Barcelona, Advanced course on Shimura varieties and L-functions; 2009/10: 19-20-21.

Lecture 1.

1.

<u>The A-O conjecture</u>: let S be a Shimura variety, over $\mathbb{C}$, and $\Sigma \subset S$ any subset of special points. Then all irreducible components of Σ^{Zar} are special subvarieties of S.

<u>Origin</u>. Yves André, 1989, the question on curves in S with inf. many sp. pts.
Frans Oort, 1994, the question ~~for~~ when $S = A_{g,1}$.
Ben Moonen, 1995, general formulation.

<u>Similarity</u> with Manin-Mumford conjecture / Thm of Raynaud. Let A be a complex semiabelian variety, $\Sigma \subset A$ a set of torsion points. Then each irr. comp. of Σ^{Zar} is an irreducible comp. of an algebraic subgroup.

<u>Generalisations</u>. 1. Equidistribution of Galois orbits of special points. Thm in M-M. case. ~~Ref~~ Ullmo, Zhang.
2. Pink formulated a conjecture that contains A-O, M-M, Mordell-Lang; preprints, article on his homepage.
$S = A_{g,1}$, restrictions on Σ: Moonen, 1994-5. <u>Mixed Shimura varieties!</u>

<u>Results</u>. $S = A_1 \times A_1$, Edixhoven (GRH), 1995, André, 1995/6.. <u>no GRH!</u>
HMS (I), S general, Σ^{Zar} a curve, E-Y (restr. on Σ). Y.
$(A_1)^n$ (I). Clozel-Ullmo: introduction of <u>ergodic methods</u>; no GRH
General case: Klingler-Yafaev + Ullmo-Yafaev, under GRH;
not yet published, but process seems to converge...
These 2 articles are in fact the subject of this course.
Jonathan Pila: introduced o-minimal structures, "model theory", just as it happened for M-M. No GRH; treats $(A_1)^n$. Seems (to me) hard to generalise, for the moment, because of too strong lower bds on Galois orbits are needed.
Florian Breuer: Drinfeld modular varieties.

A toy toric M-M case.

Thm. Let $C \subset \mathbb{C}^\times \times \mathbb{C}^\times$ be an irred. closed curve s.t. $\Sigma := C \cap (\mathbb{C}^\times \times \mathbb{C}^\times)_{\mathrm{tors}}$ is infinite. Then $\exists\, a, b \in \mathbb{Z}$ C is an irred. comp. of $Z(x^a y^b - 1)$, i.e., C is a translate over a torsion point of a 1-dimensional subtorus of $\mathbb{C}^\times \times \mathbb{C}^\times$.

Proof 1. We may and do assume that $1 = (1,1)$ is a nonsing. pt. of C (via translation by a nonsingular special point).

2. C is defined over a finite ext. $\mathbb{Q} \to F \to \mathbb{C}$.

3. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $(\mathbb{C}^\times \times \mathbb{C}^\times)_{\mathrm{tors}}$ via $\chi : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow \hat{\mathbb{Z}}^\times = \varprojlim (\mathbb{Z}/n\mathbb{Z})^\times$,

$\sigma \cdot (x_1, x_2) = (x_1^{\chi(\sigma)}, x_2^{\chi(\sigma)})$.

For $x \in \Sigma$: $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x = \{$generators of $\langle x \rangle\}$, $\# \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x = \varphi(\mathrm{order}(x))$,

$\# \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot x \geqslant \varphi(\mathrm{order}(x)) / \dim_{\mathbb{Q}}(F)$. $\to \infty$ as x ranges through Σ.

Take $x \in \Sigma$ with $n := \mathrm{order}(x)$ large w.r.t. $\deg(C)$ in $\mathbb{P}^2$ and w.r.t. $\dim_{\mathbb{Q}} F$.

Take p a small prime s.t. $p \nmid n$, and put $a := p^{\dim_{\mathbb{Q}} F}$.

Key observation: $x^a = (x_1^a, x_2^a) \in C \cap \varphi_a C$, $\quad \varphi_a : \mathbb{C}^\times \times \mathbb{C}^\times \to \mathbb{C}^\times \times \mathbb{C}^\times$

$(z_1, z_2) \mapsto (z_1^a, z_2^a)$.

indeed: $(\mathbb{Z}/n\mathbb{Z})^\times \supseteq \mathrm{Gal}(F(\zeta_n)/F) \ni \bar{a}$.

index $\mid \dim_{\mathbb{Q}} F$.

hence $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot x \subset C \cap \varphi_a C$

Bezout: if $C \cap \varphi_a C$ is finite, then:

$$\frac{\varphi(n)}{\dim_{\mathbb{Q}} F} \leqslant \# \mathrm{Gal}(\overline{\mathbb{Q}}/F) \cdot x \leqslant \#(C \cap \varphi_a C) = \cancel{\deg(C) \cdot \deg(\varphi_a C)}$$

$$\leqslant \# \{ c \in C \mid \varphi_a(c) \in C \}$$

$$\leqslant \# (C \cap \varphi_a^{-1} C) \leqslant \deg C \cdot \deg \varphi_a^{-1} C$$

$$\leqslant a \cdot (\deg C)^2.$$

~~$\deg(C) \cdot \deg(\varphi_a) \cdot \deg C$~~

So we want $p^{\dim_{\mathbb{Q}} F} = a < \dfrac{\varphi(n)}{\dim_{\mathbb{Q}} F \cdot \deg(C)^2}$. For large n, such p do exist!

Conclusion: for infinitely many $a \in \mathbb{Z}_{>1}$, we have $C = \varphi_a C$.

