Topics in Arithmetic Geometry

Adèles, product formula, idèles

p-adic numbers: Other completions of $\mathbb{Q}$.

Recall that if we complete $\mathbb{Q}$ with respect to the standard absolute value $|\cdot|$, we obtain $\mathbb{R}$. ~~In other words~~ We can also think of $\mathbb{R}$ in the following way

$$\mathbb{R} = \{ a_n \dots a_1 a_0 , a_{-1} a_{-2} \dots \mid n \in \mathbb{Z}_{\geq 0} , \ a_i \in \{0, \dots, 9\} \} / \sim \quad [0,999\dots = 1]$$
$$\mapsto \sum_{i \in \mathbb{Z}} a_i 10^i$$

We can think of $\mathbb{Q}_p$ as

$$\mathbb{Q}_p = \{ \dots a_1 a_0 , a_{-1} \dots a_{-n} \mid n \in \mathbb{Z}_{\geq 0} , \ a_i \in \{0, \dots, p-1\} \}$$
$$\mapsto \sum_{i \in \mathbb{Z}} a_i p^i$$
$$\cup\!|$$
$$\mathbb{Z}_p = \{ \dots a_1 a_0 , 0 \dots 0 \mid n \in \mathbb{Z}_{\geq 0} , \ a_i \in \{0, \dots, p-1\} \}$$

A better way is the following. Let $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ be given by

$$\frac{a}{b} \mapsto 0 \quad \text{if } \frac{a}{b} = 0$$
$$\frac{a}{b} \mapsto p^{-(v_p(a) - v_p(b))} = p^{-v_p(a/b)} \quad \text{otherwise.}$$

This is a non-archimedean absolute value

$$|xy|_p = |x|_p \cdot |y|_p \qquad |x+y|_p \leq \max(|x|_p, |y|_p)$$

and we can complete $\mathbb{Q}$ with respect to $|\cdot|_p$ to obtain $\mathbb{Q}_p$.

There is yet another way. We can think of $\mathbb{Z}_p$ as $\varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$. One can show that $\mathbb{Z}_p$ is a discrete valuation ring with unique maximal ideal $p\mathbb{Z}_p$; every $x \in \mathbb{Z}_p \setminus \{0\}$ can be uniquely written as $p^{v_p(x)} \cdot u$, with $u \in \mathbb{Z}_p^\times$. Now we can define $\mathbb{Q}_p$ as $\mathbb{Z}_p[\frac{1}{p}] = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}_p = \bigcup_{n \geq 0} p^{-n} \mathbb{Z}_p$.

Adèles of $\mathbb{Q}$:

~~The adèle~~ Define $\mathbb{A} := (\prod_p' \mathbb{Q}_p) \times \mathbb{R}$, where $\prod'$ denotes the restricted product, i.e. $\mathbb{A} = \{ x \in \prod_p \mathbb{Q}_p \times \mathbb{R} \mid \text{for almost all } p : x_p \in \mathbb{Z}_p \} = \hat{\mathbb{Q}} \times \mathbb{R} = \mathbb{A}^\infty \times \mathbb{A}_\infty$ where $\mathbb{A}_p$ is the factor at p, $\mathbb{A}^p$ is the factor away from p (where p may also be ∞)

We can make $\mathbb{A}$ into a $\mathbb{Q}$-algebra by

$$\mathbb{Q} \to \mathbb{A} : \underset{x}{\overset{\mathbb{Q}}{\cup\!|}} \mapsto (p \mapsto \underset{x}{\overset{\mathbb{Q}_p}{\cup\!|}} , \underset{x}{\overset{\mathbb{R}}{\cup\!|}})$$

Topological $\mathbb{Q}$-algebra

The topology of $\mathbb{A}$ is not the one induced from the product topology of $\prod_p \mathbb{Q}_p \times$. For the good topology, $\prod_p \mathbb{Z}_p \times \mathbb{R} \subseteq \mathbb{A}$ is open, and has itself the product topology. This topology has some nice properties. For example, it is locally compact; note that the p-adic norm $|\cdot|_p$ on $\mathbb{Q}$ to $\mathbb{Q}_p$ in a unique way, and then $\mathbb{Z}_p$ is the closed unit ball.

Now we can write $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, which is $\varprojlim_n \mathbb{Z}/n\mathbb{Z}$, with respect to the division relations.

