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TAG1, 2009/12/07. Some basics of Galois repr., continued.

1. The exercise from last time: maximal compact subgroups, up to conjugation, of $PGL_2(\mathbb{Q}_\ell)$?

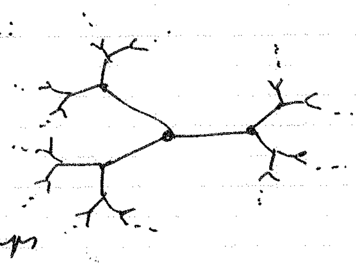
One uses the Bruhat–Tits tree T:

$V(T)$ (vertices) $= \{\mathbb{Z}_\ell\text{-lattices in } \mathbb{Q}_\ell^2\}/\mathbb{Q}_\ell^\times$

$E(T)$ (edges) $= \{\{[M_1],[M_2]\} \mid M_1 \subset M_2, \text{ index } \ell\}$

$PGL_2(\mathbb{Q}_\ell)$ acts on T, T is a tree: $\forall M_1, M_2$ $\mathbb{Z}_\ell$-lattices in $\mathbb{Q}_\ell^2$, $\exists!\, m \in \mathbb{Z}$ s.t. $\ell^m M_1 \subset M_2$ with cyclic quotient; unique path $[M_1] \rightsquigarrow [M_2]$.

T is the $\ell+1$-regular tree: $\ell = 2$:



$PGL_2(\mathbb{Q}_\ell)$ acts transitively on $V(T)$ and on $E(T)$.

<u>Claim</u>. The max. compact subgroups of $PGL_2(\mathbb{Q}_\ell)$ are the stabilisers of vertices and edges.

<u>Vertex</u>: $PGL_2(\mathbb{Z}_\ell) = GL_2(\mathbb{Z}_\ell)/\mathbb{Z}_\ell^\times$

<u>Edge</u>: $\begin{pmatrix} 0 & 1 \\ \ell & 0 \end{pmatrix}$: $\begin{cases} e_1 \mapsto \ell e_2 \\ e_2 \mapsto e_1 \end{cases}$ <u>switches</u> $[\mathbb{Z}_\ell^2]$ and $[\mathbb{Z}_\ell \oplus \ell\cdot\mathbb{Z}_\ell]$

$$\left\langle \left\{ \begin{pmatrix} * & * \\ \ell * & * \end{pmatrix} \in GL_2(\mathbb{Z}_\ell) \right\} \cup \left\{ \begin{pmatrix} 0 & 1 \\ \ell & 0 \end{pmatrix} \right\} \right\rangle / \mathbb{Q}_\ell^\times .$$

<u>Proof</u> Let $K \subset PGL_2(\mathbb{Q}_\ell)$ be a compact subgroup. Then $K' := K \cap PGL_2(\mathbb{Z}_\ell)$ is open in K, hence of finite index, hence $K \cdot [\mathbb{Z}_\ell^2]$ is a finite subset of $V(T)$. Add all the paths between the elements of $K \cdot [\mathbb{Z}_\ell^2]$. This finite tree has a central vertex or a central edge, fixed by K $\boxtimes$ (To find the center: remove end points, then end-vertices, then end points, until only 1 is left.)

2. Ramification

First finite extensions. Let $\mathbb{Q} \to K$ be a finite Galois ext., group G.

Then:
$$\begin{tikzcd} K \arrow[r, hook] & O_K \\ \mathbb{Q} \arrow[u] \arrow[r, hook] & \mathbb{Z} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}(O_K) \arrow[d, "\pi"'] \\ \mathrm{Spec}(\mathbb{Z}) \end{tikzcd}$$

$\circlearrowleft G$ (right action) quotient in the category of loc. ringed spaces (true for any ring A with action of a finite group).

Def. For p prime; $\mathbb{Z} \to O_K$ is unramified $\iff$ $\mathbb{F}_p \otimes_{\mathbb{Z}} O_K$ is reduced

$\iff$ $\overline{\mathbb{F}}_p \otimes_{\mathbb{Z}} O_K$ is reduced

For
$$\begin{tikzcd} m \arrow[r, mapsto] & O_K \arrow[r, two heads] & \mathbb{F} \\ p\mathbb{Z} \arrow[u, mapsto] \arrow[r, mapsto] & \mathbb{Z} \arrow[u, hook] \arrow[r, two heads] & \mathbb{F}_p \arrow[u, hook] \end{tikzcd}$$
: $D_m := \{g \in G \mid gm = m\}$, decomposition group of m.

