$

$$\left\{ \begin{pmatrix} \mathcal{F} \xleftarrow{\varphi} \mathcal{G} \\ X \xrightarrow{f} Y \end{pmatrix} : \begin{array}{l} \forall\, U \subset Y \text{ open:} \\ \mathcal{G}(U) \longrightarrow \mathcal{F}(f^{-1}U) \end{array} \right\}$$

## 1. Locally ringed spaces (LRS).

A loc. ringed space is a $(X, \mathcal{O}_X)$, $X$ in Top, $\mathcal{O}_X$ a sheaf of rings on $X$, (comm, 1)
s.t. $\forall x \in X$: $\mathcal{O}_{X,x} := \varinjlim_{U \ni x} \mathcal{O}_X(U)$ is a local ring: $\exists!$ max. ideal $m_x$.
$\left( \mathcal{O}_{X,x} \xrightarrow{ev_x} \kappa_x \,;\ ev_x^{-1} \kappa_x^\times = \mathcal{O}_{X,x}^\times \right)$.

Morphisms: $\begin{pmatrix} \mathcal{O}_X & \xleftarrow{f^\#} & \mathcal{O}_Y \\ X & \xrightarrow[f]{} & Y \end{pmatrix}$, s.t. $\forall x \in X$: 
$$\begin{tikzcd} \mathcal{O}_{Y,fx} \arrow[r, "f^\#_x"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{O}_{X,x} \arrow[d] \\ \kappa_{fx} \arrow[r, dashed] & \kappa_x \end{tikzcd}$$

Example 1. Let $X$ be a $C^\infty$-manifold, $C^\infty_{X,\mathbb{R}}$ : Open$(X) \to \mathbb{R}$-alg
$U \mapsto \{ f : U \to \mathbb{R} \mid f\ C^\infty \}$.
Then $(X, C^\infty_{X,\mathbb{R}})$ is in LRS/$\mathbb{R}$.

2. Let $X$ be a manifold of dim. $>0$, $\mathcal{F}(U) = \{ f : U \to \mathbb{R} \}$, then $(X, \mathcal{F})$ is not a LRS.

## 2. Schemes.

Let $A$ be a ring, then we have $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}\,A})$ in LRS;

in Set: $\mathrm{Spec}\,A = \{$prime ideals of $A\}$

Top: $\forall f \in A$: $\forall \mathfrak{p} \in \mathrm{Spec}\,A$: $f(\mathfrak{p}) :=$ image of $f$ in $\mathrm{Frac}(A/\mathfrak{p})$.
the closed subsets of $\mathrm{Spec}\,A$ are the $Z(S) = \{ \mathfrak{p} \in \mathrm{Spec}\,A \mid \forall f \in S : f(\mathfrak{p}) = 0 \}$, $S \subset A$ subset.
Note: the $D(f) = \{ \mathfrak{p} \in \mathrm{Spec}\,A \mid f(\mathfrak{p}) \neq 0 \}$ form a basis for the topology.

LRS. Let $M$ be an $A$-module, then we have a unique sheaf of $\mathbb{Z}$-mod. $\widetilde{M}$ on $\mathrm{Spec}\,A$, such that $\forall f \in A$: $\widetilde{M}(D(f)) = M_f = \{1, f, \dots\}^{-1} M$, $\forall f, g \in A$ s.t. $D(g) \subset D(f)$, $\widetilde{M}(D(f)) \to \widetilde{M}(D(g))$ is the natural morphism. (See EGA 1, I.1.3) (in particular: Thm. I.1.3.7)

One has: $\forall U \subset \mathrm{Spec}\,A$ open:
$$\widetilde{M}(U) = \left\{ s : U \to \coprod_{\mathfrak{p} \in U} M_\mathfrak{p} \;\middle|\; \begin{array}{l} \forall \mathfrak{p} \in U : s(\mathfrak{p}) \in M_\mathfrak{p}, \\ \forall \mathfrak{p} \in U\ \exists\ V \subset U \text{ open neighb. of } \mathfrak{p}, \\ m \in M,\ f \in A \text{ s.t.} \\ \forall \mathfrak{q} \in V : f(\mathfrak{q}) \neq 0,\ s(\mathfrak{q}) = \frac{m}{f} \text{ in } M_\mathfrak{q} \end{array} \right\}$$

And: $\forall \mathfrak{p} \in \mathrm{Spec}\,A : (\widetilde{M})_\mathfrak{p} = M_\mathfrak{p}$.

We put: $\mathcal{O}_{\operatorname{Spec} A} = \widetilde{A}$.

Def. A *scheme* is a loc. ringed space $(X, \mathcal{O}_X)$ s.t. $\forall x \in X$, $\exists$ open neighb. $U \subset X$ s.t. $(U, \mathcal{O}_X|_U)$ is isom. to some $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ (i.e., is an affine scheme) ~~affine~~

Def. morphisms of schemes are morphisms of LRS.