4. Take $a \in \mathbb{Z}_{>1}$ s.t. $C = \varphi_a(C)$.

$$\begin{tikzcd}
\mathbb{Z}\times\mathbb{Z} \arrow[d] & \mathbb{C}\text{-analytic subv.} & \text{irred. comp: closure of conn. comp. of } u^{-1}(C^{sm}). & \\
\mathbb{C}\times\mathbb{C} \arrow[loop left, "a\cdot"] \arrow[d, "u"'] \arrow[r, phantom, "\supset"] & u^{-1}C \arrow[d] \arrow[r, phantom, "\supset"] & X \arrow[d, dash] \arrow[r, phantom, "\ni"] & 0 \arrow[d] \\
\mathbb{C}^\times\times\mathbb{C}^\times \arrow[loop left, "\varphi_a"] \arrow[r, phantom, "\supset"] & C \arrow[rr, phantom, "\ni"] & & 1
\end{tikzcd}$$

Then $a\cdot X = X$.

We want to show that X is a complex line with rational slope.

Take any $0 \neq x = (x_1, x_2) \in X$. Then, $\forall\, n \geq 0$: $(a^{-n}\cdot x_1, a^{-n} x_2) \in X$. But then the order of contact of X and $\mathbb{C}\cdot x$ at 0 is infinite, hence $X = \mathbb{C}\cdot x$.

For rationality of slope: C dominates at least one of the two factors $\mathbb{C}^\times$, with degree $\leq \deg C$. This implies that the projection to $\mathbb{Z}$ of $(\mathbb{Z}\times\mathbb{Z}) \cap X$ has index $\leq \deg C$. $\boxtimes$.

$$\begin{tikzcd}
 & \mathbb{Z}\times\mathbb{Z} \arrow[loop below] \arrow[ddl, bend right, "\mathrm{pr}_1"'] & & & \\
 & \mathbb{C}\times\mathbb{C} \arrow[dd] & u^{-1}C \arrow[l, hook'] \arrow[dd] \arrow[ddl, phantom, "\square"] & X \arrow[l, hook'] \arrow[dd] \arrow[ddd, bend left, "\text{quotient by } (\mathbb{Z}\times\mathbb{Z})_X"] & \mathbb{Z}_Y \\
\mathbb{Z} \arrow[dr, bend left] & & & & \\
\mathbb{C} \arrow[d] & \mathbb{C}\times\mathbb{C}^\times \arrow[l] \arrow[d, "v"'] \arrow[dl, phantom, "\square"] & v^{-1}C \arrow[l, hook'] \arrow[d] \arrow[dl, phantom, "\square"] & Y \arrow[l, hook'] \arrow[dl, "\text{quot. by } \overline{\mathrm{pr}_1((\mathbb{Z}\times\mathbb{Z})_X)}"] & \text{irred.} \quad \mathbb{Z}_Y = \overline{\mathrm{pr}_1((\mathbb{Z}\times\mathbb{Z})_X)} \\
\mathbb{C}^\times & \mathbb{C}^\times\times\mathbb{C}^\times \arrow[l, "\mathrm{pr}_1"] & C \arrow[l, hook'] & &
\end{tikzcd}$$

$$\mathrm{Irr}(v^{-1}C) = \mathbb{Z}/\mathrm{pr}_1((\mathbb{Z}\times\mathbb{Z})_X)$$

$\#$ at most $\deg(v^{-1}C \longrightarrow \mathbb{C})$ $= \deg(C \xrightarrow[\mathrm{pr}_1]{} \mathbb{C}^\times)$.

Now $A_1 \times A_1$, as in CM article, but as suggested in DMJ, without reducing to $\mathbb{Q}$.
and treat the case where all $x \in \Sigma$ have the same CM field.

I've stated the Thm, and explained its meaning. (CM points, $\begin{array}{c} Y_0(n) \\ \downarrow \\ \mathbb{C}^2 \end{array}$)

I've also said: $\#\mathrm{Gal}(\overline{\mathbb{Q}}/K)\cdot x = \#\mathrm{Pic}(O_{K,f}) = |\mathrm{discr}(O_{K,f})|^{\frac{1}{2}+o(1)}$ (Siegel).
(but not explained) $x \in \mathbb{C}$, $x \sim E$, $\mathrm{End}(E) = O_{K,f}$

and: GRH bec. we want primes p split in $O_{K,f}$ and $p < |\mathrm{discr}(O_{K,f})|^{1/4-\varepsilon}$.

Lecture 2.

CM elliptic curves, some facts. ~~For~~ $\mathbb{Q} \hookrightarrow K \hookrightarrow$ imag. quadr. $\mathbb{C}$, $f \in \mathbb{Z}_{\geq 1}$.

put

$$S_{K,f} = \{(E/\overline{\mathbb{Q}}, \alpha) \mid E/\overline{\mathbb{Q}} \text{ ell. curve } \alpha: \begin{tikzcd} O_{K,f} \arrow[r, "\sim"] \arrow[dr, hook, bend right, "\text{the given emb.}"'] & \mathrm{End}(E) \arrow[d] \\ & \mathrm{End}(\mathrm{Lie}\,E) = \overline{\mathbb{Q}} \end{tikzcd} \}/\cong$$

For $\mathbb{Q} \hookrightarrow K \hookrightarrow \mathbb{C}$ imag. quadr., $f \in \mathbb{Z}_{\geq 1}$:

$$S_{K,f} := \{(E/\mathbb{C}, \alpha) \mid E/\mathbb{C} \text{ ell. curve}, \alpha: \underbrace{O_{K,f} \xrightarrow{\sim} \mathrm{End}(E) \hookrightarrow \mathrm{End}(\mathrm{Lie}\,E) = \mathbb{C}}_{\text{the given emb.}}\}/\cong$$

Then $S_{K,f} = \{ [(\mathbb{C}/\Lambda, \text{action by } O_{K,f})] \mid \Lambda \in \mathrm{Pic}(O_{K,f}) \}$

$\#S_{K,f} = \#\mathrm{Pic}(O_{K,f})$. Hence: all these $(E/\mathbb{C}, \alpha)$ are defined $/\overline{\mathbb{Q}}$,

$S_{K,f}$ is a $\mathrm{Pic}(O_{K,f})$-torsor: $(L, E) \mapsto L \otimes_{O_{K,f}} E$ $\quad$ $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ acts on $S_{K,f}$.

