The projective line over $\mathbb{Z}$, as functor.

For k a field: $(a_0 : a_1) \mapsto k\cdot\binom{a_0}{a_1} \longmapsto \left(k^2 \xrightarrow{(-a_1\ a_0)} k\right)$

$\mathbb{P}^1(k) = {}_{k^\times}\backslash(k^2 - \{0\}) = \{1\text{-dim. subsp. of } k^2\} = \{1\text{-dimensional quotients } k^2 \twoheadrightarrow L\}/\sim$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

We want to generalise this to arbitrary locally ringed spaces X.

Advantage: L has 2 given elements; the subspace in k^2 has 2 given elts in its dual.

$\{1\text{-dim. } k\text{-v.sp. } L \text{ with } s_0, s_1 \in L \text{ s.t. } L = k\cdot s_0 + k\cdot s_1\}/\sim$

where $(V, s_0, s_1) \sim (V', s_0', s_1')$

$\Updownarrow$

$\exists f : V \xrightarrow{\sim} V'$ s.t. $f(s_0) = s_0'$, $f(s_1) = s_1'$.

Def. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and $n \in \mathbb{Z}_{\geq 0}$.

Then $\mathcal{F}$ is called locally free of rank n if $\forall x \in X\ \exists U \subset X$ open, containing x, s.t. $\exists\ \varphi : \mathcal{O}_U^n \xrightarrow{\sim} \mathcal{F}|_U$.

Equivalently: $\forall x \in X\ \exists U \ni x$, $s_1, \ldots, s_n \in \mathcal{F}(U)$ s.t. $\mathcal{O}_U^n \xrightarrow{(s_1 \cdots s_n)} \mathcal{F}|_U$ is an isom.

Def. For X a LRS we put: $\mathbb{P}^1(X) = \{(\mathcal{L}, s_0, s_1) \mid \mathcal{L}$ loc. free $\mathcal{O}_X$-mod. rk 1, $s_0, s_1 \in \mathcal{L}(X)$ s.t. $\mathcal{O}_X^2 \xrightarrow{(s_0\ s_1)} \mathcal{L}\}/\sim$,

where $(\mathcal{L}, s_0, s_1) \sim (\mathcal{L}', s_0', s_1') \iff \exists\ \varphi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$, $s_0 \mapsto s_0'$, $s_1 \mapsto s_1'$.

Note: such a φ is unique, because, locally on X, s_0 or s_1 generates $\mathcal{L}$.

Note: $\forall\ U \subset X$ open: $\mathbb{P}^1(X) \to \mathbb{P}^1(U)$, gives a presheaf: $\mathrm{Open}(X) \to \mathrm{Set}$.

Thm. This presheaf is a sheaf.

Proof. Let $(U_i)_{i \in I}$ be an open cover of X, $\forall i$: $[(\mathcal{L}_i, s_{i,0}, s_{i,1})] \in \mathbb{P}^1(U_i)$, s.t.

$\forall i,j$: $\exists\ \varphi_{i,j} : \mathcal{L}_i|_{U_{i,j}} \xrightarrow{\sim} \mathcal{L}_j|_{U_{i,j}}$ s.t. $s_{i,0} \mapsto s_{j,0}$, $s_{i,1} \mapsto s_{j,1}$.

We must show that $\exists!\ [(\mathcal{L}, s_0, s_1)] \in \mathbb{P}^1(X)$ s.t. $\forall i\ \exists\ \psi_i : \mathcal{L}_i \xrightarrow{\sim} \mathcal{L}|_{U_i}$, $s_{i,0} \mapsto s_0$, $s_{i,1} \mapsto s_1$.

Let us first show the uniqueness: assume that we have $(\mathcal{L}, s_0, s_1)$ with ψ_i's and $(\mathcal{L}', s_0', s_1')$ with ψ_i''s.