We have $\operatorname{Spec} : \mathrm{Ring} \longrightarrow \mathrm{LRS}$, contravariant

$\Gamma : \mathrm{LRS} \longrightarrow \mathrm{Ring}$, $(X, \mathcal{O}_X) \longmapsto \mathcal{O}_X(X)$, contravariant.

These are adjoints: $\forall (X, \mathcal{O}_X)$, $A$:

$$\operatorname{Hom}_{\mathrm{LRS}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\mathrm{Ring}}(A, \mathcal{O}_X(X)) \quad \text{is bijective}.$$

$$(f, f^\#) \longmapsto A = \mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \xrightarrow{f^\#} \mathcal{O}_X(X).$$

(see EGA 1, I.1.6.3). For the case $X$ in Sch, see HAG.

Important: Spec induces an anti-equivalence ~~between~~

$\mathrm{Ring} \longrightarrow \mathrm{AffSch}$.

## 3. $\mathcal{O}_X$-modules.

Def. Let $(X, \mathcal{O}_X)$ be in LRS. An *$\mathcal{O}_X$-module* (or: *sheaf of $\mathcal{O}_X$-modules*) is a sheaf of $\mathbb{Z}$-modules $\mathcal{M}$ with, $\forall U \subset X$ open, an $\mathcal{O}_X(U)$-module structure on $\mathcal{M}(U)$, compatible with restrictions.

*Example.* $\mathcal{O}_X^n$, kernels, cokernels.

Let $X$ be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* if $\forall U \subset X$ open affine: $\mathcal{F}|_U = \widetilde{\mathcal{F}(U)}$; $M \longmapsto \widetilde{M}$ is an equivalence: $A\text{-modules} \longrightarrow \mathrm{QCoh}(\operatorname{Spec} A)$.

Reference: EGA 1, I.1.4, but see also EGA 1, 0.5.

## 4. Representable functors approach.

Let $\mathbb{G}_m := \mathrm{Spec}(\mathbb{Z}[x,y]/(xy-1))$.

$\forall$ $X$ in Sch: $\mathrm{Hom}_{Sch}(X, \mathbb{G}_m) = \mathrm{Hom}_{Ring}(\mathbb{Z}[x,y]/(xy-1), \mathcal{O}_X(X))$

$$\begin{array}{cc} \mathrm{Hom}_{Ring}(\mathbb{Z}[x,y]/(xy-1), \mathcal{O}_X(X)) & \varphi \\ \downarrow\wr & \downarrow \\ \mathcal{O}_X(X)^\times & \varphi(x). \end{array}$$

In particular, $\forall$ ring $A$:

$\mathrm{Hom}_{Sch}(\mathrm{Spec}\, A, \mathbb{G}_m) \xrightarrow{\sim} A^\times$, funct. in $A$:

isomorphism of functors: $\mathrm{Hom}_{Sch}(\mathrm{Spec}(\cdot), \mathbb{G}_m) \longrightarrow (\cdot)^\times$.

Terminology: $\mathbb{G}_m$ <u>represents</u> the functor $X \longmapsto \mathcal{O}_X(X)^\times$, $Sch \longrightarrow Grp$.

<u>Yoneda Lemma.</u> Let $\mathcal{C}$ be any category. For $X$ in $\mathcal{C}$, define $h_X : \mathcal{C} \to Set$, $Y \mapsto h_X(Y) := \mathrm{Hom}_{\mathcal{C}}(Y, X)$, contrav. in $Y$.

Let $X$ vary; we have: $h : \mathcal{C} \longrightarrow \mathrm{Hom}_{contra}(\mathcal{C}, Set)$, <u>covariant</u>.

Then $h$ is fully faithful, i.e., $h$ induces an equivalence to the essential image of $h$; the representable functors. In fact, $\forall$ $F : \mathcal{C} \to Set$ contrav., and $\forall$ $X \in \mathcal{C}$:

$$\mathrm{Hom}(h_X, F) \longrightarrow F(X), \quad \varphi \mapsto \varphi(\mathrm{id}_X), \text{ is bijective}$$

See EGA 1, I.0.1. We may as well write $X$ for $h_X$.

Hence: $X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$.

<u>Example</u> 1. Let $n \geq 0$. The contravariant functor $Sch \to Set$,

$\mathbb{P}^n$: $S \longmapsto \{\text{sub-}\mathcal{O}_S\text{-modules } \mathcal{E} \subset \mathcal{O}_S^{n+1} \text{ s.t. } \mathcal{O}_S^{n+1}/\mathcal{E} \text{ is loc. free of rk } 1\}$

is representable; the representing scheme, unique up to unique isom., is denoted $\mathbb{P}^n$. (EGA I. 9).

2. Let $\begin{array}{ccc} & & X \\ & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$ in Sch, then the functor $Sch \to Set$,

$$S \longmapsto \{(P, Q) \in X(S) \times Y(S) \mid f(P) = g(Q) \text{ in } Z(S)\}$$

is representable; $\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$.

See HAG. II. 3.