Consider the map $\varphi : \mathbb{Z}_2 \to \mathbb{R} : \Sigma_{i\geq 0} a_i \cdot 2^i \mapsto \Sigma_{i\geq 0} a_i \cdot 3^{-i}$ and we claim that φ is a homeomorphism onto C, the standard Cantor set in $[0,1]$, and hence it is not locally constant.
In practice, we want to consider locally constant functions.

Product formula

Let K be a number field. Then define

$$\mathbb{A}_K := (\textstyle\prod'_{\mathfrak{p}\subseteq \mathcal{O}_K} K_{\mathfrak{p}}) \times (\mathbb{R}\otimes_{\mathbb{Q}} K),$$

and note that $K = \mathbb{Q}[X]/(f)$, f irreducible, so $\mathbb{R}\otimes_{\mathbb{Q}} K = \mathbb{R}[X]/(f)$, which is isomorphic to $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$, if f has r_1 real roots and $2r_2$ complex. Also $\mathbb{R}\otimes_{\mathbb{Q}} K = \prod_{v|\infty} K_v = \prod_{\sigma: K \hookrightarrow \mathbb{C} \text{ up to conjugation}} K_\sigma$. We can also think of $\mathbb{A}_K$ as

$$\mathbb{A}_K = \textstyle\prod'_p (\mathcal{O}_p \otimes_{\mathbb{Q}} K) \times \mathbb{R}$$

$$\textstyle\prod_p' (\mathcal{O}_p \otimes_{\mathbb{Q}} K) = \varprojlim_{0\neq I \subseteq \mathcal{O}_K} \mathcal{O}_K/I = \varprojlim_n \mathcal{O}_K/n\mathcal{O}_K .$$

Idèles

The idèles of K is $\mathbb{A}_K^\times$, and we want to define a topology on $\mathbb{A}_K^\times$ such that it becomes a . As a set

$$\mathbb{A}_K^\times = \{x \in \mathbb{A}_K \mid \text{for almost all } \mathfrak{p} : |x|_{\mathfrak{p}} = 1 \text{ for all } v : x_v \neq 0\}.$$

The topology we want is not the topology induced from $\mathbb{A}_K$, as for that topology, the inverse map is not continuous. Also, $\mathbb{A}_K^\times \subseteq \mathbb{A}_K$ is not open. The right topology is given by the map

$$\mathbb{A}_K^\times \hookrightarrow \mathbb{A}_K \times \mathbb{A}_K, \quad x \mapsto (x, x^{-1})$$

and then the image is the closed subset $\{(x,y) \in \mathbb{A}_K^2 \mid x\cdot y = 1\}$. Then take the induc topology. Then $\mathbb{A}_K^\times$ is locally compact and contains $\hat{\mathcal{O}}_K^\times \times K_{\mathbb{R}}$ as an open subset, and $\hat{\mathcal{O}}_K^\times$ has its product topology, where $\hat{\mathcal{O}}_K^\times = \prod_p \mathcal{O}_{K,\mathfrak{p}}^\times$.

Product formula:

There ~~are~~ are a natural norms $\|\cdot\|_v : K_v \to \mathbb{R}$ such that for all $x \in K$: $\prod_v \|x\|_v = 1$.
For $x \in K$, $\|x\|_v$ is the factor by which $x\cdot : K_v \xrightarrow{\sim} K_v$ changes any Haar measu

Ex.: $\mathbb{R} \xrightarrow{\|\cdot\|} \mathbb{R}$, $x \mapsto |x|$
$\mathbb{C} \xrightarrow{\|\cdot\|} \mathbb{R}$, $x \mapsto |x|^2$
$K_{\mathfrak{p}} \xrightarrow{\|\cdot\|} \mathbb{R}$, $x \mapsto |x|^{[K_v:\mathbb{Q}_p]}$
$\cup\!|$
$\# \mathcal{O}_{K_{\mathfrak{p}}}/p\cdot\mathcal{O}_{K_{\mathfrak{p}}} = p^{[K_v:\mathbb{Q}_p]}$

$$\underset{\text{discrete}}{K} \hookrightarrow \mathbb{A}_K \twoheadrightarrow \underset{\text{compact}}{\mathbb{A}_K/K} \qquad \underset{\text{discrete}}{K^\times} \hookrightarrow \mathbb{A}_K^\times \twoheadrightarrow \mathbb{A}_K^\times/K^\times$$

ker $\|\cdot\|$ compact

$$\mathbb{A}_K^\times/K^\times \xrightarrow{\|\cdot\|} \mathbb{R}_{>0}^\times, \quad [x] \mapsto \textstyle\prod_v \|x\|_v$$

~~x 2 preserves~~