More explicitly: if $I \subset O_{K,f}$ inv. ideal, $E = \mathbb{C}/\Lambda$,

then $\Lambda \rightarrowtail \mathbb{C} \twoheadrightarrow E$, $I \otimes_{O_{K,f}} -$:

$$\begin{tikzcd} & & & E & \\ I\cdot\Lambda \arrow[rr, tail] \arrow[dr, hook] & & \mathbb{C} \arrow[rr, two heads] \arrow[ur] & & I \otimes_{O_{K,f}} E \arrow[ul] \\ & \Lambda \arrow[ur] \arrow[rrr] & & & \Lambda/I\cdot\Lambda \arrow[u, hook] \end{tikzcd}$$

$\Lambda/I\cdot\Lambda =$ free $O_{K,f}/I$-module rank!

$\mathrm{Gal}(\overline{\mathbb{Q}}/K) \times \mathrm{Pic}(O_{K,f})$ acts on $S_{K,f}$, hence: $\mathrm{Gal}(\overline{\mathbb{Q}}/K) \longrightarrow \mathrm{Pic}(O_{K,f})$.

For $m \subset O_K$ max. ideal prime to f: $\mathrm{Frob}_m = [m^{-1}]$: it makes the lattice a bit bigger "at m".

So:

$$\begin{tikzcd} \mathrm{Gal}(\overline{\mathbb{Q}}/K)^{ab} \arrow[r, two heads] & \mathrm{Pic}(O_{K,f}) \\ K^\times \backslash \hat{K}^\times \arrow[u, "\text{CFT}", "\wr"'] \arrow[r] & K^\times \backslash \hat{K}^\times / \hat{O}_{K,f}^\times \arrow[u, "\wr"'] \end{tikzcd}$$

Now the proof of the Thm. for $A_1 \times A_1 = \mathbb{C}^2$: $C \hookrightarrow \mathbb{C}^2$ irred. curve ~~each~~ with $\Sigma = C \cap \mathbb{C}^2_{\text{special}}$ infinite.

Step 1. Reduce to C dominates both factors $\mathbb{C}$.

Step 2. C is defined over a finite ext. $\mathbb{Q} \to F \to \mathbb{C}$.

Step 3. $\mathbb{C}^2 \hookrightarrow \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, $\mathrm{Pic}(\ ) = \mathbb{Z} \times \mathbb{Z}$: $|$ & $—$; degrees of the 2 projections.

Take $x = (x_1, x_2) \in \Sigma$ s.t. ~~$|\mathrm{discr}(O_{K_1,f_1} \otimes O_{K_2,f_2})|$~~
$\max(|\mathrm{discr}(O_{K_1,f_1})|, |\mathrm{discr}(O_{K_2,f_2})|)$ large w.r.t. $(\overset{d_1}{\overset{\shortparallel}{\deg_1}} C, \overset{d_2}{\overset{\shortparallel}{\deg_2}} C)$
and $\dim_{\mathbb{Q}} F = d$.

Take p a small prime, split in O_{K_1,f_1} & in O_{K_2,f_2}.

Let $T_{p^d} :=$

$$\begin{tikzcd} & Y_0(p^d)^2 \arrow[dl] \arrow[dr] & \\ \mathbb{C}^2 & & \mathbb{C}^2 \end{tikzcd}$$

$: (\underset{\underset{E_1}{\wr}}{Y_1}, \underset{\underset{E_2}{\wr}}{Y_2}) \mapsto \sum_{\substack{G_1 \subset E_1 \text{ order } p^d \text{, cyclic} \\ G_2 \subset E_2 \text{ order } p^d \text{, —}}} (E_1/G_1, E_2/G_2)$: a degree-$(p^d(1+\frac{1}{p}))^2$ ~~[illegible]~~ eff. ~~divi.~~ 0-cycle.

Key observation: $\mathrm{Gal}(\overline{\mathbb{Q}}/K_1K_2F) \cdot x \subset C \cap T_{p^d} C$.

$$\# \mathrm{Gal}(\overline{\mathbb{Q}}/K_1K_2F) \cdot x \geqslant \left(\max_{i=1,2} |\mathrm{discr}(O_{K_i,f_i})| \right)^{\frac{1}{2}+o(1)} / d.$$

If $C \cap T_{p^d} C$ finite, then $\#(C \cap T_{p^d} C) \leqslant 2(p^d(1+\frac{1}{p}))^2 \cdot d_1 d_2$.

So, if $2 \cdot (p^d \cdot (1+\frac{1}{p}))^2 \cdot d_1 d_2 < (\max |\mathrm{discr}(O_{K_i,f_i})|)^{\frac{1}{2}+o(1)} / d$,
then $C \subset T_{p^d} C$.

Thm (effective Chebotarev; Lagarias-Odlyzko-Montgomery): Assume GRH for all number fields; $\exists B \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R}_{>B}$,
$\forall$ ~~[illegible]~~ $\mathbb{Q} \to M$ finite, Galois :

$$\left| \pi_{M,1}(x) - \frac{1}{\dim_{\mathbb{Q}} M} \cdot \mathrm{Li}(x) \right| \leqslant \frac{1}{3 \cdot \dim_{\mathbb{Q}} M} \cdot x^{1/2} \cdot \left(\log |\mathrm{discr}\, O_M| + (\dim_{\mathbb{Q}} M) \log x \right)$$

Simple application: $\exists$ split primes p of size $\approx (\log |\mathrm{discr}(O_M)|)^{2+\varepsilon}$.