Then, $\forall i$ we have: $\mathcal{L}|_{U_i} \xrightarrow[\sim]{\psi_i^{-1}} \mathcal{L}_i \xrightarrow[\sim]{\psi_i'} \mathcal{L}'|_{U_i}$, s.t. $s_0|_{U_i} \mapsto s_0'|_{U_i}$, $s_1|_{U_i} \mapsto s_1'|_{U_i}$.

So, $\forall i,j$: $\psi_i' \circ \psi_i^{-1}|_{U_{i,j}} = \psi_j' \circ \psi_j^{-1}|_{U_{i,j}}$ bec. both do $s_0 \mapsto s_0'$, $s_1 \mapsto s_1'$.

Therefore $\exists!\ \psi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ s.t. $\forall i$ $\psi|_{U_i} = \psi_i' \circ \psi_i^{-1}$, and $s_0 \mapsto s_0'$, $s_1 \mapsto s_1'$.

This concludes the uniqueness proof.

Now we prove the existence of a $(\mathcal{L}, s_0, s_1) \in \mathbb{P}^1(X)$ s.t.

$\forall i\ \exists\, \varphi_i : \mathcal{L}_i \xrightarrow{\sim} \mathcal{L}|_{U_i}$, $s_{i,0} \mapsto s_0$, $s_{i,1} \mapsto s_1$.

Here is it: for $U \subset X$ open, put:

$$\mathcal{L}(U) = \left\{ \left(s_i \in \mathcal{L}_i(U \cap U_i)\right)_{i \in I} \;\middle|\; \forall\, i,j \in I : \varphi_{ij}\left(s_i|_{U \cap U_{ij}}\right) = s_j|_{U \cap U_{ij}} \right\}$$

$s_0 \in \mathcal{L}(X)$: $(s_{i,0})_{i \in I}$, $\quad s_1 \in \mathcal{L}(X)$: $(s_{i,1})_{i \in I}$.

One checks that $\mathcal{L}$ is a sheaf of $\mathcal{O}_X$-modules, loc. free of rank 1, generated by s_0 & s_1. Note: $\mathcal{L}|_{U_i}$ "=" $\mathcal{L}_i$. $\square$

<u>Remark</u>. For $[(\mathcal{L}, s_0, s_1)] \in \mathbb{P}^1(X)$, we set $\mathcal{L}^\vee := \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) = \left(U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{L}|_U, \mathcal{O}_U)\right)$ as "line" in $\mathcal{O}_X^2$ via: $\mathcal{L}^\vee \xrightarrow{\begin{pmatrix} -s_1 \\ s_0 \end{pmatrix}} \mathcal{O}_X^2 \xrightarrow{(s_0\ s_1)} \mathcal{L}$.

Let us indeed prove that for A a ring, $a_0, a_1 \in A$ s.t. $A = A\cdot a_0 + A \cdot a_1$, the sequence $0 \to A \xrightarrow{\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}} A^2 \xrightarrow{(-a_1\ a_0)} A \to 0$ is exact.

Take b_0, b_1 in A s.t. $1 = a_0 b_0 + a_1 b_1$, and consider the <u>homotopy</u>:

$$\begin{tikzcd}
A \arrow[r, "\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}"] \arrow[d, "\mathrm{id}"'] & A^2 \arrow[r, "(-a_1\ a_0)"] \arrow[d, "\mathrm{id}"] \arrow[dl, "(b_0\ b_1)"'] & A \arrow[d, "\mathrm{id}"] \arrow[dl, "\begin{pmatrix} -b_1 \\ b_0 \end{pmatrix}"'] \\
A \arrow[r, "\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}"'] & A^2 \arrow[r, "(-a_1\ a_0)"'] & A
\end{tikzcd}$$

This proves that $0 \to \mathcal{L}^\vee \to \mathcal{O}_X^2 \to \mathcal{L} \to 0$ is locally ~~split~~ exact, and therefore exact.