Note: $D_{gm} = g D_m g^{-1}$; given p, the D_m form a conj. class of subgroups.

Then: $I_m \hookrightarrow D_m \twoheadrightarrow \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p) = \mathbb{Z}/(\dim_{\mathbb{F}_p} \mathbb{F})\cdot\mathbb{Z}$

$\mathrm{Frob}_m : \; x \mapsto x^p \quad \longleftarrow\!\!| \; 1$

I_m: inertia group at m

$I_m = \{1\} \iff p$ unramified

need these later, to define/understand "conductor".

Filtration by higher ramification subgroups:

for $i \in \mathbb{Z}_{\geq 0}$: $G_i := \{ g \in D_m \mid g = 1 \text{ on } O_K/m^{i+1} \}$

Hence $G_0 = I_m$, $G_0 \supset G_1 \supset \cdots$, $G_i = \{1\}$ for $i \gg 0$.

And: G_1 is a p-group, $G_0/G_1 \hookrightarrow \mathbb{F}^\times$, prime to p.

Examples. 1. $K = \mathbb{Q}(i)$ $\quad \mathbb{Z} \to \mathbb{Z}[x]/(x^2+1) = \mathbb{Z}[i]$, $G = \{1, \sigma\}$, $\sigma: i \mapsto -i$.

Unram. at all $p \neq 2$.

Take $p = 2$, unif. at m: $\epsilon := 1+i$ $\quad (m = (1+i))$. $\quad G_0 = G$.

Then $\sigma\epsilon - \epsilon = (1-i) - (1+i) = -2i$, $v_m(-2i) = 2$, $\sigma = 1$ on $\mathbb{Z}[i]/m^2$.

hence (for $j \geq 0$): $\sigma \in G_j \iff j \leq 1$. $\quad G_0 = G_1 = G$, $\quad G_2 = \{1\}$.

2. $K = \mathbb{Q}(\sqrt{2})$, $\mathbb{Z} \to \mathbb{Z}[x]/(x^2-2) = \mathbb{Z}[\sqrt{2}]$, $\sigma: \sqrt{2} \mapsto -\sqrt{2}$, $m = (\sqrt{2})$, $\mathbb{F} = \mathbb{F}_2$

$v_m(\sigma(\sqrt{2}) - \sqrt{2}) = v_m(2\sqrt{2}) = 3$.

So, here: $G_0 = G_1 = G_2 = G$, $\quad G_3 = \{1\}$.

$\exists$ Nice relation with $\Omega^1_{O_K/\mathbb{Z}}$.

Def. For p unram. in $\mathbb{Z} \to O_K$, $m \mid p$: $\quad D_m \xrightarrow{\sim} \mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$

$\mathrm{Frob}_m \longleftarrow\!\!| \; (x \mapsto x^p)$

So we get, from p, a conj. class of Frobenius elements. (arithmetic Frobenius.)

Infinite extensions.

$$\begin{array}{ccc} \overline{\mathbb{Q}} & \supset & \overline{\mathbb{Z}} \\ \uparrow & & \uparrow \\ \mathbb{Q} & \supset & \mathbb{Z} \end{array}$$

not noetherian...

$$\overline{\mathbb{Z}} = \varinjlim_{K} \mathcal{O}_K$$

$$\mathrm{Spec}(\overline{\mathbb{Z}}) = \varprojlim \mathrm{Spec}(\mathcal{O}_K)$$

Let p be prime.

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \circlearrowright (\mathrm{Spec}\,\overline{\mathbb{Z}})(\overline{\mathbb{F}_p})$, transitively. (follows from finite level case).

$$(\mathrm{Spec}\,\overline{\mathbb{Z}})(\overline{\mathbb{F}_p}) = \mathrm{Hom}(\overline{\mathbb{Z}}, \overline{\mathbb{F}_p})$$

For $\mathfrak{m} \rightarrowtail \overline{\mathbb{Z}} \twoheadrightarrow \overline{\mathbb{F}_p}$:

$$I_\mathfrak{m} \rightarrowtail D_\mathfrak{m} \twoheadrightarrow \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) = \hat{\mathbb{Z}}.$$

$$D_\mathfrak{m} = \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \quad : \quad \overline{\mathbb{Q}_p} = \varinjlim_{K} K_{\mathfrak{m}\cap K}.$$

The filtration with ramification subgroups must be <u>renumbered</u> in order to pass to the limit $D_\mathfrak{m}$: <u>upper numbering</u> (indices in $\mathbb{Q}$).