$\Gamma = SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \subset G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$

Step 4:

$$\begin{tikzcd} \mathbb{H} \times \mathbb{H} \arrow[d, "\pi := (j{,}j)"'] & \pi^{-1} C \arrow[l, hook'] \arrow[d] & X \arrow[l, hook'] \arrow[dl] \\ \mathbb{C} \times \mathbb{C} & C \arrow[l, hook'] & \end{tikzcd}$$

irred. comp.

some $\gamma \cdot (n^{-1/2} \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}, n^{-1/2} \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}) \cdot \gamma^{-1}$, $\gamma \in \Gamma$.

$G_X \neq$ discrete ----

$\Updownarrow$

$\exists g \in GL_2(\mathbb{Q})^+$ s.t. $G_X =$ graph of conj. by g.

g unique up to $\mathbb{Q}^\times$; may assume $g \cdot \mathbb{Z}^2 \supset \mathbb{Z}^2$, $X = \{(\tau, g \cdot \tau) \mid \tau \in \mathbb{H}\}$.

$(g\mathbb{Z}^2)/\mathbb{Z}^2$ cyclic, order n $\rightsquigarrow$ $C =$ image of $Y_0(n)$. $\square$

From now on: preparation for Andrei's part. Reference: mostly Deligne's 1979 article "Variétés de Shimura --" in "Corvallis".

Hodge structures.

Let V be an $\mathbb{R}$-vect. space, f.d.. A Hodge structure on V is a decomp.

$$V_{\mathbb{C}} := \mathbb{C}\otimes_{\mathbb{R}} V = \bigoplus_{p,q\in\mathbb{Z}} V_{\mathbb{C}}^{p,q}, \quad \text{s.t. } \forall p,q: \overline{V_{\mathbb{C}}^{p,q}} = V_{\mathbb{C}}^{q,p},$$

where $\overline{z\otimes v} = \bar{z}\otimes v$, for $z\in\mathbb{C}$, $v\in V$.

For V a f.d. $\mathbb{Q}$-vect. sp., a HS on V is one on $V_{\mathbb{R}}$.

for X proj. smooth $/\mathbb{C}$: $H^n(X,\mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$

Example. Let $A = V/\Lambda$ be a complex ab. var: V a $\mathbb{C}$-vect. sp., $\Lambda\subset V$ a lattice, s.t. V/Λ is algebraic.

Then: $V = \mathbb{R}\cdot\Lambda = \mathbb{R}\otimes_{\mathbb{Z}}\Lambda$. Put $V_{\mathbb{Q}} = \mathbb{Q}\otimes\Lambda = \mathbb{Q}\cdot\Lambda\subset V$.

Then $V_{\mathbb{R}} = V$, and $V_{\mathbb{C}} = \mathbb{C}\otimes_{\mathbb{R}} V$ = a $2\cdot g$-dim. $\mathbb{C}$-vect. sp.

But V is already a $\mathbb{C}$-vect. space. where $g = \dim A = \dim_{\mathbb{C}} V$.

Hence $V_{\mathbb{C}}$ is a $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$-module. $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C} \xrightarrow{\sim} \mathbb{C}\times\mathbb{C}$

So: $V_{\mathbb{C}} = V_{\mathbb{C}}^{-1,0}\oplus V_{\mathbb{C}}^{0,-1}$, $\quad x\otimes y \longmapsto (xy, x\bar{y})$ idempotents $(1,0)$ and $(0,1)$.

$x\otimes y$: $\quad xy \quad x\bar{y}$.

$(1,0)\cdot V_{\mathbb{C}} \quad (0,1)\cdot V_{\mathbb{C}}$.

subsp. of $V_{\mathbb{C}}$ on wich $x\otimes 1$ acts as $1\otimes x$, subsp. — $x\otimes 1$ as $1\otimes\bar{x}$.

$1\otimes y$: $y\otimes 1 \quad 1\otimes y$: $\bar{y}\otimes 1$.

We can get A back from $V_{\mathbb{Q}}$ + HS + $\Lambda\subset V_{\mathbb{Q}}$: the $\mathbb{C}$-str. on $V = V_{\mathbb{R}}$ is given by

$$\begin{tikzcd} V_{\mathbb{R}} \arrow[r, hook] \arrow[rr, bend right, "\sim"'] & V_{\mathbb{C}} \arrow[r, "pr"] & V_{\mathbb{C}}^{-1,0}. \end{tikzcd}$$

Polarisation on A: a pos. def. hermitian form $H: V\times V\to\mathbb{C}$, s.t.

$E := \operatorname{Im} H: V\times V \longrightarrow \mathbb{R}$, (antisym., symplectic!), $\quad E: V_{\mathbb{Q}}\times V_{\mathbb{Q}}\to\mathbb{Q}$

$$\begin{array}{ccc} V\times V & \longrightarrow & \mathbb{R} \\ \vee & & \vee \\ \Lambda\times\Lambda & \longrightarrow & \mathbb{Z} \end{array}$$

symplectic, H = expressed in E.

Translate into group theory: let $\mathbb{C}^\times$ act on V as V is a $\mathbb{C}$-vect. sp.

Then $\mathbb{C}^\times$ acts on $V_{\mathbb{C}}$: y as $1\otimes y$, y as y on $V_{\mathbb{C}}^{-1,0}$, $\bar{y}$ on $V_{\mathbb{C}}^{0,-1}$.