<u>Example</u>. Let $A = \mathbb{Z}[x]/(x^2+5) = \mathbb{Z}[\sqrt{-5}] = \mathcal{O}_{\mathbb{Q}(\sqrt{-5})}$, all non-zero ideals are invertible. Let $L = (2, \sqrt{-5}-1)$.

Then we have $[(\tilde{L}, 2, \sqrt{-5}-1)] \in \mathbb{P}^1(A)$. This point cannot be written with the usual homogeneous coordinates, because L is not free of rank 1.

Thm. $\mathbb{P}^1 : \mathrm{LRS} \to \mathrm{Set}$ is represented by a scheme, denoted $\mathbb{P}^1$, or $\mathbb{P}^1_{\mathbb{Z}}$.

~~Proof.~~ There are various ways to construct such a scheme.

HAG: "Proj $\mathbb{Z}[x_0, x_1]$"; glue two $\mathbb{A}^1$'s along $\mathbb{G}_m$. The last constr. does not respect the symmetry ($GL_2(\mathbb{Z})$ acts on $\mathbb{P}^1$. in fact $\mathrm{Aut}(\mathbb{P}^1) = PGL_2(\mathbb{Z})$)

The first constr. is nice, but should be better motivated by relating it to ${}_{k^\times}\backslash(k^2 - \{0\})$, which is what I want to do now.

Recall: $\mathbb{G}_m(A) = A^\times$. $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$. $\mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[x_0, x_1]$.

Let $U = \mathbb{A}^2 - Z(x_0, x_1)$, open subscheme of $\mathbb{A}^2$, not affine.

For $(a_0, a_1) \in \mathbb{A}^2(A) = A^2$:

$$(a_0, a_1) \text{ factors through } U \iff (a_0, a_1)^{-1} Z(x_0, x_1) = \emptyset \iff \underset{\operatorname{Spec} A\ \cup}{Z(a_0, a_1)} = \emptyset$$

$$\iff Aa_0 + Aa_1 = A.$$

$\mathbb{G}_m$ acts on $\underset{\mathbb{A}^2}{\overset{\cap}{U}}$: $\lambda \cdot (a_0, a_1) = (\lambda a_0, \lambda a_1)$

We can take the naive quotient: $A \mapsto Q(A) = A^\times \backslash U(A)$, but this is not a sheaf.

We have $U = U_0 \cup U_1$, $U_0 = D(x_0)$, $U_1 = D(x_1)$, $U_0(A) = A^\times \times A$, $U_1(A) = A \times A^\times$.

$\mathbb{G}_m$ acts on U_0 and on U_1, and <u>their</u> naive quotients are representable:

$$\begin{tikzcd} U_0 \arrow[r, shift left, "q_0"] & Q_0 \arrow[l, shift left, "r_0"] \arrow[loop below, "\mathrm{id}"] \end{tikzcd} : \quad A^\times \times A \to A, \ (a_0, a_1) \mapsto a_0^{-1} a_1$$

$$(1, a) \longleftarrow\!\shortmid\ a$$

These are affine; rings: $\mathbb{Z}[x_0, x_1, \frac{1}{x_0}] \leftarrow \mathbb{Z}[x_{0,1}]$

$$\frac{x_1}{x_0} \longleftarrow\!\shortmid\ x_{0,1}$$

$= \mathbb{Z}[x_0, x_1, \frac{1}{x_0}]_0$: degree zero part; the $\mathbb{G}_m$-action gives the $\mathbb{Z}$-grading; quotient ~ invariant functions.

More generally: for $f \in \mathbb{Z}[x_0, x_1]_d$:

$$U \supset D(f) \xrightarrow{q} D_+(f) = \operatorname{Spec}\left(\mathbb{Z}[x_0, x_1, \tfrac{1}{f}]_0\right)$$

$$\mathcal{L}(D_+(f)) = \mathbb{Z}[x_0, x_1, \tfrac{1}{f}]_1, \quad s_0 = x_0, \quad s_1 = x_1.$$

This is the universal $(\mathcal{L}, s_0, s_1)$.