$\mathbb{C}$-vect. space via $x\otimes 1$, the tautological one.

Equiv. of cat's: ab. var. $/\mathbb{C}$ $\longleftrightarrow$ polarisable $\mathbb{Q}$-HS of type $(-1,0)$, $(0,-1)$ with a lattice.

By construction, $\mathbb{C}^\times \circlearrowright V_{\mathbb{C}}$ commutes with $\overline{(\ )}$, bec. it comes from V.

~~$= (\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m)(\mathbb{R})$~~

Let V be a f.d. $\mathbb{R}$-v.sp.

Bijection: $\{\text{HS. on } V\} \rightleftarrows \{$actions of $\mathbb{C}^\times$ on V s.t. on $V_\mathbb{C}$ it is given by char's $\mathbb{C}^\times \longrightarrow \mathbb{C}^\times,\ z \mapsto z^{-p}\cdot\bar{z}^{-q}\}$

$$\updownarrow$$

$\mathbb{R}$-HS $\rightleftarrows$ f.d. reps. of $\mathbb{S}$.

$\{$actions of $\underset{\mathbb{S}:=}{}\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m\mathbb{C}}$ on $V\}$

$\forall\, \mathbb{R}\to A$: $\mathbb{S}(A) = (\mathbb{C}\otimes_\mathbb{R} A)^\times = \left\{\begin{pmatrix} a & -b \\ b & a\end{pmatrix} \,\middle|\, \begin{matrix} a,b\in A \\ a^2+b^2\in A^\times\end{matrix}\right\}$

$X(\mathbb{S}) = \mathrm{Hom}(\mathbb{S}_\mathbb{C}, \mathbb{G}_{m,\mathbb{C}}) = \mathrm{Hom}(\mathbb{S},\mathbb{S}) \underset{\text{take } \mathbb{R}\text{-points}}{\hookrightarrow} \mathrm{Hom}(\mathbb{C}^\times,\mathbb{C}^\times)$ ✓

$\| \quad \{z^p\bar{z}^q \mid p,q\in\mathbb{Z}\}.$

$\left(\begin{matrix} \mathbb{R}^\times \hookrightarrow \mathbb{C}^\times : \ \mathbb{G}_{m,\mathbb{R}} \hookrightarrow \mathbb{S} \\ \mathbb{C}^\times \xrightarrow{\text{Norm}} \mathbb{R}^\times : \ \mathbb{S} \longrightarrow \mathbb{G}_{m,\mathbb{R}} \end{matrix}\right)$

Mumford-Tate groups. Let V be a f.d. $\mathbb{Q}$-vect.sp. and $h: \mathbb{S}\to \underline{GL}(V_\mathbb{R})$ a HS.

Then $MT(h) :=$ smallest sub-alg.gp. M of $\underset{\text{alg gr. }/\mathbb{Q}!}{\underline{GL}(V)}$ s.t.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "h"] \arrow[dr] & \underline{GL}(V_\mathbb{R}) \\ & M_\mathbb{R} \arrow[u, phantom, "\vee"] \end{tikzcd}$$

$=$ stabiliser of the lines $\mathbb{Q}\cdot t \subset \bigoplus_i V^{\otimes n_i}\otimes(V^\vee)^{\otimes m_i}$ with t of some type (p,p). (Hodge classes!)

Example. Let $A = V/\Lambda$ be a $\mathbb{C}$-ab.var. Then A is of CM-type $\Longleftrightarrow$ $MT(V_\mathbb{Q})$ is a torus.

M-T conjecture. Let $A/\mathbb{Q}$ be an ab. var.

Fix $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$, $\Lambda := H_1(A(\mathbb{C}),\mathbb{Z})$. Then $\Lambda_{\mathbb{Q}_\ell} = \mathbb{Q}\otimes T_\ell(A(\overline{\mathbb{Q}})) \overset{\rho_\ell}{\circlearrowleft} \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Conj: $(\rho_\ell\, \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}))^{\mathrm{Zar},0} = MT(\Lambda_\mathbb{Q})_{\mathbb{Q}_\ell}$ (sub alg. of $\underline{GL}(\Lambda_{\mathbb{Q}_\ell})$).

Exercise Let $E_1,\dots,E_n$ be complex ell. curves. Compute MT of $H^1(E_1\times\dots\times E_n,\mathbb{Q})$.

Lecture 3.

Last time, I explained: $\mathbb{R}\text{-HS} \overset{\text{equiv.}}{\rightleftarrows}$ f.d. reps. of $\mathbb{S}$, $\quad \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$.

$\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$.

Mumford-Tate groups. For V a f.d. $\mathbb{Q}$ vect. sp. and $h : \mathbb{S} \to \underline{GL}(V_{\mathbb{R}})$:

$MT(h) :=$ intersection of all closed subgrps. $H \subset \underline{GL}(V)$ s.t.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "h"] \arrow[dr, dashed] & \underline{GL}(V_{\mathbb{R}}) \\ & H_{\mathbb{R}} \arrow[u, hook] \end{tikzcd}$$

$$= \bigcap \mathrm{Stab}_{\underline{GL}(V)}(t).$$

$t \in \bigoplus V^{\otimes n_i} \otimes (V^\vee)^{\otimes m_i}$ of type $(0,0)$, i.e. fixed by $\mathbb{S}$

Notice: 1. for (V_1, h_1) and (V_2, h_2), $f : V_1 \to V_2$: $\quad f$ is morph. of HS $\Leftrightarrow$ $f \in V_1^\vee \otimes V_2$ is of type $(0,0)$.

2. $MT(h)$ connected.

Example For $A = V/\Lambda$ ab. var.: A is of CM type $\iff MT(V_{\mathbb{Q}})$ is a torus.

A is "generic" $\iff MT(V_{\mathbb{Q}}) = \underline{GSp}(V_{\mathbb{Q}}, \overset{\text{a polar. on } A.}{\psi})$

For E_1, E_2 non CM, non isogeneous ell. curves $/\mathbb{C}$:

$$MT(E_1 \times E_2) \cong \{(g_1, g_2) \in GL_2(\mathbb{Q})^2 \mid \det(g_1) = \det(g_2)\}^{\text{Zar}}.$$

($g \in GL(V_1 \oplus V_2)$: on all tensor constr., g commutes with morphisms of HS;

$V_1 \oplus V_2 : (id, 0), (0, id) : \; g = (g_1, g_2) \in GL(V_1) \times GL_2(V_2)$.

$V_1 \otimes V_1 = \mathrm{Sym}^2(V_1) \oplus \det(V_1)$ $\; \det(V_1) \xrightarrow[\text{any isom.}]{\sim} \underbrace{\det(V_2)}_{\mathbb{Q}(\pm 1) : \text{ type } (-1,-1)} : \det(g_1) = \det(g_2)$.)

MT conjecture : relation between MT and ℓ-adic Galois images.

Hodge generic & exceptional locus. Let S be a nonsing. complex alg. var., and V a polarisable variation of $\mathbb{Q}$-HS on S^{an}. (V loc. const. sheaf of $\mathbb{Q}$-v.sp. …) Then $\exists\, \Sigma \subset S$ countable union of proper algebr. subvar's s.t. $s \mapsto MT(V_s)$ is locally const. on $S - \Sigma$. $S_{exc} :=$ smallest such Σ, $S_{gen} := S - S_{exc}$. For $s \in S_{gen}$, $MT(V_s) \subset GL(V_s)$ is called the generic MT grp. of V on S.

(Cattani-Deligne-Kaplan)

Example: A_1 : $(A_1)_{exc} = j\Big(\bigcup\limits_{\substack{\mathbb{Q} \subset K \subset \mathbb{C} \\ \text{quadr.}}} K \cap \mathbb{H}\Big)$.

Reductive groups. (char. 0). (affine)

Let $k \supset \mathbb{Q}$, G a connected alg. grp. / k (automatically ~~smooth~~ nonsingular).

TFAE: 1. $\mathrm{Rep}(G)$ is semi-simple

2. G has no nontrivial normal subgroup of the form $\mathbb{G}_{a,k}^n$.

Such G are called reductive. Can be described in terms of root data.

Examples. $\mathbb{G}_a$ is not: $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. $\mathbb{G}_a \rtimes \mathbb{G}_m$ is not: $\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$, $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ is not.

GL_n is reductive, tori are: $\mathrm{Rep}(\mathbb{G}_m) = \{k\text{-vect.sp} + \mathbb{Z}\text{-grading}\}$.

For $k \to k'$: G red. $\Leftrightarrow$ $G_{k'}$ red.

Over $\mathbb{C}$: G red. $\Leftrightarrow$ $\exists$ model $G_{\mathbb{R}}$ over $\mathbb{R}$ s.t. $G_{\mathbb{R}}(\mathbb{R})$ is compact.

For V a polarisable $\mathbb{Q}$-HS: $MT(V)$ is reductive.

$T := Z(G)^\circ$ is a torus, $G^{der} \times T \twoheadrightarrow G$, $G^{der} \rightarrowtail G \twoheadrightarrow C$ torus, $Z(G) \rightarrowtail G \twoheadrightarrow G^{ad}$.

Example: $\mu_2 \rightarrowtail SL_2 \twoheadrightarrow SL_2^{ad}$; $\{\pm 1\} \rightarrowtail SL_2(k) \to SL_2^{ad}(k) \twoheadrightarrow H^1(\mathrm{Gal}(\bar{k}/k), \pm 1)$

Reference: Deligne, Variétés de Shimura ..., "Corvallis".

Shimura data & varieties.

Def. A Shimura datum is a pair (G, X), with G a red. gr. / $\mathbb{Q}$ and X a $G(\mathbb{R})$-orbit in $\mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$, that satisfies:

(1) $\forall h \in X$ the HS. on $\mathrm{Lie}(G_{\mathbb{R}})$ is of type $\{(-1,1), (0,0), (1,-1)\}$.

(2) $\forall h \in X$: $\mathrm{inn}_{h(i)} \circlearrowright G^{ad}_{\mathbb{R}}$ is a Cartan involution, i.e., $\{g \in G^{ad}(\mathbb{C}) \mid h(i)\,\bar{g}\,h(i)^{-1} = g\}$ is compact

(3) write $G^{ad} = \prod_i G_i$ with G_i simple; then $\forall i, \forall h$: $\mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \to G_i$ is not trivial.

Reason for (1) (2) & (3): X has a unique complex str. s.t. each repr. V of $G_{\mathbb{R}}$ gives a polarisable variation of $\mathbb{R}$-HS on X. (hence, satisf. Griffiths transv.)

Examples: 1. $(GL_{2,\mathbb{Q}}, \mathbb{H}^{\pm})$ where $\mathbb{H}^{\pm}$ is the $GL_2(\mathbb{R})$-orbit of $\mathbb{S} \rightarrowtail GL_{2,\mathbb{R}}$ "$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$".

2. $(GSp_{2g}, \mathbb{H}_g^{\pm})$, symplectic similitudes; $\psi = \left(\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array}\right)$, orbit of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto \left(\begin{array}{c|c} a & b \\ \hline -b & a \end{array}\right)$.

3. $(T, \{h\})$, where T is any torus / $\mathbb{Q}$, $h: \mathbb{S} \to T_{\mathbb{R}}$ any morphism.

Exercise: Show that $\nexists$ Shim. datum with $G = SL_{2,\mathbb{Q}}$.

Facts. $\pi_0(X)$ is finite (as $\pi^0 G(\mathbb{R})$ is), the conn. comp. are hermitian symm. domains, notation: X^+. $G(\mathbb{Q}) \subset G(\mathbb{R})$ is dense.

For (G,X) a Shim. datum, and $K \subset G(\hat{\mathbb{Q}})$ compact open:

$$Sh_K(G,X)^{an} := G(\mathbb{Q}) \backslash \left(X \times G(\hat{\mathbb{Q}})/K\right) = \coprod_i \Gamma_i \backslash X^+, \quad \underline{\text{finite union}}.$$

where (X^+, g_i) are representatives for $\underset{G(\mathbb{Q})}{} \backslash \left(\pi_0(X) \times G(\hat{\mathbb{Q}})/K\right)$,

and $\Gamma_i = G(\mathbb{Q})_{X^+} \cap g_i K g_i^{-1}$: arithm. subgroups in $G(\mathbb{Q})$. (congruence subgr.)

Example: $GSp_{2g}(\mathbb{Q}) \backslash \left(\mathbb{H}_g^{\pm} \times GSp_{2g}(\hat{\mathbb{Q}}) / GSp_{2g}(\hat{\mathbb{Z}})\right) = A_{g,1}^{an}$.

Baily–Borel: $Sh_K(G,X)^{an}$ is the analytification, canonically, of a complex quasi-pr. alg. variety $Sh_K(G,X)_{\mathbb{C}}$; proof by compactification; these are functorial in the Shim. datum

$$Sh(G,X)_{\mathbb{C}} := \varprojlim_K Sh_K(G,X)_{\mathbb{C}} \circlearrowleft G(\hat{\mathbb{Q}}), \quad Sh_K(G,X)_{\mathbb{C}} = Sh(G,X)_{\mathbb{C}}/K.$$

↑ proj. system with finite transition maps.

$\mathbb{C}$-schemes, not of finite type unless ... (G trivial?) (even if $G = T$ nontriv. torus, zero-dim'l but still not finite type).

Hecke correspondences. For (G,X), K, K', $g \in G(\hat{\mathbb{Q}})$:

$$\begin{tikzcd}
\circlearrowleft^{K}\ g^{-1}K'g \arrow[r, leftrightarrow] & \circlearrowleft K' \\
Sh(G,X) \arrow[r, "\cdot g", "\sim"'] \arrow[d] \arrow[dr] & Sh(G,X) \arrow[d] \\
Sh_K(G,X) & Sh_{K'}(G,X)
\end{tikzcd}$$

$$\begin{tikzcd}
 & Sh_{K \cap g^{-1}K'g}(G,X)_{\mathbb{C}} \arrow[dl] \arrow[dr] & \\
Sh_K(G,X)_{\mathbb{C}} & & Sh_{K'}(G,X)_{\mathbb{C}}.
\end{tikzcd}$$

Special subvarieties. 2 equivalent ways of looking at it: Hodge class loci (or: MT groups), image of morphisms.

1. (morphisms) A closed irred. subvar. $Z \subset Sh_K(G,X)_{\mathbb{C}}$ is *special* if
$\exists\ (G',X')$, $\exists\ f: G' \to G$ s.t. $\forall h \in X'$ $(h: \mathbb{S} \to G'_{\mathbb{R}})$ $f \circ h \in X$,
$\exists\ g \in G(\hat{\mathbb{Q}})$ s.t.: Z is an irreducible component of the image of $Sh(G',X')_{\mathbb{C}} \xrightarrow{f} Sh(G,X)_{\mathbb{C}} \xrightarrow[\sim]{\cdot g} Sh(G,X)_{\mathbb{C}} \xrightarrow{quot} Sh_K(G,X)_{\mathbb{C}}$.
Special points: 0-dim'l special subv.

2. **MT groups.**
For $h \in X$: $MT(h) \subset G$ smallest $H \subset G$ s.t. $h: \mathbb{S} \to \overset{H_{\mathbb{R}} \subset}{G_{\mathbb{R}}}$. For $z \in Sh_K(G,X)_{\mathbb{C}}$: $MT(z)$ with $G(\mathbb{Q})$ conj. class of emb. in G.

Let $Z \subset Sh_K(G,X)_{\mathbb{C}}$ be a closed irred. subvar. Then $\exists\ Z_{exc} \subset Z$ countable union of proper closed subvarieties where $MT(z)$ is smaller than on $Z_{gen} = Z - Z_{exc}$ where it is loc. constant.

Let $z \in Z_{gen}$,

$$\begin{tikzcd}(x,g) \arrow[d, mapsto] & \in & X \times G(\hat{\mathbb{Q}}) \arrow[d] \\ z & \in & Sh_K(G,X)_{\mathbb{C}}\end{tikzcd}$$

The *smallest special* $S \subset Sh_K(G,X)_{\mathbb{C}}$ containing Z is the image of [illegible] $MT(x)(\mathbb{R})^+ \cdot x \times \{g\}$.

(other way: use a suitable rep'n of G to make a VHS on $Sh_K(G,X)$ if K "neat".)

Galois action, canonical models.

$Sh(G,X)_{\mathbb{C}} \circlearrowleft G(\hat{\mathbb{Q}})$ has a canonical model over the *reflex field* $E(G,X) \subset \mathbb{C}$:

for $h \in X$:

$$\begin{tikzcd}\mathbb{G}_{m,\mathbb{C}} \arrow[r] \arrow[rr, bend left, "\mu_h"] & \mathbb{S}_{\mathbb{C}} \arrow[r, "h"] & G_{\mathbb{C}} \arrow[r] & GL(V_{\mathbb{C}})\end{tikzcd}$$

$V_{\mathbb{C}} = \bigoplus_{p,q} V_{\mathbb{C}}^{p,q}$ $\quad \mu_h(z)$: z^{-p} on $V_{\mathbb{C}}^{p,q}$.

$E(G,X)$ = field of definition of the $G(\mathbb{C})$-orbit of μ_h in $\mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}})$.

Note: $G \backslash \underline{\mathrm{Hom}}(\mathbb{G}_m, G)$ is an étale $\mathbb{Q}$-scheme, μ_h is a $\mathbb{C}$-valued point of it, and $E(G,X)$ the residue field at that point.

Example. For $G = GL_n$: $\mathbb{Z}$-gradings on $\mathbb{Q}^n$, up to isomorphism

$$\coprod \mathrm{Spec}\ \mathbb{Q}. \quad \{f: \mathbb{Z} \to \mathbb{N},\ \sum_{i \in \mathbb{Z}} f(i) = n\}$$

For T a torus $/\mathbb{Q}$, $\underline{\mathrm{Hom}}(\mathbb{G}_m, T) =$ the étale $\mathbb{Q}$-scheme corresponding to the discrete $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-set $X_*(T) = \mathrm{Hom}(\mathbb{G}_{m\overline{\mathbb{Q}}}, T_{\overline{\mathbb{Q}}})$.

Let $(T, \{h\})$ be a Shim. datum with T a torus, $E \overset{\mathbb{C}}{\hookrightarrow} := $ the reflex field.

$$\mathrm{Sh}_K(T,\{h\})_E(\overline{\mathbb{Q}}) = T(\mathbb{Q})\backslash T(\hat{\mathbb{Q}})/K = T(\hat{\mathbb{Q}})/\underbrace{T(\mathbb{Q}).K}_{\text{}} \quad \text{open subgr, cont. } T(\mathbb{Q})$$

$$\mathrm{Sh}(T,\{h\})_E(\overline{\mathbb{Q}}) = \varprojlim_K T(\hat{\mathbb{Q}})/T(\mathbb{Q}).K = T(\hat{\mathbb{Q}})/\overline{T(\mathbb{Q})} \leftarrow \text{closure.}$$

$$= T(\mathbb{A})/$$

Over E: $\mathbb{G}_{m,E} \xrightarrow{\mu} T_E$

hence over $\mathbb{Q}$: $\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{m,E} \xrightarrow{\mathrm{Res}\,\mu} \mathrm{Res}_{E/\mathbb{Q}} T_E \xrightarrow{\text{"Norm"}} T$

for $\mathbb{Q} \to A$: $(\mathrm{Res}_{E/\mathbb{Q}} T_E)\, A = T(E \otimes_{\mathbb{Q}} A) \hookrightarrow T(\overline{\mathbb{Q}} \otimes_{\mathbb{Q}} A) \xrightarrow{\cdot} T(A)$ $\quad \mathrm{Hom}(E, \overline{\mathbb{Q}})$

Take $\mathbb{A}$-points:

$$\begin{tikzcd}
\mathbb{A}_E^\times \arrow[r, "\mathrm{rec}"] \arrow[d] & T(\mathbb{A}) \arrow[d, "\mathrm{quot}"] \\
\mathrm{Gal}(\overline{\mathbb{Q}}/E)^{ab} \arrow[r] & T(\hat{\mathbb{Q}})/\overline{T(\mathbb{Q})}
\end{tikzcd}$$

gives the $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$-action on $\mathrm{Sh}(T,\{h\})_E(\overline{\mathbb{Q}})$.

<u>General case</u>. $\pi_0(\mathrm{Sh}_K(G,X)^{an}) = \pi_0(G(\mathbb{Q})\backslash X \times G(\hat{\mathbb{Q}})/K)) = G(\mathbb{Q})_+ \backslash G(\hat{\mathbb{Q}})/K$

$$= \overline{G(\mathbb{Q})_+} \backslash G(\hat{\mathbb{Q}})/K$$

$$= \text{quotient of } G(\hat{\mathbb{Q}})/\rho\, \tilde{G}(\hat{\mathbb{Q}})$$

$$= \text{an comm. gp.}$$

$$\begin{tikzcd}
G(\mathbb{R}) \arrow[r] \arrow[d, phantom, "\cup"] \arrow[dr, phantom, "\supset"] & G^{ad}(\mathbb{R}) \arrow[d, phantom, "\cup"] \\
G(\mathbb{R})_+ \arrow[r] & G^{ad}(\mathbb{R})^+
\end{tikzcd}$$

, $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G(\mathbb{R})_+$ in $G(\mathbb{R})$

$$\begin{tikzcd}
 & \tilde{G} \arrow[d, "\rho"', "\text{univ. cover}"] \\
G & G^{der} \arrow[l, hook']
\end{tikzcd}$$

$\tilde{G}(\mathbb{R})$ connected,

hence $\rho\,\tilde{G}(\mathbb{Q}) \subset G(\mathbb{Q})_+$

Now $\overline{\tilde{G}(\mathbb{Q})} = \tilde{G}(\hat{\mathbb{Q}})$ (strong approx.)

(uses cond. (3) of Shim. datum)

$\pi_0(\mathrm{Sh}(G,X)_{\overline{\mathbb{Q}}})$ is a $G(\hat{\mathbb{Q}})/\overline{G(\mathbb{Q})_+}$-torsor, but the rec. map is more difficult to describe.

But one has $G \twoheadrightarrow G/G^{der} = C$, $(G,X) \to (C,\{\cdot\})$ and that seems to suffice.